CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2014

Contents

9:00am Classes

Moore Method Point-Set Topology. (🌶🌶🌶🌶, Alfonso, Tuesday–Saturday)
This is the third of our four weeks of Moore Method point-set topology. We will explore compactness, the product topology, and various of their cousins. If you did not attend during the first two weeks but have some background in topology, you may be able to join at this point. Talk to me if you are interested.

Homework: Recommended

Prerequisites: None

Related to (but not required for): Scandalous Curves

On Beyond i. (🌶🌶, Steve, Tuesday–Saturday)
There is a nice progression of number systems, $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$: we start with the natural numbers, and at each stage we fix some problem. So, for example, we go from $\mathbb{Q}$ to $\mathbb{R}$ to "fill in the holes," and we go from $\mathbb{R}$ to $\mathbb{C}$ so that equations like $x^2 + 1 = 0$ will have solutions. Once we get to $\mathbb{C}$, though, we seem to be done: there are no holes as in the case of $\mathbb{Q}$, and the fundamental theorem of algebra tells us that every polynomial which is not constant already has a root over $\mathbb{C}$. So there's no need to keep going. So let's keep going! Having only *one* square root (up to $\pm$) is boring. We want more! There are number systems past the complex numbers - strange things like the quaternions, the octonions, and the sedenions - which satisfy this perfectly normal craving. In this class, we'll begin by playing around with these systems, and then turn to the underlying bit of abstract mathematics, the *Cayley-Dickson construction*, which lets us build these and many others. Oh, and we'll also look at reasons why someone might be interested in these systems, other than curiosity and a love of the bizarre.

Homework: Recommended

Prerequisites: Familiarity with complex numbers

Related to (but not required for): Universal Algebra, Introduction to Ring Theory

Random Graphs. (🌶🌶🌶, Misha, Tuesday–Saturday)
Pick a graph on n vertices by flipping a (possibly biased) coin for each possible edge between them, and including the edges for which the coin lands heads. What will this graph look like?

That may seem like a silly question: any graph you could possibly picture can be obtained from this process. But some properties of a graph will hold almost always for it: as n goes to infinity, their probability approaches 1. For example, if our coin is fair, we can almost always choose $1.99 \log_2 n$ vertices for which there's an edge between every pair; we can almost never choose $2 \log_2 n$ vertices in this way.

In this class, we'll use techniques of the probabilistic method to prove results of this flavor, and answer questions such as the following: if we add randomly chosen edges to a graph one at a time, how many will we need to add before it contains a Hamiltonian cycle?

Homework: Recommended

Prerequisites: Graph theory (being comfortable with the definitions)

Related to (but not required for): Introduction to Graph Theory, Problem Solving: The Probabilistic Method

Polynomial Fermat's Last Theorem. (🌶🌶🌶, Sachi, Tuesday–Saturday)
Fermat's Last Theorem famously states that there are no nontrivial integer solutions to the equation $x^n + y^n = z^n$ for $n > 2$. I know a truly marvelous proof of this, which is too long to show in five classes. Instead, we will work out the answer to a related problem: is it possible to solve the equation $x(t)^n + y(t)^n = z(t)^n$ where $x(t), y(t)$ and $z(t)$ are polynomial equations in t? This is a fantastic new problem, which is much easier to solve. We will give a proof that no non-constant polynomials $x(t), y(t)$, and $z(t)$ can satisfy this equation. Furthermore, the proof is incredibly geometric in nature: this polynomial Fermat problem reduces to the problem of whether there are any nice functions from a sphere to a torus. Also in this class, we will talk about other results in algebraic geometry that yield to analytic and topological methods. Two other topics we will touch on are hyperbolic spaces and elliptic curves (a particularly nice kind of degree 3 curve). For example, there is a way of making sense of how to do addition with points on an elliptic curve, but it is not possible to have a consistent way of adding points on any other degree curve.

Homework: Optional

Prerequisites: Complex Analysis Week 1

Related to (but not required for): Complex Analysis, Analytic NT

A Hat Problem. (🌶, Alex, Tuesday)
To start, here's a problem you've probably heard before: n prisoners stand in a circle, each wearing a hat that is either black or white. Each prisoner can see the hats of all the others, but not her own. Starting with some first prisoner and traveling around the circle, each prisoner is asked to guess her own hat color: if she guesses correctly, she's set free, but if she guesses incorrectly, she's killed. They're allowed to invent a strategy beforehand; what strategy should they invent so that in the worst case scenario, the most prisoners are saved?

