CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2016

Contents

9:10 Classes

Dynamical Systems. (🌶🌶🌶, Jane, Tuesday–Saturday)
Dynamical systems are spaces that evolve over time. Some examples are the movement of the planets, the bouncing of a ball around a billiard table, or the change in the population of rabbits from year to year. The study of dynamical systems is the study of how these spaces evolve, their long-term behavior, and how to predict the future of these systems. It is a subject that has many applications in the real world and also in other branches of mathematics.

In this class, we'll survey a variety of topics in dynamical systems, exploring both what is known and what is still open. Each day of class will focus on a different area of dynamical systems. We'll think about billiard trajectories on polygonal and non-polygonal tables. We'll learn about how fractals like the Mandelbrot set and Sierpinski's triangle arise from dynamics. We'll study cellular automata and how they can be used to model real-life situations. We'll also explore the connections between dynamical systems and number theory, such as what dynamics can tell us about the distribution of the leading digits of the powers or two.

Homework: Recommended.

Prerequisites: None.

Field Extensions and Galois Theory (Week 1 of 2). (🌶🌶🌶🌶, Mark, Tuesday–Saturday)
We'll begin by defining what field extensions are, and seeing some of what they can be used for. As an example, if you were never comfortable with defining the complex numbers by postulating the existence of a square root of -1, we'll see early on how we can "make" a square root of -1 using polynomials. We'll also see why some ancient and (in)famous construction problems, such as squaring the circle, are provably impossible.

You may know the story of the brilliant mathematician and societal misfit Galois, who died tragically young in a duel after developing an exciting new area of mathematics that was beyond what most of his contemporaries could even follow. In this class we will build up to the fundamental theorem of Galois theory, which gives an unexpected and beautiful correspondence allowing us to find and describe field extensions in terms of subgroups of a certain group.

If time permits, we'll move on (perhaps with some side trips to scenic overlooks of other material) to sketch a proof that there is no general way of solving polynomial equations of degree 5 or more by radicals; that is, there is no analog of the quadratic formula for degree 5 and higher. (There are such analogs for degree 3 and 4.)

Homework: Recommended.

Prerequisites: Linear algebra, group theory, ring theory (familiarity with polynomial rings will be especially useful).

Required for: Algebraic Number Theory (W4).

Cluster: Rings and Fields.

Model Theory. (🌶🌶🌶, *Steve Schweber*, Tuesday–Saturday)
We talk about math a lot. But *how* do we talk about it? The formal language we use to describe mathematical structures such as groups, rings, and fields is interesting in itself—and, in fact, can be viewed as a mathematical structure! In this class we'll explore the mathematics of talking about mathematics:

- How can we describe various structures in math?
- When are two structures "indistinguishable" using a given language?
- How can we use the study of mathematical languages to help us understand the structures we care about?

We'll focus on the main formal language of mathematics—*first-order logic*—but we'll also see some of its close relatives, including (as time permits) equational logic, second-order logic, and infinitary logic.

Homework: Recommended.

Prerequisites: None.

Neural Networks. (, Kevin, Tuesday–Saturday)
Here's some math you won't learn at Mathcamp[1]:

> The following lemma surjective restrocomposes of this implies that $\mathcal{F}_{x_0} = \mathcal{F}_{x_0} = \mathcal{F}_{\mathcal{X},\dots,0}$.
>
> **Lemma 0.2.** *Let X be a locally Noetherian scheme over S, $E = \mathcal{F}_{X/S}$. Set $\mathcal{I} = \mathcal{J}_1 \subset \mathcal{I}'_n$. Since $\mathcal{I}^n \subset \mathcal{I}^n$ are nonzero over $i_0 \leq \mathfrak{p}$ is a subset of $\mathcal{J}_{n,0} \circ \overline{A}_2$ works.*
>
> **Lemma 0.3.** *In Situation ??. Hence we may assume $\mathfrak{q}' = 0$.*
>
> *Proof.* We will use the property we see that $\mathfrak{p}$ is the mext functor (??). On the other hand, by Lemma ?? we see that
> $$D(\mathcal{O}_{X'}) = \mathcal{O}_X(D)$$
> where K is an F-algebra where δ_{n+1} is a scheme over S. □

In fact, you won't learn this math anywhere, because it's complete and utter nonsense. But it *looks* like math at a distance, and I bet you could convince a lot of non-mathematicians that it's real!

