CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2014

Contents

09:00am Classes

Infinite Trees. (🌶🌶🌶, Susan, Tuesday–Saturday)
Continuation of Week 1 class.

Homework: Recommended

Prerequisites: Infinite Trees, Week 1

Related to (but not required for): Continuum Hypothesis

Evasiveness. (🌶🌶🌶🌶, Tim!, Tuesday–Saturday)
In this class, we'll explore a conjecture in computer science that has been open for over 40 years, concerning the complexity of graph properties. Just as the complexity of a problem (like the problem of determining whether a given number is composite) can be quantified by how long it takes a computer to solve it, one can also come up with a good way of measuring the complexity of a graph property (like "is planar" or "contains a 3-cycle"). The graph properties with maximum complexity are called evasive, and the conjecture is that a huge class of graph properties — specifically, all those that are nontrivial and monotone — are evasive.

We'll trace the story of this conjecture through time from its conception 1973 to the present day, including research from the past few years. Along the way, we'll see decision-tree complexity, scorpion graphs, clever counting, collapsible simplicial complexes, transitive permutation groups, and hypergraph properties.

This class is directly related to my research, and in class we'll see a result I proved this year, and elements of its proof.

Homework: Recommended

Prerequisites: Group Theory (normal subgroups, quotient groups)

Combinatorial Game Theory. (🌶🌶, Mira, Tuesday–Saturday)

Let's play a berry-eating game (my favorite kind!). We have a plate with 14 blueberries, 20 blackberries, and 17 raspberries, and we're going to take turns eating them. On your turn, you may eat as many berries as you want (as long as you eat at least one), but they all have to be berries of the same type. Then it is my turn. The winner of the game is the person who eats the last berry. Who's going to win?

This is the game of Nim. It turns out that this innocent-looking game has enormous significance: all other games where the two players have the same options (unlike, say, chess, where you only move the black pieces and I only move the white ones) are equivalent to Nim! In this class, you'll discover how to win at Nim, prove that it is the universal impartial game, and use this fact to develop strategies for lots of other games. I will do my best to guide you through this process rather than giving away the punchline: discovering this beautiful theory by yourself is half the fun.

If you know the strategy for Nim, but not the other stuff in the blurb, you can skip the *second* day of the class (Wednesday). By the way, combinatorial game theory has very little to do with regular game theory, which deals with games of incomplete information (like the famous Prisoner's Dilemma). If you're interested in games like that, go to Joe Halpern's course this week. Better yet, go to both courses! ☺

Electricity and Graphs. (🌶🌶, Paddy, Tuesday–Saturday)
Oh nyo! You went out for a walk around campus and are now lost. Conveniently, you are a mathematician! As such, you have decided to generalize your specific problem of being lost to a *slightly* broader problem:

- Space: instead of being lost on campus, why not be lost on $\mathbb{Z}^d$, the d-dimensional integer lattice?
- Behavior: instead of trying to remember which way home is, why not wander randomly?

Surely these generalizations will help us.

So: will we find our way back to the dorms? Or is there a chance that we will wander forever? And how does this connect to electrical circuits and graphs?

Come to this class to find out!

Homework: Recommended

Prerequisites: Graph theory, or at least the knowledge of what a graph is. Some gullibility with respect to probability.

Related to (but not required for): All the other graph theory classes.

Complex Analysis. (🌶🌶🌶, Kevin, Tuesday–Saturday)
This class is a continuation of Week 1's Complex Analysis class. Whereas the first week sprinted towards the powerful technique of computing contour integrals via residues, this week will focus on proving further phenomena exhibited by complex functions. These surprising results often have no analog in the real setting. For example, as a function of a real variable, $f(x) = \sin x$ is differentiable everywhere and bounded. But any complex function which is differentiable everywhere and bounded is automatically constant! We'll also prove the Fundamental Theorem of Algebra at least twice as corollaries of these results.

It's difficult to jump into this class without the first week, but if you have some background in contour integration, you may be able to manage. Talk to me if you're interested!