If the prisoners are allowed to hear previous guesses and factor those into their own guesses, the best strategy saves $n - 1$ prisoners. (I won't spoil the strategy here, for those who've not heard it before)

In this class, we'll ask the question: what if the prisoners are deaf? Can they save anyone? How many can they save? Come and find out!

N.B. I haven't been able to find results on this problem anywhere online. If you can solve the homework problems, you'll know more about this than I do!

Homework: Optional

Prerequisites: None

Mathematical Magic. (🌶, Don, Wednesday–Saturday)
The art of magic frequently relies on sleight of hand and misdirection in order to fool the audience into believing that the impossible has happened before their eyes. In this class, we will learn about magic tricks that require no such deception; rather, they work because they are based on mathematical principles.

Take, for example, the Gilbreath principle: arrange a deck so that the colors of the cards alternate red, black, red, black, and so on. Deal out around half the deck, then riffle shuffle the two halves together: you may be surprised to find that every pair of cards, taken from the top of this shuffled deck, consists of a red card and a black card.

At the start of each class, I will present a magic trick that works via math, instead of deception. It will then be your job to "solve" the trick, and formulate a principle that explains how it works. We'll then work together to prove this principle, and see how else it could be applied.

While we'll be learning about all of this math in the context of magic tricks, much of it has far more serious applications; the Gilbreath principle, for example, has applications to the Mandelbrot set and to Penrose tiles. Other tricks will have connections to computer algorithms and machine navigation. Ultimately, though, isn't magic the best application you could ever ask for?

Homework: Optional

Prerequisites: None

10:00am Classes

Grading Proofs Made Easy: Probabilistically Checkable Proofs of Proximity. (🌶🌶🌶, Tim!, Tuesday–Saturday)

Say you are teaching calculus, and your students turned in a pile of proofs that you have to grade. But you don't really want to read all those papers; you'd rather just glance at each of them in a couple places then write a score.

If you require your students to write their proofs in a very specific format, then your job becomes much easier! You can just check three letters of the proof at random — the 8th letter is a "b", the 59th letter is a "q", and the 512th letter is a "t". If the proof is correct, you'll mark it right with certainty, and if it's incorrect, you'll mark it wrong with probability proportional to how far off the proof is from a correct proof.

In this class, we'll discuss property testing and develop this format of proof (although as a disclaimer, the format is too cumbersome to use in real calculus classes). An indispensable piece of the puzzle is an unusual proof of Arrow's Theorem, which says that there are no fair election systems. Although it seems like an unrelated topic, it will show up in the Mirror of Erised as exactly what we need.

Homework: Recommended

Prerequisites: None

Analytic Number Theory. (🌶🌶🌶🌶, Kevin, Tuesday–Saturday)
Mathematicians have studied the integers and the prime numbers for thousands of years, and Euclid's proof that there are infinitely many prime numbers is still the standard one that everyone knows.

But that's not the number theory we're doing in class. With the development of modern analysis and algebra, we can prove tremendously stronger results. For example, we will start by proving Dirichlet's Theorem that, given two relatively prime positive integers a and n, there are infinitely many primes congruent to a mod n, a vast generalization of the simple fact that there are infinitely many primes.

To do this, we'll use complex-analytic techniques and ideas to study functions like the Riemann zeta function, and we'll see how the analytic properties of these functions tell us about integers and the primes.

Depending on how much time we have after Dirichlet's Theorem, we may see how the Riemann hypothesis tells us about the distribution of primes. We may even get to state and discuss consequences of the generalized Riemann hypothesis!

Homework: Recommended

Prerequisites: Complex Analysis Week 1

Related to (but not required for): When Factoring Goes Wrong, Polynomial Fermat's Last Theorem

Quantum Mechanics in Pictures. (🌶🌶🌶, Aaron, Tuesday–Saturday)
Quantum mechanics has developed a reputation for being esoteric and counterintuitive, but it doesn't have to be that way! Over the last decade or two, mathematical physicists have discovered new ways of thinking about and working with physical theories that reveal some of the rhyme and reason behind the quantum world's sometimes strange-looking behavior. Taking advantage of this modern point of view, we'll learn the conceptual basics of quantum mechanics, use it to analyze some simple lab procedures, find out why quantum models are sometimes more useful than classical ones, and hopefully decide that quantum mechanics isn't quite as bizarre as it's often made out to be. (Behind the scenes, this class will be powered by category theory—in particular, the theory of bicategories. If you're interested in applications of category theory in physics, you might enjoy the class, although I won't talk about categories at all. I don't understand all the categorical details of what we'll be doing, but I'd love to spend time puzzling through it together at TAU!)

