CLASS DESCRIPTIONS—WEEK 5, MATHCAMP 2015

Contents

9:10 Classes

Building a machine that learns like a human. (🌶🌶, *Josh Tenenbaum*, Thursday)
People can learn a new concept almost perfectly from just a single example, yet machine learning algorithms typically require tens or hundreds of examples to perform similarly. People can also use what they learn in richer ways than most machines do — to guide their action, imagination, and explanation. I will talk about how we can build machine learning algorithms that better capture these human learning abilities. Our approach represents concepts as simple programs that can generate observed examples, and learning can be described as a search for the program that best explains what you see. I will show results from a series of "visual Turing test" experiments probing this model's ability to classify as well as to create new instances of concepts, showing that in many cases it is indistinguishable from the performance of humans. Time permitting, I will also talk about other exciting recent work in AI that aims to build machines with human-level learning abilities, such as the Atari Video Game player from Google DeepMind.
Homework: None
Prerequisites: Previous classes on probability, statistics, machine learning, Bayesian inference, or monte carlo would be helpful but are not necessary.

From Counting to a Theorem of Fermat. (🌶🌶, Mark, Friday)
A standard theorem stated by Fermat (it's actually uncertain whether he had a proof) states that every prime p congruent to 1 modulo 4 is the sum of two squares. (On the other hand, if p is 3 modulo 4, it has *no* hope of being the sum of two squares.) There are many proofs of this theorem, but perhaps the weirdest one, due to Heath-Brown and simplified by Zagier, uses just counting —

no "number theory" at all! In this class we'll see at least that proof, and maybe some others and/or related proofs of other things.

Homework: None

Prerequisites: None!

Learn to code! (Asilata, Thursday)

This class will be a gentle introduction to programming and basic principles of coding, in one of my favourite programming languages: Haskell. If you know how to code in another language, this class is probably not for you. (You can feel free to show up to learn the basics of Haskell, but remember that it will be gentle.)

If you haven't coded much or never coded at all before, this class is definitely for you! We'll start with some basic general principles of coding and do hands-on examples before moving on to Haskell-specific features. Haskell is a language that is particularly satisfying if you are mathematically inclined, and we'll see why. We'll also discover that Haskell is lazy, and how we can exploit that to our advantage.

Here is some bonus pretty code taken straight from the Haskell homepage. The following lines generate an *infinite list* (yes you read that right!) called "primes". This is exactly a list of all the prime numbers, obtained by implementing Eratosthenes' sieve!

```haskell
primes = filterPrime [2..]
  where filterPrime (p:xs) = p : filterPrime [x | x <- xs, x `mod` p /= 0]
```

Homework: Recommended

Prerequisites: None

Primitive roots. (Mark, Wednesday)
Suppose you start with 1 and keep multiplying by a modulo n, where a and n are relatively prime positive integers. As it turns out, you will always get back to 1. But will you have seen all the integers k with $\gcd(k, n) = 1$ by then, as part of the "number wheel" you just made? In this class we'll explore when (that is, for what values of n) you can find an a such that every integer modulo n that's relatively prime to n shows up on that single wheel (such an a is called a *primitive root* mod n). We may not get much beyond the case that n is prime, but even in that case the analysis is interesting. In particular, we'll be able to show that a exists in that case without having any idea of how to find a, other than the flat-footed method of trying $2, 3, \ldots, n-1$ until we find a number that works.

Homework: None

Prerequisites: A little elementary number theory.

Problem Solving: Tetrahedra. (Misha, Friday)
In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

Homework: Optional

Prerequisites: None.

Qualifying Quiz Problem 6. (Jalex, Friday)
Problem 6 on the Qualifying Quiz this year asked you to find an optimal strategy for an n-player game when n was a power of 2. It turns out that the induction part of the intended solution was

nonessential — many campers found that more-or-less the same strategy works for any even n. More surprisingly, somebody submitted a nonconstructive proof that there is an optimal strategy for all n. The proof is clean — it's just a few key lemmas about directed multigraphs.
Homework: Recommended
Prerequisites: Know what a directed graph is.