What twisted brain produced this abomination? An artificial one! Taking (loose) inspiration from our own brains, neural networks are more or less simply a jumble of interconnected nodes that fire or not based on the nodes connected to them. With proper training, this tangle can learn to do many things unreasonably effectively, from arithmetic to handwriting recognition to Shakespeare imitation to doing a remarkably good impression of a crackpot mathematician.

In this class, we'll see a lot of demos and we'll study the math and algorithms behind increasingly complex systems, from the lowly perceptron to the mighty recurrent neural network. This will *not* be a programming class, however—our goal is to focus on the ideas underpinning neural networks rather than details of implementation.

Homework: None.

Prerequisites: None.

Cluster: Mathematics and Its Applications.

Problem Solving: Induction. (, Misha, Tuesday–Saturday)
You probably first saw induction in the context of proving a result like

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}.$$

Such a proof is fairly straightforward and maybe your main worry was "Can my last sentence just be 'by induction, we're done' or do I need something fancier?"

In this class, we'll see how these proofs can get much more complicated. Our induction will start out strong, and on each day of class it will get stronger than all the previous days combined. You'll see examples of crazy induction in algebra, game theory, graph theory, number theory, and other theories. You'll learn how to use induction (and how *not* to use it) to solve problems of your own, olympiad and otherwise.

[1] Taken from `http://cs.stanford.edu/people/jcjohns/fake-math/4.pdf`.

In class, we will spend time solving problems together; I will focus less on answering the question "why is this claim true?" and more on answering the question "why would we think of solving a problem this way?" There will be plenty of problems left for homework, and you will not get much out of a problem-solving class unless you spend time solving those problems.

Homework: Required.

Prerequisites: None.

Cluster: Problem Solving.

10:10 Classes

Almost Planar. (, Marisa, Wednesday–Saturday)
Draw a network with 120 Mathcampers, and add an edge between two campers if they have ever taken a class together before. I would bet you \$5 that the resulting graph is not planar (as in, can't be drawn without crossing some edges), but I'm so confident about winning that the question just isn't very interesting. So let's ask a more subtle question: how *far* is our graph from planar? In this class, we'll look at several different ways of measuring "closeness" to planarity, from the structural to the space-bending. The format will be inquiry-based, so you'll be discovering and proving results throughout the week.

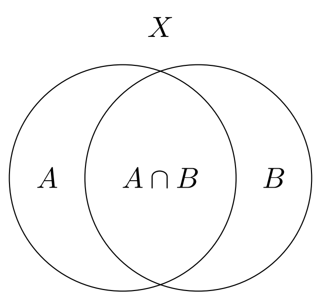
Homework: Optional.

Prerequisites: Introduction to Graph Theory or equivalent.

Cluster: Maps, Graphs, Colors, Walks.

Extending Inclusion-Exclusion. (, Jeff, Wednesday–Saturday)
A general principle in mathematics is that if you want understand some complicated mathematical object, you break it up into simpler objects, and then reassemble the larger object from the simple pieces, and see how those pieces fit together.

A simple example comes from combinatorics and sets. Let X be a set, and let $X = A \cup B$, as drawn in the following picture.



If we are trying to compute the size of X, we can instead compute it as

$$|X| = |A| + |B| - |A \cap B|$$

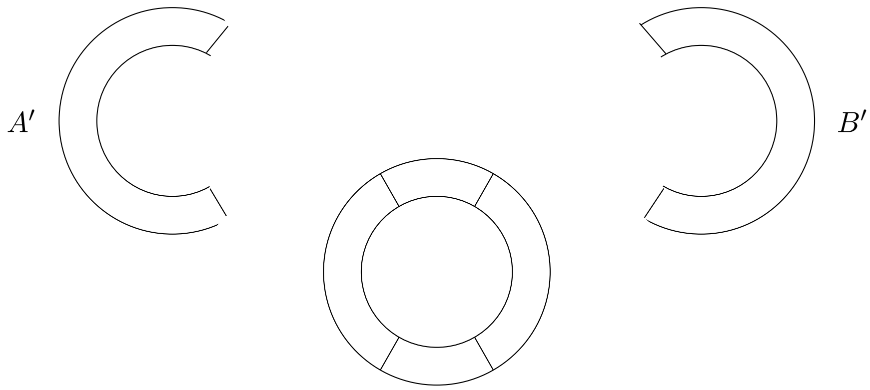
Notice that X is "glued" together from A and B, so we had to subtract the contribution from the intersection of $A \cap B$ when computing the size (or we would make the error of overcounting.)