Homework: Recommended

Prerequisites: Complex Analysis Week 1

Related to (but not required for): Real Analysis, Analytic Number Theory, Polynomial Fermat's Last Theorem

10:00am Classes

Moore Method Point-Set Topology. (🌶🌶🌶🌶, Alfonso, Tuesday–Saturday)
This is the second of our four weeks of Moore Method point-set topology. If you did not attend during the first week but have some background in topology, you may be able to join at this point. Talk to me if you are interested.

Homework: Required

Prerequisites: None

Category Theory. (🌶🌶🌶🌶, Don, Tuesday–Saturday)
Many different topics in math have a notion of a map between two objects: sets have functions, groups have group homomorphisms, rings have ring homomorphisms, topological spaces have continuous functions, posets have order-preserving maps, graphs have graph homomorphisms. Some of these also have a notion of a product: Cartesian product of sets, direct product of rings or groups, the product topology on the cartesian product of topological spaces. If you've seen two or three of these before, you may have noticed some similarities between the different definitions.

Category theory is a language used to describe these kinds of similarities. It's a framework that almost any kind of math fits into, and often by putting a subject into the language of category theory, interesting questions arise. Additionally, you can use category theory to prove many theorems at once — by proving them in the general case. In this class, we'll learn about categories, functors (maps between categories), and limits and colimits (constructions that can exist across different categories, like products).

Homework: Required

Prerequisites: Familiarity with abstract algebra; the more topics from Ring Theory, Universal Algebra, Group Theory, Linear Algebra you're comfortable with, the better.

The Geometry of Spacetime. (🌶🌶, Aaron Fenyes, Tuesday–Saturday)
General relativity, Einstein's famous theory of gravity, was the first manifestation of an idea that has become a major theme in modern physics: the idea that the laws of physics are intimately connected to the geometry of the universe. Just as you might introduce yourself to the geometry of space by exploring simple spaces like planes, spheres, and hyperboloids, we'll get a feel for the geometry of spacetime by exploring some of the simplest geometries spacetime can have.

In lectures, I'll paint a soft, intuitive picture of how the spacetimes we're playing with look and behave. Details will be left to the homeworks, where I'll help you prove and calculate to your heart's content. The lectures should be understandable without the homework, although they may not feel very solid.

Homework: Recommended

Prerequisites: Linear Algebra

Linear and Nonlinear Systems of Differential Equations. (🌶🌶🌶, Mark, Tuesday–Saturday)
Many models have been devised to try to capture the essential features of phenomena in economics, ecology, and other fields using systems of differential equations. One classic example is given by the Volterra-Lotka equations from the 1920s:

$$\frac{dx}{dt} = -k_1 x + k_2 xy\,; \qquad \frac{dy}{dt} = k_3 y - k_4 xy\,,$$

in which x, y are the sizes of a predator and a prey population, respectively, at time t, and k_1 through k_4 are constants. There are two obvious problems with such models. Often the equations are too hard to solve (except, perhaps, numerically); more importantly, they are not actually correct (they can only hope to approximate what really goes on).

On the other hand, if we're approximating anyway and we have a system

$$\frac{dx}{dt} = f(x,y)\,; \qquad \frac{dy}{dt} = g(x,y)\,,$$

why not approximate it by a *linear* system such as

$$\frac{dx}{dt} = px + qy\,; \qquad \frac{dy}{dt} = rx + sy\,?$$

Systems of that form can be solved using eigenvalues and eigenvectors, and usually (but not always) the general behavior of the solutions is a good indication of what actually happens for the original (nonlinear) system if you look near the right point(s). If this sounds interesting, come find out about concepts like trajectories, stationary points, nodes, saddle points, spiral points, and maybe Lyapunov functions. Expect plenty of pictures, and probably an opportunity for some computer exploration using *Mathematica* or equivalent. (If you don't want to get involved with computers, that's OK too; most homework will be doable by hand.)

Homework: Recommended

Prerequisites: Linear algebra (eigenvectors and eigenvalues), calculus, a little bit of multivariable calculus (equation of tangent plane)

Related to (but not required for): Some applications of mathematics.

Pythagorean Triples and Unsolvable Equations. (🌶🌶, David Roe, Tuesday–Saturday)
You've seen the definition of modular arithmetic, either in Week 1 or outside of Mathcamp. But what is it good for?