Homework: Recommended

Prerequisites: Linear Algebra, experience with complex numbers

IMO P6 Research. (🌶🌶🌶, Po–Shen Loh, Tuesday–Wed)
This year, problem 6 on the International Math Olympiad turned out to be a semi-open research problem. The originally proposed problem is solvable using basic methods, but it is still open to determine the best possible result. A set of lines in the plane is in *general position* if no two are parallel and no three are concurrent. A set of lines in general position cuts the plane into *cells*, some of which have finite area; we call these *finite cells*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite cells has a completely blue boundary. What's the best bound that you can get? We'll improve the stated bound using techniques from probabilistic combinatorics.

Homework: None

Prerequisites: None

Related to (but not required for): Problem Solving: The Probabilistic Method

The Sylow Theorems. (🌶🌶🌶, Mia, Thu–Saturday)
Suppose I give you a mystery group and all I tell you is its order. What can you tell me? A surprising amount, actually! For example, if I tell you that my group has order 77, you can tell me that it has exactly one subgroup of order 11 and exactly one of order 7. In fact, you can even tell me that my group has to be cyclic, isomorphic to $\mathbb{Z}_{77}$. All you need are the Sylow theorems. (And in case you think this follows in some straightforward way from the fact that 77 is a product of two distinct primes, note that it doesn't work for 55.)

So how to we go about this "group sleuthing"? You probably already know Lagrange's theorem, which states that the order of a subgroup divides the order of the group. You might wonder, does the converse hold? That is, given a number n that divides the order of the group, is there necessarily a subgroup of order n? The answer is no. Despite this disappointment, the Sylow theorems renew our hope, giving us some conditions for which the converse does hold. And so, with the Sylow theorems in hand, we can continue our quest to understand our mystery groups.

Homework: Recommended

Prerequisites: Group Theory

Voting theory. (🌶, Alfonso, Tuesday–Saturday)
When a large group of people have to make a decision together, bad things can happen. For example, suppose that a group of 10 campers is trying to decide which game they want to play tonight. Suppose further that 3 of them want to play Dominion, and the remaining 7 would prefer to play any game they can possibly think of other than Dominion. If the remaining 7 are divided between 5 or 6 different games, a strict plurality election system will force them to play Dominion, even though a majority of the 10 campers would be thoroughly unsatisfied. It seems, then, that the plurality election system is unfair. What could we do to make it fair? Which election system is the most fair? What does "fair" mean, anyway?

In this class we will try to formalize the question of "what is a fair voting system" mathematically, and we will analyze actual voting systems used in the world accordingly. We will use real data from recent elections in various countries!

Warning: Your faith in democracy may vanish.

Homework: Optional

Prerequisites: None

11:00am Classes

Latin Squares. (🌶🌶, Marisa, Wednesday–Saturday)
Do you remember Sudoku, the popular little logic puzzle? It goes like this: take a 9×9 grid and fill it with the numbers 1 through 9 so that each digit appears once in every row and every column. (There is also a constraint about subsquares, which I will ignore.) If I give you two different Sudokus, with different fills of the 9x9 grid, you can superimpose one on top of the other and look at the pairs of symbols you've produced from the matching squares. Here's a deeper puzzle: can you find two Sudoku grids that are *so* different from each other that when you superimpose them, you get all 9^2 possible pairs? Great. Now can you find *eight* different Sudoku grids, with the property that every pair of grids produces all 9^2 possible pairs? No need to do it by hand – we can hit this with the hammer of Group Theory. Then, as a bonus, we can construct affine and projective geometries out of our pile of mutually orthogonal puzzle grids!

Homework: Recommended

Prerequisites: At a minimum, you'll need modular arithmetic and finding the inverse of an $n \times n$ matrix. If you also know the multiplication table of the Klein four-group and the definition of a finite field, then you'll catch more of the details.

Related to (but not required for): NP–Completeness and Latin Squares

Congruent Numbers and Elliptic Curves II. (🌶🌶🌶🌶, Ruthi, Wednesday–Saturday)
This class is a continuation of the Week 2 class "Congruent Numbers and Elliptic Curves". We will continue where we left off, probably by discussing more about elliptic curves over the rationals, introducing L-functions of elliptic curves, what the Birch and Swinnerton Dyer conjecture actually says, and what all of this has to do with Tunnell's Theorem.