Quantum factoring. (, Pesto, Wednesday)
With the best currently known algorithms, all the world's supercomputers working together for a year couldn't factor a 4000-digit integer into prime factors. Even a small[1] computer that takes advantage of quantum mechanical phenomena that aren't well approximated by what classical computers do could factor such numbers using "Shor's algorithm". We'll race through a description of how we model quantum computation (without assuming any knowledge of physics), then see Shor's algorithm.
Homework: None
Prerequisites: Linear algebra (in some sense, almost everything we do in this class will be multiplication of matrices), Number Theory (Fermat's Little Theorem).

Quaternions and rotations. (, Chris + Alfonso, Wednesday)
See the other blurb
Homework: Recommended
Prerequisites: None.

The Hoffmann Singleton Theorem. (, Pesto, Thursday)
"I'm thinking of a set of four integers. Four of them are 2, 3, and 7. What's the other one?"

"How should I know? You could have picked any other integer."

"No, these are the answers to a perfectly natural mathematical question: as natural as "If you can get from any other vertex of a graph to any other vertex in at most two steps, but there are no triangles or squares, and every vertex has the same degree, what's that degree?" "

"Seems like a natural enough question. Maybe 5, like the first four primes?"

"Nope, 57."

"No way! How could 2, 3, 7, and 57 be the answer to *anything* that natural?"
Homework: Recommended
Prerequisites: Linear algebra (enough to understand the statement "if A is a matrix and $A^2 = I$, then every eigenvalue of A is 1 or -1" and either know its proof or be willing to accept it without proof).

The tale of summation is in order! (, *Paweł Piwek*, Thursday)
Don't you agree?

Let p be Zach's favorite prime (or yours). Take an integer $x \in \{0, 1, \ldots, p-1\}$ and start computing its powers $x, x^2, x^3, \ldots \bmod p$. The minimal exponent d for which $x^d \equiv 1 \bmod p$ is called the *order* of $x \bmod p$ (If there is no d with $x^d \equiv 1 \bmod p$, then you took $x = 0$; unlucky for you.)

Now suppose you *fix* d and add up (mod d) all the $x \in \{0, 1, \ldots, p-1\}$ which have that order d. What will you get? And what does this computation mod p have to do with complex roots of unity? To find out, come to this class!
Homework: None
Prerequisites: Some experience with polynomials; Fermat's Little Theorem (or Lagrange's Theorem from group theory)

[1] "Small" according to some arbitrary attempt to compare quantum computer sizes to classical computer sizes, but still millions of times bigger than any currently-constructible quantum computer.

Wagner's Theorem. (🌶🌶🌶, Pesto, Friday)
Draw five points on a piece of paper. Try to connect every pair of them without any edges crossing. You can't do it.

Three utilities each need to connect to each of three houses. Then some two of the utility lines must cross.

Stated graph-theoretically, these results say that K_5 and $K_{3,3}$ aren't planar. (We'll prove so, quickly.)

What's really surprising is that these are in some sense the *only* non-planar graphs: you can find one of them in every nonplanar graph if your vision's fuzzy enough that you can make any connected set of vertices blur into one. This is Wagner's Theorem, which we'll state and prove.

Homework: Optional

Prerequisites: Graph theory: if G is a connected planar graph drawn with f faces, and G has v vertices and e edges, then $v - e + f = 2$.

What's Up With e? (🌶🌶, Susan, Wednesday)
The continued fraction expansion of e is

$$1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots$$

What's up with that? Come find out!

Homework: None

Prerequisites: Calculus (derivatives and integrals).

10:10 Classes

Abelian Groups. (🌶🌶🌶, Don, Thursday–Friday)
I love group theory, but I don't love when elements of groups don't commute with each other; that's not good for the environment!

Even with all of our groups being abelian, though, algebraists are still far from a complete understanding. What we can do is look at a variety of different extra restrictions, and classify the abelian groups meeting those restrictions; say, the finitely generated abelian groups, or the divisible abelian groups.

Homework: None

Prerequisites: Group Theory

Related to (but not required for): Introduction to Groups (W1); Galois Cohomology (W2); Profinite groups (W5); What is the deal with Homology? (W5)

Cyclotomic Polynomials and Extensions. (🌶🌶🌶, Milica, Wednesday)
Definition: The n-th cyclotomic polynomial is the polynomial whose roots are the primitive n-th roots of unity.