The field of Diophantine equations, where you look for integer solutions to polynomial equations, is one of the oldest in mathematics. Modular arithmetic gives us insight into the difference between $x^2 + y^2 = z^2$ (there are infinitely many Pythagorean triples) and $x^2 + 2y^2 = z^2$ (the only solution is $(0, 0, 0)$).

We'll spend most of the week on such quadratic equations, learning how to tell whether or not they have nontrivial solutions and understanding those solutions when they do. We'll close with a brief discussion of the (much trickier) case when the degree is bigger than two.

Homework: Recommended

Prerequisites: A Quick Introduction to Number Theory or a good understanding of modular arithmetic, including division modulo primes.

Related to (but not required for): A Quick Introduction to Number Theory, When Factoring Goes Wrong

11:00am Classes

Models of Computation as Strong as Programming. (🌶🌶, Pesto, Tuesday–Saturday)
All programming languages are basically the same: even a language as wonderful as (insert your favorite programming language here) can't do anything (insert your least favorite programming language here) couldn't, if you gave it enough extra computation time, extra memory, and extra programmer time.

So, let's ignore all the programming languages and just talk about what computers *can* do ("decide") and what they can do feasibly ("in P"). For instance, here's a problem of the sort we'll consider:

Consider functions from $\{0, 1\}^{2014}$ (the vertices of a 2014-dimensional cube) to the integers. Suppose you can, given a formula for such a function f, find out whether there's any input x such that $f(x) = 0$ (more quickly than by checking all 2^{2014} possible values of x). Can you find out (also quickly) any particular input x for which $f(x) = 0$?

Independent of last week's class.
Homework: Recommended
Prerequisites: None
Related to (but not required for): Models of Computation Simpler than Programming

Fractal Zoo. (🌶🌶, Jeff, Tuesday–Saturday)
Let's take a safari! Our journey will focus on strange and beautiful creatures called fractals. With luck, you will see the nearly invisible Cantor dust, the magnificent Sierpinski gasket, and the dangerously large Peano curve — all sorts of wildlife that defy common intuition. With every encounter, you'll be better prepared to handle the unusual metric spaces you may find in the wilderness as a mathematician. All of these fractals are metric spaces — which is a shape that comes with a notion of distance. Oftentimes these spaces are familiar and well-behaved. . . however in this course we will take a tour of the more exotic spaces that one can find in metric topology.
Along the way, we'll develop tools to better understand and classify fractals, topologically define dimension, and see a metric space that consists of metric spaces. At the end of our trip, we will build a fractal zoo to contain all of these mysterious beasties.
Homework: Recommended
Prerequisites: None
Related to (but not required for): Moore Method Point-Set Topology, Real Analysis

Counting Conics. (🌶🌶🌶, Sachi, Tuesday–Saturday)
In 1848, Jacob Steiner posed the question: "Given five conics in the plane, are there any conics tangent to all five of them, and if so how many?" Problems that count the number of geometric objects with a particular property are known as enumerative problems in algebraic geometry, and solving these problems often requires deep geometrical insights. Steiner gave the answer 7776, which turns out to be incorrect. We will solve Steiner's problem (correctly) and pose a number of other questions about counting conics that contain specific points and are tangent to given lines and conics. In order to solve these problems, we will have to make sense of how to count conics, and convince ourselves that these problems are actually well posed. For this, we will delve into the world of algebraic geometry: we will use projective geometry and the concept of a moduli space to solve some easier counting problems, and then use more advanced tools like intersection theory and blow-ups to find the correct solution to Steiner's problem.
Homework: Recommended
Prerequisites: Some linear algebra helpful but not required
Related to (but not required for): Cubic Curves, Polynomial Fermat's Last Theorem

Galois Theory. (🌶🌶🌶🌶, Mark, Tuesday–Saturday)
You may have heard the story of the brilliant mathematician and societal misfit Galois, who died tragically young in a duel after developing an exciting new area of mathematics that was beyond what most of his contemporaries could even follow. In the first week of this class we will build up to the fundamental theorem of Galois theory, which gives an unexpected and beautiful correspondence allowing us to find and describe field extensions in terms of subgroups of a certain group. (We'll first go over what field extensions are, and what they can be used for. As an example, if you were never comfortable with defining the complex numbers by postulating the existence of a square root of -1, you'll see early on how we can "make" a square root of -1 using polynomials. You'll also see why some ancient and (in)famous construction problems, such as squaring the circle, are provably impossible.) In the second week we'll finish the proof of the fundamental theorem, and then move on (perhaps with some side trips to scenic overlooks of other material) to sketch the proof that there is no general

way of solving polynomial equations of degree 5 or more by radicals; that is, there is no analog of the quadratic formula for degree 5 and higher. (There are such analogs for degree 3 and 4.)