Homework: Required

Prerequisites: Week 1 of Congruent Numbers and Elliptic Curves

The Intermediate Value Theorem and Chaos. (🌶🌶🌶, Paddy, Wednesday–Saturday)
A dynamical system, roughly speaking, is just a way to model how something changes or evolves over time: consider water sloshing about in a tub, interest accruing on a bank account, or rabbit populations on an island.

With any notion of a dynamical system comes a corresponding concept of **chaos**; a property that a dynamical system can have that (in a sense) says its long-term behavior can be very sensitive to initial inputs. For example, it is not hard to study the long-term behavior of money in a bank account; even if we don't know the exact amount we initially deposited, we can approximate where our investment will go as time passes. Conversely, imagine taking a tub full of turbulent water and putting a rubber ducky somewhere in it. As time passes, it is not at all obvious where the duck will go; moreover, its later location may depend very precisely on where the duck starts!

Using tools from calculus, namely the concepts of continuity and the intermediate value theorem, we're going to study chaos in this class. It'll be fun!

Homework: Recommended

Prerequisites: Being comfortable with the rigorous definitions of continuity.

Related to (but not required for): Real Analysis

The John Conway Hour. (🌶→🌶🌶🌶🌶, John Conway, Tuesday–Saturday)
Not to be Announced.

Homework: None

Prerequisites: None

Knots, Games, Coloring, and Algebra... Oh My! (🌶🌶, Allison Henrich and Sam Nelson, Wednesday–Saturday)

Knot theory is one of the most active areas of study in the mathematical field of topology. What's more, the way we learn about knots is often by using seemingly unrelated mathematical tools in fields like combinatorics, algebra, analysis, and geometry. Knots can be used to model DNA, molecules, and many other objects in physics and biology, so they are incredibly useful objects to study, in addition to being fascinating in their own mathematical right!

In this course, we will begin with a guided exploration of knots by playing with these objects to gain intuition, learn about fundamental questions, and formulate conjectures. Our conjectures will often either directly or indirectly be related to the central question in knot theory: given two knots, how can we tell if they are the same or different?

We will see firsthand that, to tell knots apart, we need to find knot "invariants". These are functions we can compute which are the same for all diagrams of a given knot. We will start with two easy invariants, the Gaussian linking number and Fox tricolorability. The coloring idea leads us to develop an algebraic structure called "kei".

In addition to learning about methods for showing either that two knots are the same or that they are different, we will explore different methods of unknotting. This will lead us quite naturally into the study of games that can be played on knots!

Homework: Recommended

Prerequisites: None

Related to (but not required for): Combinatorial Topology

1:00pm Classes

How to cut a Sandwich. (🌶🌶🌶, Jeff, Tuesday–Saturday)
Suppose you go to a store, and decide to split a ham sandwich with your friend. The sandwich is made of bread, ham and cheese, and you would like to cut it in such a way that both parties get an equal amount of each of the three ingredients. While it is obvious that you can do this if you are allowed to make the cut as you wish, can you do it if you make the cut along a single planar slice? Even if the sandwich has a highly non-symmetric shape?
The answer (which we will prove in this class) is yes. In this introduction to algebraic topology and applications, we will define the fundamental group of a space— which, as the name suggests, is one of the most important tools in algebraic topology. After developing some tools to help us understand the fundamental group, we will then explore as many of the following mysteries as possible:

- Why is there always a pair of opposite points on the Earth with the same temperature and air pressure?
- Why is it always possible to split a cake fairly between 3 people, even if they value parts of the cake differently?
- Why does a map of the world, no matter where you put it and no matter how you wrikle it, always have a point which lies exactly on top of the place it describes?
- Why does every polynomial have a complex root?