Theorem: The regular n-gon can be constructed by straightedge and compass if and only if n is the product of a power of 2 and distinct Fermat primes.

Find out how these two things are related!

Homework: Recommended

Prerequisites: Familiarity with field extensions and with complex roots of unity.

Differentials and higher differentials. (🌶🌶, *Anti Shulman*, Tuesday)
A differential is a thing like dx or dy. You've probably seen them appearing in pairs in Leibniz'

notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the derivative, but appearing singly they are rarely given much attention in calculus classes. However, a highfalutin' generalization of them called "exterior differential forms" plays an important role in differential geometry.

A *higher* differential is a thing like d^2y or $\mathrm{d}x^2$. You may have seen them appearing pairwise in Leibniz' notation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for the second derivative, but for the most part they have been banished from appearing singly in modern mathematics. However, this prejudice is unjustified! I'll explain about how to make sense of them, and why they make life much easier when computing higher derivatives and integrals. In particular, we'll see that Leibniz' notation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is actually *wrong*!

Homework: None

Prerequisites: Calculus. Multivariable calculus recommended, but not required.

Extensions. (, Asilata, Tuesday)
Take any two abelian groups, for instance $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Then the product group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has the property that $\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is an injective homomorphism, and $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is a surjective homomorphism, and the kernel of the second map is exactly the image of the first map.

Based on this example, we define an *extension* of H by G to be any new abelian group E for which there is an injective homomorphism $G \hookrightarrow E$ and a surjective homomorphism $E \twoheadrightarrow H$, and the kernel of the second homomorphism is precisely the image of the first.

Are there any other extensions than the "product extension" shown above? What does "other" mean? How many are there, and what do they all look like?

We'll think about all of this. But first, hang on tight, it's about to get meta: not only are there (sometimes) lots of different extensions, but the *set of extensions* itself forms an abelian group! Doesn't that sound pretty? Let's learn about it!

Homework: Recommended

Prerequisites: None.

Martin's Axiom and Ramsey Ultrafilters. (, Steve + Susan, Thursday–Friday)
Lets play with the complete graph on $\mathbb{N}$![2] We like colors, so were going to color the edges — each edge gets colored either "red" or "blue"[3] It turns out that no matter how we do this, we can find an infinite set of vertices H (called a "homogeneous set") such that all the edges between vertices in H have the same color. For instance, maybe we color the edge between a and b red if and only if a and b have the same parity (even or odd); then we can take H to be the set of odd numbers.

But that's graph theory, and we want to do logic. Also, we're lazy; we want a machine that will magically find homogeneous sets for every coloring. Also also, we like ultrafilters. Can has useful ultrafilter please?

It turns out that, if we assume Martin's axiom[4], there is an ultrafilter $\mathcal{U}$ such that — for *any* coloring at all — there is a homogeneous set H for that coloring such that $H \in \mathcal{U}$. Time remaining, well show that *every* ultrafilter has an ultrafilter like $\mathcal{U}$ living inside it.[5]

Homework: Recommended

Prerequisites: Ultrafilters, Martin's Axiom

Related to (but not required for): LARGE cardinals! (W5)

[2]Excitement, not factorial.
[3]We had more exciting colors, but Susan lost them.
[4]We had a great proof just from the axioms of ZFC, but Steve ate it, so now its independent of ZFC. Bad Steve.
[5]Yo dawg, we heard you like ultrafilters.

Multiplicative Functions. (🌶🌶, Mark, Tuesday–Wednesday)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such multiplicative functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.
Homework: Optional
Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)

Quadratic Reciprocity. (🌶🌶–🌶🌶🌶, Mark, Thursday–Friday)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

Q1: "Is q a square modulo p?"
Q2: "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square mod 9973, you'll be able to answer a lot more quickly, whether or not you use technology!
Homework: Optional
Prerequisites: Number theory through Fermat's little theorem.

Stupid Games on Uncountable Sets. (🌶🌶, Susan, Tuesday–Wednesday)
Let's play a game. You name a countable ordinal number. And then I name a bigger countable ordinal number. We keep doing this forever. When we're done, we'll see who wins. In this class we'll be discussing strategies for winning an infinite game played on ω_1. In particular, we'll talk about how to set up the game so that at any point, *neither* player has a winning strategy.
Homework: Optional
Prerequisites: None.