Inclusion-exclusion works well with some other quantities too—for instance, if we had replaced "cardinality" with "area", we would see the same principle holds.

There are a lot of places where the principle of Inclusion-exclusion seem to hold, but are a little bit off. For instance, let $C(X)$ count the number of connected components of a shape. So, for our example, $C(X) = C(A) = C(B) = C(A \cap B) = 1$. Then we again have that

$$C(X) = C(A) + C(B) - C(A \cup B).$$

However, if we look at the shape drawn here



$Y = A' \cup B'$

it is no longer the case that we have this inclusion-exclusion principle. We get

$$C(Y) = C(A') + C(B') - C(A' \cap B') + 1.$$

However, the error to our principle measures the fact that Y has a hole punched in the middle of it. To get the full generalizations of this, we'll have to explore the relationships between combinatorics, linear algebra and topology.

Homework: Required.

Prerequisites: You should be able to explain that if $f : X \to Y$ is a linear map between vector spaces, why $\dim \ker(f) + \dim \mathrm{im}(f) = \dim(X)$.

Cluster: Algebraic Topology.

Multilinear Algebra. (🌶🌶🌶🌶, Nic, Wednesday–Saturday)

If you've seen a lot of linear algebra before, you've probably encountered the definition of an inner product. (Sometimes it's called a "dot product" or a "scalar product.") This is a way of taking two vectors and producing a number, but unlike a lot of the other functions you encounter in linear algebra, it's not a linear transformation. Instead, it's what's called a "bilinear map"—it's linear in each of its coordinates separately.

This class is an exploration of a construction called a tensor product which is critical to analyzing functions like this one. Along the way we'll also produce a very clean and well-motivated definition of the determinant, and introduce the notion of a "universal property," which is foundation of the modern, categorical approach to abstract algebra.

Homework: Recommended,

Prerequisites: Linear algebra, up through the definitions of linear independence, spanning, bases, dimension, and kernels, and the relationship between the dimensions of the image and kernel of a linear transformation. The first week of Linear Algebra should be enough, but you're encouraged to take the second week at the same time as this class.

The Word Problem for Groups. (🌶🌶, Assaf, Wednesday–Saturday)

A group is a collection of reversible operations, with some rules about how they relate to each other. For example, we can think of the operation of rotating a circle by $\frac{1}{3}$. This operation has order 3, meaning that doing it 3 times is like doing nothing, which is codified as $(\text{rotation})^3 = 1$. In this manner, we can write down all of our operations (generators) and what kinds of rules (relators) they follow in order to describe the group. This is called a group presentation. In some cases, writing down all the rules is redundant; for example, the rule: (do nothing)$^2 = 1$ is a consequence of (do nothing) $= 1$, since doing nothing twice is the same as doing nothing once.

Now, given a group presentation with a finite set of generators and relators, can we always tell if some other sequence of operations will be the "do nothing" operation? In other words, does there exist an algorithm which decides if a sequence of operations is the trivial operation? This is called the Word Problem, and it was posed by Max Dehn in 1911.

The surprising answer, shown by Pyotr Novikov in 1955, is no—there does not exist such an algorithm. In this course, we will prove this result by studying how we can embed incomputable sets in groups. This will give us a candidate for a "bad" group for which a solution to the word problem would contradict the incomputability of the set.

Homework: Required.

Prerequisites: None.

Cluster: Algebraic Novelties.

Why Are We Learning This? A seminar on the history of math education in the U.S.. (🌶, Sam, Wednesday–Saturday)
Have you ever sat in class and thought: who the @%#@ thought to teach math this way? What about: it's so obvious that to fix math education, all we need to do is x? Or: is this class just an exercise in obedience training?

This seminar will provide historical context for those types of thoughts. We'll try to understand that context for both the actual content in the curriculum and the way that content is communicated. Most importantly, we'll pay attention to how math education reform movements have succeeded—or failed. If you've ever wanted to try and fix math education, this class should provide the context to help. Oh, and we'll also see some fun and crazy examples of math problems that have plagued students for centuries.