Homework: Recommended

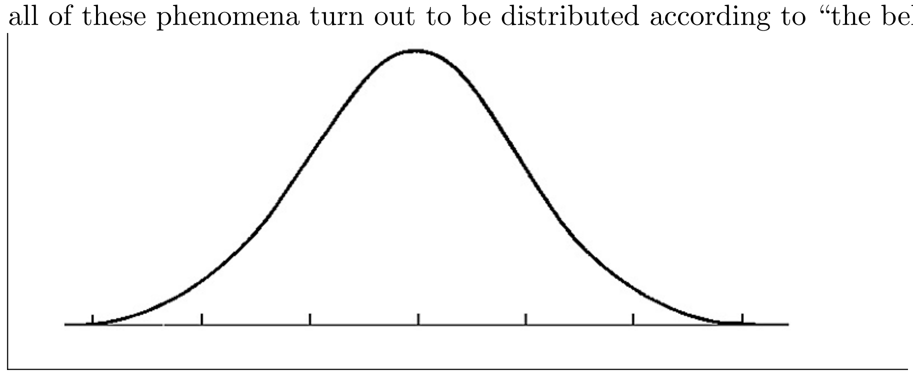
Prerequisites: You should be able to define "Continuous function" using open sets, and name at least 3 groups.

Related to (but not required for): Other topology

The Bell Curve. (🌶🌶🌶, Mira, Tuesday–Saturday)

> I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason.
>
> Sir Francis Galton, 1889

Human heights; SAT scores; errors in scientific measurements; the number of heads you get when you toss a million coins; the number of people per year who forget to write the address on a letter they mail. . . . what do all of these (and numerous other phenomena) have in common?
Empirically, all of these phenomena turn out to be distributed according to "the bell curve":



This curve, known in the 19th century as the "Law of Error", is now usually called the *normal* or *Gaussian* distribution. It is the graph of the function $\frac{e^{x^2/2}}{\sqrt{2\pi}}$ (scaled and translated appropriately). We will see how Gauss derived this function from a completely backward argument – a brilliant leap of intuition, but pretty sketchy math. We'll see how the great probabilist Laplace explained the curve's ubiquity through the Central Limit Theorem. (Maybe you've learned about CLT in your statistics class ... but do you know the proof?) We'll talk about how the normal distribution challenged the nineteenth century concept of free will. Finally, we'll look at some other mathematical contexts in which the normal distribution arises – it really is everywhere!

Homework: Required

Prerequisites: Integral calculus: there will be a lot of integrals

Related to (but not required for): Bayesian Statistics

The Continuum Hypothesis. (Susan, Tuesday–Saturday)
In 1874, Georg Cantor published a revolutionary article called "Über eine Eigenschaft des Inbegriffes aller reelen algebraischen Zahlen," or "On a Property of the Collection of All Real Algebraic Numbers." In this article he proved, for the first time, that the real numbers are in a precise mathematical sense larger than the natural numbers.[1] This begs the question: how much bigger? Is it the next cardinality up, or is there some infinity between the infinities? Or perhaps more than one—an infinity of infinities between the infinities!

The continuum hypothesis states that there is no set whose cardinality is strictly between that of the reals and that of the natural numbers. Cantor believed the continuum hypothesis to be true, but he never managed to prove it. In 1900, the problem remained open, and was listed as one of Hilbert's problems: does there exist a cardinality that is larger than countable, but smaller than the continuum?

The question was resolved in 1963, almost a hundred years after the initial problem was posed, when a mathematician named Paul Cohen proved that the continuum hypothesis can be neither proved nor disproved from the standard axioms of set theory.[2] The methods used by Cohen in this work are stunningly beautiful. And in this class, you'll get to see them. Come with me and explore this gorgeous proof of the lack of a proof.

Homework: Required

Prerequisites: None

Required for: None

Related to (but not required for): Infinite Trees

Graph Coloring. (Mo & *Matt Stamps*, Tuesday–Saturday)

In kindergarten you learned how to color... or so you think. Do you know how many ways there are to color the vertices of a graph G with r colors so that no two adjacent vertices receive the same color? Do you know how to give bounds on the minimum number of colors required to color G at all? Come to this course and REALLY learn to color properly!

Homework: Required

Prerequisites: Linear Algebra, Graph Theory

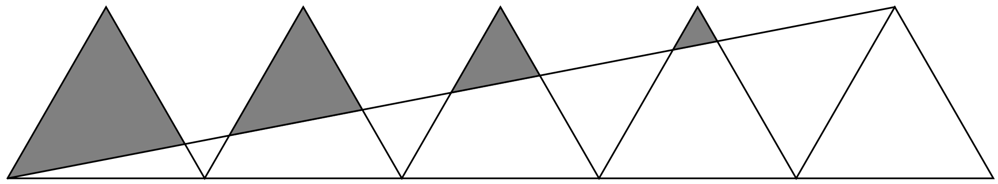
Related to (but not required for): Introduction to Graph Theory

[1] This article presents a lovely proof that is entirely different from the diagonalization argument that many of you have seen.