The Cake is a Lie. (🌶🌶, Sachi, Thursday–Friday)
It's my birthday, and I want to share cake with my friends.

Susan and Kevin are attending my party, but they're still having a feud, so I need to make sure I give them the same amount of cake. Unfortunately, Susan really like the pink frosted flowers, and Kevin likes the gel icing writing. So, cutting the cake in thirds will not look fair to them, depending on the distribution of the icing.

How can I cut the cake so that Kevin, Susan, and I each believe that we have at least 1/3 of the cake and would not like to trade with anyone else?
Homework: Recommended
Prerequisites: None.

11:10 Classes

Abel's Theorem. (🌶🌶🌶, Mira, Tuesday–Friday)
Let's prove this thing!
Homework: Required

Prerequisites: Abel's Theorem, Weeks 1-4

Abstract Nonsense. (🌶🌶🌶🌶, Chris, Wednesday–Friday)
So you've learned some category theory but you still feel like that's too concrete for you? Then this class is the answer! Abstract Nonsense is the semi-official title of categorical proving-techniques that on first sight feel completely non-sensical, but apply to a wide range of situations due to their abstract nature.

One aspect of this are so-called simplicial sets — the most abstract and weird way to describe topological spaces. In this class we will be studying these simplicial sets and maybe in the end use them to define the classifying space of a category. Yes, that's right – we will turn a category into a space. Why? Because we can! And also because we then can define K-Theory (maybe).

Homework: Recommended

Prerequisites: Category Theory (Categories, Functors)

Combinatorial Game Theory. (🌶🌶, Alfonso, Tuesday–Saturday)
We have a plate with blueberries, blackberries, and raspberries. We take turns eating them. In your turn, you may eat as many berries as you want, at least one, but they all have to be of the same type. Then it is my turn. Unless you are Marisa, the goal is not to eat a lot of berries: the winner is the person who eats the last one. Will you beat me?

The above is an example of an impartial combinatorial game. There are tons where it came from, and you have probably encountered some. To solve them all, there are basically only two skills that you need to learn. If you want enlightenment, my young grasshopper, avoid spoilers and come to this class. I will motivate the two skills and I will guide you all to

figure them out. The beauty of this topic is as much in the final results as it is in the journey, and I do not want to deprive you of the pleasure of discovering it slowly. We will also attack many examples, from easy ones to actual open problems.

Homework: Recommended

Prerequisites: None.

Combinatorial Topology. (🌶🌶🌶, Jeff, Tuesday–Friday)
So, you want to be a topologist. But, you've never taken point-set topology[6]. How much can you prove about topology?
Turns out, quite a bit. In this class, we'll be developing simplicial complexes, which give combinatorial representations of topological spaces. Then, we'll look at discrete Morse theory, a combinatorial representation of a construction from differential topology. Along the way, we'll draw lots of pictures and diagrams, and get a feel for what topology *should* do, without messing around with all of those icky open sets.

Homework: Recommended

Prerequisites: None.

The Fast More Four Do Over. (🌶🌶🌶🌶, Yüv, Tuesday)
If I give you a real a and one more real b, how fast can you find out $a \cdot b$? We need to do this all the

[6]Disclaimer: I've never taken point-set topology

time, so we want to be able to do it fast. And if we try to do it the way we were told when we were kids, it will be too slow. Can we make it more fast?

The More Four Do Over is a math idea that does take some data and does turn it into a set of data in such a way that the new data has a lot of info on how the old data does vary. It's a tool that is very full of use, and we love it. Also, it lets us find out $a \cdot b$, and that is very good.

But wait! If we try to do the More Four Do Over, it is also too slow. We want to do it fast! Very very fast! What do we do? Well, luck is on our side. It does turn out that we can find a sly way to do the More Four Do Over that is way more fast than what we did in the past. And if we know how to do the More Four Do Over fast, then we can find out $a \cdot b$ fast too, and it all just gets more fast.

In this talk, we will find out how to do the Fast More Four Do Over and how to use that to find out $a \cdot b$. And we will do it all in the Game of Four.[7]

Homework: None

Prerequisites: None.