This class is two weeks long and will continue in Week 4.

Homework: Recommended

Prerequisites: Linear algebra, basic group theory, ring theory (including the First Isomorphism Theorem for rings); some experience/comfort with polynomial rings would help.

The Banach-Tarski Paradox. (🌶🌶🌶, Mark Sapir, Tuesday–Thursday)
A continuation of the Week 1 class.

Homework: Recommended

Prerequisites: Week 1 of The Banach-Tarski Paradox

Möbius Transformations and the Night Sky (or, an Ape Pointing at the Stars). (🌶🌶🌶🌶, Aaron, Friday)

The night sky looks like a sphere. (I'll explain why you should be surprised by this, and also why it's true.) If you change your velocity, the constellations in the night sky change shape. (I'll explain why you shouldn't be surprised by this, and also why it's true.) These two facts lead to a beautiful and unexpected connection between the Lorentz transformations (the symmetries of spacetime in special relativity), and the Möbius transformations (the symmetries of the sphere in complex analysis).

Prerequisites: Familiarity with linear transformations and complex numbers is necessary. Familiarity with the Lorentz transformations (covered on the first two days of Geometry of Spacetime) is nice, but not required.

Related to (but not required for): Geometry of Spacetime

Surreal Numbers. (🌶, Alex, Saturday)

If you've ever constructed the reals, then you know it's kind of a drag:

- Start with 0.
- Put in 1.
- Use addition to get $\mathbb{N}$
- Use subtraction to get $\mathbb{Z}$.
- Put in fractions to get $\mathbb{Q}$.
- Complete the Cauchy sequences to get $\mathbb{R}$.

That's a big number of steps! In this class, we'll give a single definition that yields not only the reals, but a much richer extension that has a proper subfield of every cardinality. (In other words, it's *huuuge*.) If you've never seen transfinite induction, you're in for a treat!

Homework: None

Prerequisites: None

1:00pm Classes

Compressed Sensing. (🌶🌶🌶, Soledad Villar, Tuesday–Saturday)
Compressed sensing is a recently developed mathematical breakthrough that explains how one can reconstruct some data from a very small portion of it (sometimes as little as 2% of it). Wait... what? What about the pigeonhole principle? Well, the trick is that most data that we deal with (like pictures and videos) are not just random data; they have some underlying structure that causes them to lie

in a lower dimensional space. Compressed sensing techniques identify and exploit that structure. But it is not just about compressing. It also allows us to acquire an already compressed version of the data by taking a few measurements, making magnetic resonance imaging possible. In this class we are going to learn the mathematical principles behind compressed sensing and how we can apply them to facial recognition.

Homework: Recommended

Prerequisites: Linear algebra.

Congruent Numbers and Elliptic Curves. (🌶🌶🌶🌶, Ruthi, Tuesday–Saturday)
Let n be a positive integer. We call n a congruent number provided that there is a right triangle that has all rational sides and area n. Can we find an easy way of determining whether n is a congruent number? Tunnell's theorem gives us a way, but it uses hard and modern mathematics: the theory of elliptic curves and modular forms, which are fundamental objects of study in advanced number theory. It is also dependent on a weak form of an unsolved conjecture called the Birch and Swinnerton-Dyer Conjecture. (Cool fact: a proof — or counterexample — of this conjecture is worth a million dollars!)

In this class we will: discuss congruent numbers and Tunnell's theorem, define elliptic curves and their basic properties, explain why answering questions about elliptic curves will tell us about congruent numbers, and get some idea about how this leads us to Tunnell's result.

This class is two weeks long and will continue in Week 3.

Homework: Required

Prerequisites: Modular arithmetic. It will help to be familiar with abelian groups and fields, but I will introduce everything you need.