[2] It had already been established that it could not be disproved by Gödel in 1940. The new result in the 1963 paper was that it could not be proved.

Problem Solving: Geometry. (🌶🌶, Misha, Tuesday–Saturday)

In the picture below, what is the ratio of the shaded area to the area of one of the five (congruent) triangles?



This is the sort of question we'll answer in this class. We will take easy ideas like similar triangles and ratios of areas and use them to solve hard olympiad geometry problems (and also some easy olympiad geometry problems). I will occasionally tell you how this works but mostly you will just work on geometry problems in class. Then when class is over you will still have geometry problems and you should work on them out of class as well. Lots of geometry problems!
Homework: Required
Prerequisites: None
Related to (but not required for): Other problem solving

Colloquia

NTBA. (John Conway, Tuesday)
Not to be Announced.

The Internet. (Po-Shen Loh, Wednesday)
The Internet was miraculously created through the individual actions of a multitude of agents, rather than by a central authority. In this colloquium, we introduce some elements of the game theoretic perspective on network formation, and explore its relationship with extremal graph theory.

A Tale of Knots and Games. (Allison Henrich, Thursday)
Do you like knots and links? Do you like games? In this talk, I will show how these two areas of research can be combined into an incredibly fun kind of mathematics. We'll play several games using knots and links and discuss ways we can "stack the deck" and guarantee ourselves a win.

Circle Packings and Penrose Tilings. (Matt Stamps, Friday)
Circle packings provide a way to approximate (and manipulate) surfaces in a manner that captures (and reveals) interesting geometric properties. They have far-reaching applications in several fields of mathematics — including complex analysis, differential geometry, and number theory — but they are also used by brain surgeons to navigate the human cerebral cortex! In this talk, I'll teach you the circle packing basics and how they can be employed to shine a light on Penrose tilings: the amazing tilings of the plane that exhibit zero translational symmetries.

Visitor Bios

John Conway. One of the most creative thinkers of our time, John Conway is known for his ground-breaking contributions to such diverse fields as knot theory, geometry of high dimensions, group theory, transfinite arithmetic, and the theory of mathematical games. Outside the mathematical community, he is perhaps best known as the inventor of the "Game of Life."

Po-Shen Loh. Po-Shen Loh studies questions that lie at the intersection of two branches of mathematics: combinatorics (the study of discrete systems) and probability theory. Randomness can manifest itself in the construction of a combinatorial system, as in the case of a so-called "random graph", but may also be artificially introduced as a proof technique to solve problems about purely deterministic systems, as was pioneered by Paul Erdős in what is now known as the Probabilistic Method.

Research Pair # 1.

Allison Henrich. Allison Henrich is a researcher in the field of knot theory. She has published on questions related to the complexity of unknotting a knot as well as the mathematical study of games you can play with knots. Currently, she's interested in learning more about a newly discovered type of knot called a pseudoknot. Pseudoknots may be useful in the study of DNA replication. In her spare time, Allison practices aerial acrobatics.

Sam Nelson. I am a combinatorial and algebraic topologist specializing in low-dimensional topology. That is, I study the algebraic properties of knots. Some kinds of knots don't fit in 3-dimensional space! We will look at these kinds of objects and how to use abstract algebra to tell them apart. I'm not just a topologist; I'm also a dance club DJ specializing in electronic music genres like EBM and Industrial.

Research Pair #2.

Matt Stamps. Matt Stamps applies ideas from geometry and topology to problems in combinatorics, mechanical engineering, and biology. This often involves studying spaces whose points correspond to more complicated objects, like graphs or robot arms or proteins... Aside from math, Matt loves to travel and write short stories in coffee shops around the world; he's visited fourteen countries outside of the US and Canada within the last three years.

Mohamed Omar. Mo's research interests lie in the interplay between algebra, combinatorics, and optimization. Canada/USA Mathcamp is near and dear to his heart; he was an academic coordinator and a mentor in past years and earned fame by trolling anagram games with 4-letter crustacean larvae. For fun he plays in a competitive LGBT basketball league, creates math contest problems, frequently visits a cafe in Los Angeles housing over 700 board games, promotes all things Canadian (minus Rob Ford) with emphatic pride, and screams impatiently while watching Wheel of Fortune.