Related to (but not required for): Algorithms (W1); Algorithms in Number Theory (W1); Time-Frequency Analysis (W1); The Hidden Dance of Partial Differential Equations (W4)

1:10 Classes

A Card Trick, and a Set. (🌶️🌶️, Don, Friday)
The Gilbreath principle says that when you shuffle a deck a certain way, much of its structure is preserved; you can use the similarity of the shuffled deck to the original to perform a number of magic tricks.

The Mandelbrot set is a marvelously complicated set in $\mathbb{C}$ that is infinitely self-similar yet not regular.

These two notions of self similarity are, in a certain sense, the same! We'll see why in this class.

Homework: None

Prerequisites: None

Related to (but not required for): Mathematical Magic (W3)

Bad History: A Crash Course in Historiography and how not to do the History of Math. (🌶️, Sam, Tuesday)
It turns out that, when mathematicians try to do research in the history of mathematics, they sometimes do awful things. We'll talk about some of the common pitfalls (i.e., whig history) and how to avoid them. That way, if you want to ever think seriously about the history of math, you don't accidentally garner the scorn of the history of mathematics community! Yay!!

Homework: Recommended

Prerequisites: None.

[7] If I give you two numbers a and b, how fast can you calculate their product $a \cdot b$? Since this is such an important operation, we want to be able to do it quickly, and the algorithm that we learned in school is actually too slow. Can we make it faster? The (Discrete) Fourier Transform is an operation that converts a sequence of numbers into another sequence that remembers the structure of the original sequence. It's an extremely useful tool; in particular, it lets us calculate $a \cdot b$.
However, there's a problem: if we try to calculate the Discrete Fourier Transform naively, then it's also too slow. But luckily, there is a fast algorithm for computing it, which also means that we can multiply quickly.
In this class, we'll learn about the Discrete Fourier Transform and about the Fast Fourier Transform algorithm, and we'll see how we can use this to multiply efficiently. And we will do it all in the Game of Four.

Elliptic Curves Day 1. (🌶, Ruthi, Tuesday)
Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful. They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

In this class, we will learn what the definition of an elliptic curve over the rationals is and how to add points on one.

Homework: Recommended

Prerequisites: None.

Elliptic Curves Day 2. (🌶🌶, Ruthi, Wednesday)
Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful. They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

In this class, we'll generalize our definition of elliptic curves, and discuss what their abelian group structure looks like. (In particular, we'll state Mordell-Weil Theorem.)

Homework: Recommended

Prerequisites: Elliptic Curves Day 1, know what an abelian group is

Elliptic Curves Day 3. (🌶🌶🌶, Ruthi, Thursday)
Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful. They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

In this class, we'll discuss heights on curves and prove Mordell's Theorem.

Homework: Recommended

Prerequisites: Elliptic Curves Day 2

Elliptic Curves Day 4. (🌶🌶🌶🌶, Ruthi, Friday)
Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful. They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

In this class, I'll try to talk about some of modern questions of interest about elliptic curves, and recent research and results.

Homework: None

Prerequisites: Elliptic Curves Day 3

Finite Fields. (🌶🌶🌶, Mark, Tuesday)
You may well know that the integers modulo p form a field (basically, a set in which all four arithmetic operations are possible, including division limited only by Rule 4) if and only if p is prime. However, there are other finite fields whose sizes are not prime. Finite fields have applications in areas such as coding theory and cryptography as well as in more abstract mathematics, and they are elegant algebraic objects that are well worth exploring. In this class we'll see how to construct finite fields, and if time permits, show that we have found them all.

Homework: Optional

Prerequisites: Some ring theory (in particular, polynomials, and rings modulo ideals) and a bit of linear algebra (the idea of dimension)

Latin Squares. (🌶–🌶🌶, Marisa, Wednesday–Thursday)
Do you remember Sudoku, the little popular logic puzzle? It went like this: take a 9×9 grid and fill it with the numbers 1 through 9 so that each digit appears once in every row and every column. (And there was some constraint about subsquares, which I will ignore.) If I give you two different Sudokus, with different fills of the 9×9 grid, you can superimpose one on top of the other and look at the pairs of symbols you've produced from the matching squares. Heres a deeper puzzle: can you find two Sudoku fills that are *so* different from each other that when you superimpose them, you get all 9^2 possible pairs? Great. Now can you find *eight* different Sudoku grids, with the property that *every* pair of grids produces all 9^2 possible pairs? No need to do it by hand — we can hit this with the hammer of modular arithmetic. The result is a set of *mutually orthogonal Latin Squares* (MOLS), a combinatorial object that turns out to be related to finite geometries, designing schedules, building codes, and all kinds of other combinatorial objects.
Homework: Optional
Prerequisites: None