Related to (but not required for): Cubic Curves, Polynomial Fermat's Last Theorem, Introduction to Ring Theory

exp. (🌶🌶, *Mike Hall*, Wednesday–Saturday)

At some point in school, you happily began writing down things like $2^{\sqrt{2}}$ and $e^\pi - \pi^e$, but did your teacher actually define what it means to take irrational exponents? 'Normally', one learns many abstract calculus and analysis theorems before this really gets resolved entirely, but who wants to wait that long?

In this class, as an excuse to play around in the real number system, we'll try to define the exponential function with as little machinery as possible. Along the way, we'll discuss least upper bounds and limits a bit, prove the arithmetic mean-geometric mean inequality, and to learn how to pronounce 'Cauchy'.

This class will start at the 1-chili level, rising to 2-chili by the end of the week.

Homework: Recommended

Prerequisites: None

Related to (but not required for): Real Analysis, Fractal Zoo, Scandalous Curves

Reasoning about Knowledge and Uncertainty, with a Bit of Game Theory. (🌶🌶, *Joe Halpern*, Wednesday–Saturday)
This course provides a general discussion of approaches to reasoning about knowledge and uncertainty, and its applications to distributed systems, artificial intelligence, and game theory. I'll start by considering a simple yet powerful formal semantic model for knowledge and a language for reasoning about knowledge whose underlying idea is that of "possible worlds". I'll show how that approach gives lots of insight into puzzles like the muddy children puzzle and coordinated attack, as well as famous theorems

in economics like "we can't agree to disagree". I'll extend these ideas to allow probabilistic reasoning as well. Finally, I'll give a brief introduction to game theory and apply these ideas to defining some standard solution concepts in game theory, like Nash equilibrium and rationalizability.
Homework: Recommended
Prerequisites: None
Related to (but not required for): Bayesian Statistics, The Laws of Thought

Problem Solving: Number Theory. (🌶🌶🌶, Misha, Tuesday–Saturday)
If n, k are positive integers with $k \leq n$, then the product

$$\binom{n}{k}\binom{n}{k-1}$$

of binomial coefficients is divisible by n.

In this class we will spend almost all our time solving and discussing olympiad problems in number theory, such as the proof of the statement above (which came from the 1986 British MO).

I will sometimes talk about techniques and important theorems, but problems that can be solved just by applying these are boring. When I do decide to talk instead of making you do math, I'll mainly focus on good ways to think about number theory problems.
Homework: Required
Prerequisites: Familiarity with modular arithmetic
Related to (but not required for): Other problem solving

Colloquia

Knowledge and Common Knowledge in Multi-Agent Systems. (*Joe Halpern*, Tuesday)
Reasoning about knowledge – what I know about what you know about what I know ... – is the type of reasoning that is often seen in puzzles and paradoxes, and has been studied at length by philosophers. Recently, it has been shown to play a key role in a surprising number of other contexts, from understanding conversations to the analysis of distributed computer algorithms. I'll start the talk by considering a number of well-known puzzles and paradoxes, which both illustrate the subtleties of reasoning about knowledge and the advantages of having a good framework in which to make this reasoning precise. These puzzles also turn out to be closely related to important problems in distributed computing and game theory. In particular, they emphasize the importance of the notion of common knowledge, which turns out to be essential for reaching agreements and coordinating action. Unfortunately, we can prove that in practical multi-agent systems, common knowledge is not attainable. This seems somewhat paradoxical. How can common knowledge be both necessary and unattainable? The paradox gets resolved (to some extent) by examining a number of variants of common knowledge that turn out to be both attainable and sufficient for many applications.

Dance of the Astonished Topologist. (Alfonso, Wednesday)
A covering space of a topological space (for example a surface or a curve) is what you get when you "unfold" it. For instance, you can unfold a circle entirely and get a line, or unfold it partially and get... another circle. You could also unfold a torus, and get another torus, a cylinder, or a plane. You can unfold almost anything, like a Klein bottle or $GL(n)$, but you cannot unfold a Hawaiian ring. Interestingly, when we unfold a topological space, paths that started and ended at the same point end up wandering in space, creating something called monodromy.