Optimization Problems on Graphs. (🌶🌶, Sam, Wednesday–Thursday)
Suppose you were in charge of running the first chess-sprinting world championship. Because the sport is so popular, you have a ton of money to run the championship and, because you like the beach, choose to host the tournament in Bora Bora. Nike has also sponsored the tournament, and has given you funding to fly competitors to Bora Bora (so long as everyone in the tournament wears Nike's chess-sprinting-shoe-line). All that remains is to book the flights.

Now suppose also that you like a challenge (that is, after all, how you got into the chess-sprinting business in the first place). You thus try to book flights without using any online tools: all you let yourself have is a map that shows, for each pair of airports, the cost of the cheapest flight between those airports (if no such flight is available, the cost is infinite). If you just had this map, how could you figure out what flights to purchase?

It turns out that this type of problem is ubiquitous, and actually occurs in significantly less contrived scenarios. More formally, suppose you have a graph G, and for each pair of vertices u, v, the edge between u and v is labeled with some sort of cost (say, corresponding to the cost to get from u to v). If there is no edge between u and v, we can represent that cost with a really large number. In the shortest path problem, we're interested in finding the path from u to v of minimum cost, where the cost of the path is defined as the sum of the costs along all edges in that path.

In this class, we'll introduce the shortest path problem then talk about a few algorithms for solving that problem (like Bellman-Ford and Dijkstra's). We'll also talk about a few related problems! We will not, alas, get to go to Bora Bora.

NOTE: this is going to be a course with a relatively practical flavor. My goal will be to introduce certain types of optimization problems that occur on graphs and then to show clever ways of solving those problems. I'll justify some of the techniques, but we won't necessarily be rigorously proving all of them. But, the techniques we will talk about are all pretty awesome, and they're great techniques to have in your toolkit!
Homework: Recommended
Prerequisites: A smattering of basic graph theory: know what a graph is and what a path between two vertices is.

Sylow Theorems. (🌶🌶🌶, Nancy, Wednesday–Friday)
Sylow Theorems $\subset$ Group theory $\cap$ Number theory

The Sylow Theorems form a fundamental part of finite group theory and have very important applications in the classification of finite simple groups.

Basically, here's what the Sylow theorems do:

Suppose you have a finite group G of size m. Depending on the prime decomposition of m, you can determine how many subgroups G will have of various sizes.

Homework: Recommended

Prerequisites: Group theory

The Projective Plane. (🌶🌶, Sachi, Tuesday)
We can construct an object called the "projective plane" by taking a sphere and allowing teleportation between two antipodal points. Alternatively, we could wrap the normal plane $\mathbb{R}^2$ in a line "at infinity" and require that all parallel lines meet at one point. Or, we could take a Möbius strip and glue a disc to it. If we happened to already have a projective plane lying around, we could switch all its points with all its lines and get a projective plane back.

We'll talk about what kinds of geometry we can do with this weird shape, and why all these notions are the same.

Homework: Optional

Prerequisites: None.

The Robinson-Schensted Correspondence. (🌶–🌶🌶, Asilata, Friday)
The Schensted algorithm is a simple and beautiful recipe that turns permutations into pictures, also known as standard Young tableaux.

The algorithm is tricky, but simple to describe. Amazingly, it can be turned into a reversible procedure that sends any permutation to a *pair* of Young tableaux and vice-versa. This is known as the Robinson–Schensted correspondence, which has applications into some unexpectedly deep mathematics.

Expect a fun hour with delightful combinatorial surprises and lots of examples. It'll be a treat.

Homework: Recommended

Prerequisites: None. Group theory helpful but not required.

2:10 Classes

A game you can't play (but would win if you could). (🌶🌶, *Stefan Banach + Alfred Tarski*, Friday)
Once upon an infinity, in the Kingdom of Aleph, King Alephonso decided to put his 100 advisors to a test. He had 100 identical rooms constructed in his palace. In each room the king placed an infinite sequence of boxes; in each box he put a real number. The sequence of numbers was exactly the same in each room, but otherwise completely arbitrary.

The king told his advisors that when they were ready, each of them would be locked in one of the 100 rooms. Each of them would be allowed to open all but one of the boxes in the room. (This, of course, would take an infinite amount of time, but in the Kingdom of Aleph, theyre pretty cavalier about infinity.) Finally, each advisor would be required to name the number in the box that he or she did not open. If more than one advisor names the wrong number, they would lose their jobs and their lives.

There is no reason for anyone to hurry in the Kingdom of Aleph, and the king gives his advisors an infinite amount of time to work out a strategy. Do they have any hope of making it through his cruel test alive? What should they do?

Homework: None

Prerequisites: None

A Mathematician Goes to Vegas. (Don, Tuesday)
Many would say that someone who knows math would never gamble.

Some would say that someone who knows math would only gamble if there's a positive expected value.

I would say that simply by existing, we have no choice but to constantly gamble, and so you might as well learn how to do it right.

Learn how to do it right. Come to this class.

Homework: None

Prerequisites: Calculus

Burnside's Lemma. (*William Burnside*, Friday)
How many different necklaces can you build with 6 pebbles, if you have a large number of black and white pebbles? Notice that you won't be able to tell apart two necklaces that are the same up to rotation or reflection. You probably can answer the above question by counting carefully, but what if we are building necklaces with 20 pebbles and we have pebbles of 8 different colours?

There is a nifty little result in group theory that allows us to solve this type of problem very quickly and without a calculator. Come and learn it!

Homework: Recommended

Prerequisites: Basic group theory

Cosine Waves on Musical Staves. (J-Lo, Wednesday)
J-Lo's first class on the connections between math and music. The two days are independent; you can come to this one even if you don't go to the second day.

You will learn how to hear a trigonometric identity (I used this one to tune the piano in the main lounge), why musical harmony should be thought of as a branch of number theory, and how the continued fraction expansion of $\log_2 3$ contains a summary of the development of musical scales.

Homework: Recommended

Prerequisites: Exposure to trig functions and logarithms

Related to (but not required for): Introduction to Groups (W1); Time-Frequency Analysis (W1); Continued Fractions (Week 1 of 2) (W2); Continued Fractions (Week 2 of 2) (W3); Can You Hear the Shape of a Drum? (W3)

Fractal dimensions. (*Hermann Minkowski*, Friday)
A line has dimension 1, a plane has dimension 2, and the space we live in has dimension 3. Can you think of something of dimension 1.5? What does it mean to have dimension 1.5? Actually, what does it even mean to have dimension 2?

In this class, I will give you one possible definition of dimension and we will compute the dimension of a few objects, including some with non-integer dimensions.

Homework: Recommended

Prerequisites: You need to be comfortable with logarithms and limits.

Group operations' harmonic implications. (J-Lo, Thursday)
J-Lo's second class on the connections between math and music. The two days are independent; you can come to this one even if you don't go to the first day.

Discover how the four-chord harmonic progression you hear in almost every pop song can be interpreted via the symmetries of a regular dodecagon, why minor really is just negative major, and how Beethoven's Ninth Symphony almost traces out one of the generators of the fundamental group of a torus.

Homework: Recommended

Prerequisites: Introduction to Groups

Related to (but not required for): Introduction to Groups (W1); Time-Frequency Analysis (W1); Continued Fractions (Week 1 of 2) (W2); Continued Fractions (Week 2 of 2) (W3); Can You Hear the Shape of a Drum? (W3)

LARGE cardinals! (, Steve, Tuesday–Thursday)
Some numbers are so big we need special notation to describe them. Well, it turns out that some *sets* are so big, you need special *axioms* to even show they exist! These are called *large cardinals* and are a major focus of current research in set theory.

Large cardinals come in two flavors: the smaller(!) ones tend to arise from taking nice theorems in combinatorics and going "But what if INFINITY?!?!" The larger ones appear when we take the entire universe, stretch it, and put it inside itself for no obvious reason. Yay science!

Over time, set theorists have discovered many species of large cardinals; in this class, we'll tour the zoo.