Covering spaces have many applications in daily life, such as Lie groups, quantum field theory, or square dancing. What does square dancing have to do with covering spaces? Usual square dances have 8 dancers, but there is a 12-dancer variant called "hexagon squares" (sorry, I did not name it).

SD callers often go through a lot of trouble to explain the rules for hexagon squares, and are usually at a loss to figure out when a choreography that resolves in regular squares will resolve in hexagon squares. Their lives would be so much simpler if they simply said "hexagon squares are a triple cover of the quotient of regular square dancing by a $\mathbb{Z}/2\mathbb{Z}$ symmetry and a choreography that resolves in regular squares also resolves in hexagon squares if and only the path of every boy composed with the path of his girl has winding number around the center congruent to 0 mod 3." In other words, this is a real-life application: a question posed by dancers that algebraists managed to solve.

This talk will be illustrated with shiny animations, courtesy of Ryan Hendrickson. No topology or dancing knowledge will be assumed.

Analogue Black Holes and Burning Pumpkins. (*Mike Hall*, Thursday)
Black holes are regions of space-time from which light rays can not escape. Scientists have never had the opportunity to observe a black hole up close, and probably never will. To better understand black holes, one could still hope to find something else that behaves similarly enough to provide insight, but is possible to study in a laboratory. As it turns out, a model called the linear wave equation, well known in classical physics and mathematics, can display many of the same features as relativistic black holes, and also provides a model for such physical phenomena as acoustic waves in a moving fluid and light propagation in an inhomogeneous moving medium. While no one has yet done so, many people hope it will be possible to construct a 'sonic black hole' or an 'optical black hole' in a lab. With chalk, hand gestures, and pictures of burning pumpkins, I will describe such models in intuitive terms, explain some of the mathematics involved in making our intuitive ideas rigorous, and explain why these 'analogue' black holes may have event horizons with corners.

From Binary Quadratic Forms to $2 \times 2 \times 2$ Integer Cubes. (*David Roe*, Friday)
Can any number of the form $29m^2+34mn+10n^2$ be written as the sum of two squares? In 1798, at the age of 21, Gauss published *Disquisitiones Arithmeticae*, the seminal work on such "binary quadratic forms." In 2004, Bhargava gave a new interpretation of a central part of this theory, which has proven instrumental in some amazing recent progress in number theory (last week Bhargava, Skinner and Zhang posted a preprint "solving" one of the Clay million dollar prizes in 66% of the cases). I will show you some highlights from this 216 year old story.

Visitor Bios

David Roe. David has been involved with Mathcamp almost every summer since 1999, as a camper, JC and mentor. This year, his wife Victoria will be coming with him! They both study number theory (he's a postdoc and she's a graduate student), and love rock climbing, hiking, and meeting awesome people like you.

Joe Halpern. Joe Halpern is a professor and department chair of the Computer Science department at Cornell. He works at the intersection of computer science, math, economics, philosophy, and psychology, on topics like game theory, decision theory, and causality. Before getting his Ph.D. (in Math), he spent two years as head of the Math department at Bawku Secondary School in Ghana. He loves thinking about puzzles and paradoxes (and will talk about a few at Mathcamp). Check out his website at `http://www.cs.cornell.edu/home/halpern`

Mike Hall. Mike Hall (a.k.a. Mmmmmike) has been coming to Mathcamp since 2001, as a camper, JC, and mentor. He has just finished a post-doc at UCLA and will be spending next year in Paris. His first exposure to analysis was at Mathcamp 2001, and now he does semi-classical microlocal analysis. What have we wrought?!

Soledad Villar. Soledad used to be a math olympiad kid back in the day in Uruguay. Like every other math olympiad kid, she loves number theory and even knows a little about it. Now she is a math PhD student in Austin, Texas, studying applied mathemtics. Her research deals with problems that are known to be computationally intractable (NP-hard) but can sometimes be approximated using convex optimization. Some examples of this are clustering algorithms (where you have some data that you want to partition given a similarity criterion), and compressed sensing (which you could find out about in my class). Did you ever notice that Facebook suggests to you who to tag in a picture? Yes, they can recognize your face! She is going to talk about how to apply compressed sensing to face recognition.