Homework: Optional

Prerequisites: Familiarity with basic set theory: ordinals, cardinals, and the axiom of choice.

Related to (but not required for): Measure and Martin's Axiom (W1); Fun with Compactness in Logic (W1); Making Cantor Super Proud (W1); Ordinal Arithmetic (W3); Ultrafilters (W4); Category Theory in Sets (W4); Martin's Axiom and Ramsey Ultrafilters (W5); The Lowenheim-Skolem Theorem (W5)

Posets. (, Kevin, Tuesday–Thursday)
What do the principle of inclusion-exclusion, divisors, graph colorings, and polytopes have in common? We can study and prove results about these things using partially ordered sets! And the list goes on and on: subgroup structures, partitions, permutations... so many things come naturally with a partial ordering!

Unlike a total ordering, we can't always compare two elements in a poset. For example, it's natural to order sets based on containment, but then the sets $\{1, 2\}$ and $\{1, 3\}$ are incomparable—neither contains the other. Even though it might seem like a partial order doesn't provide much structure, we'll see that we can do quite a lot of magic with what seems to be very little!

Homework: Optional

Prerequisites: None.

Problem Solving: Linear Algebra. (, Misha, Wednesday–Thursday)
Most high school math contests (the IMO included) do not use any topic considered to be too advanced for high school, such as linear algebra. This is a shame, because there have been many beautiful problems about linear algebra in undergraduate contests such as the Putnam Math Competition.

In this class, we will look at linear algebra from a new perspective and use it to solve olympiad problems.

Homework: Recommended

Prerequisites: Linear algebra

Rapid Fire Problem Solving. (, Misha, Tuesday)
"Compute the ordered pair of integers (a, b) that minimizes $|a^4 + a^b - 2015|$."

This problem comes from the ARML 2015 Tiebreaker round. You can probably solve it. But can you solve it in less than a minute?

This class is about solving math problems very quickly. We will practice solving such problems and discuss tricks to speed up problem solving.
Homework: None
Prerequisites: None.

The mathematics of polygamy (and bankruptcy). (🌶, *Judah the Prince*, Friday)
Here is a passage from the *Mishnah*, the 2nd century codex of Jewish law:

A man has three wives; he dies owing one of them 100 [silver pieces], one of them 200, and one of them 300.

If his total estate is 100, they split it equally.

If the estate is 200, then the first wife gets 50 and the other two get 75 each.

If the estate is 300, then the first wife gets 50, the second one 100, and the third one 150.

Similarly, any joint investment with three unequal initial contributions should be divided up in the same way.

For 1800 years, this passage had baffled scholars: what could possibly be the logic behind the Mishnah's totally different ways of distributing the estate in the three cases? Then, in 1985, a pair of mathematical economists produced a beautifully simple explanation based on ideas from game theory. They showed that for any number of creditors and for any estate size, there is a unique distribution that satisfies certain criteria, and it turns out to be exactly the distribution proposed in the Mishnah. The proof is very cool, based on an analogy with a simple physical system! See if you can figure out this ancient puzzle for yourself, or come to class and find out.
Homework: None
Prerequisites: None

Colloquia

Future of You. (Staff, Come learn about your future! In this colloquium well tell you about the differences between liberal arts colleges and state universities, about study abroad and options for taking time off before college, about what life in college is like and what factors to consider when making a choice of school. Everyone is welcome, whether youve already decided on a school or havent even begun thinking about it.)

Future of Mathcamp. (Staff, This is an event for all of us to come together as a community and discuss our favorite moments at Mathcamp, as well as suggestions for future years. Be part of a creative process that will build a better camp!)

Vistor Bios

"Anti" Mike Shulman. Since first coming to Mathcamp in 1996, Anti has been a camper, a JC, a mentor, an AC, and a visitor; this will be his 15th Mathcamp. In his day job at the University of San Diego, Anti teaches students and computers to understand mathematics and each other, and works on a research project called Homotopy Type Theory that hopes to reshape the foundations of mathematics with category theory, homotopy theory, and programming.

Josh Tenenbaum. Josh is a professor of cognitive science and a member of the MIT Computer Science and Artificial Intelligence Lab (CSAIL). In his research, he builds mathematical models of human and machine learning, reasoning, and perception. His interests also include neural networks, information theory, and statistical inference.