Note: For this class, *seminar* and *homework required* mean that you'll have to read some history of math papers and look at parts of math textbooks from ages past. In class we'll devote a considerable amount of time to discussing that stuff, and you should expect approximately 45 minutes of reading for each day.

Homework: Required.

Prerequisites: None.

11:10 Classes

Building Mathematical Structures. (🌶, Zach, Tuesday–Saturday)
Come transform ordinary items into extraordinary geometric sculptures! In these intricate, large-scale, collaborative construction projects, we will work together to assemble room-filling rubber-band knot-webs, bouncy paper polyhedra, gorgeous drinking straw jumbles, and more! Browse `http://zacharyabel.com/sculpture/` for examples of the types of projects this course may feature. Assembling these mathematical creations requires scrutiny of their mathematical underpinnings from such areas as geometry, topology, and knot theory, so come prepared to learn, think, and build!

Homework: None.

Prerequisites: None.

Graph Minors. (🌶🌶🌶, Pesto, Tuesday–Saturday)
You may have seen two classic graph theory problems: "if you have five cities, can you route roads between each pair of them without crossings?" and "if you have three houses and three utilities, can you build utility lines from each utility to each house without crossings?"; that is, whether K_5 and $K_{3,3}$ are planar graphs. The answers are (spoiler alert!) both no.

In fact, those two graphs are *the* fundamental nonplanar graphs: every graph is planar unless and only unless it "contains" one of those two graphs, if we have the right definition of "contains", which we'll develop.

With the right definition of "contains", we can prove a similar result for graphs drawn on any surface (say, on a torus instead of on the plane) and generalize the four-color theorem.

Homework: Recommended.

Prerequisites: Know what connected graphs, planar graphs, and Euler's formula are.

Cluster: Techniques in Graph Theory.

History of Math. (🌶, *Moon Duchin*, Tuesday–Wednesday)
The two days of this class cover two topics.

Day 1: Math and the Body.
Math is done by people with bodies! And they use their bodies in various ways, like finger-reckoning, writing, gesture, and model-building. I've got some fun examples for us to think through, from Baghdad to Berkeley.

Day 2: Mind Reading with Artifacts.
By a happy fluke, the materials that some ancient Babylonians used for their math work were curiously permanent, and we've got a trove of clay tablets to work with. I'll talk about readings and mis-readings of artifactual evidence, and how to reconstruct a mathematical culture.

Homework: Recommended.

Prerequisites: None.

***K*-Theory.** (🌶🌶🌶🌶, Don, Thursday–Saturday)
Proving that two things aren't the same is really hard. In algebraic topology, there are homotopy and homology groups, which can let you do so for topological spaces, but they are often fiendishly difficult to compute. In ring theory, there are K-groups, which aren't just hard to compute—they were historically difficult to define.

In this class, we'll look at just the first few K-groups. $K_0(R)$ is defined in terms of all $R-$modules, so we'll study modules and resolutions to understand it. $K_1(R)$ is defined in terms of $\mathrm{GL}_\infty(R)$, so we'll need to brush up on colimits in order to define it. Finally, $K_2(R)$ is already too much unless R is a field—but we might be able to see how $K_2(\mathbb{Q})$ is inextricably linked to quadratic reciprocity.

Homework: Optional.

Prerequisites: Ring Theory.

Cluster: Rings and Fields.

Linear Algebra (Week 2 of 2). (🌶🌶🌶, Mark, Tuesday–Saturday)
This is a continuation of the week 1 class. If you would like to join and you're not sure what has been covered, check with Mark (and/or with someone who has been taking the class and who has good notes).

Homework: Recommended.

Prerequisites: Week 1 of linear algebra.

The Banach–Tarski Paradox. (🌶🌶🌶, Chris, Tuesday–Saturday)
You have heard of the Banach–Tarski paradox: take a ball, break it into a few pieces, shuffle those pieces, glue them back in a different way, and now we have two balls of the same size as the original one! Nifty trick, but how does it work? In this course you get to develop the whole mathematical theory behind this construction and prove that it actually works!

This is a superclass that meets for two class hours a day (and possibly the first hour of TAU if we need it). You will be doing most of the work: we will provide worksheets with the right definitions and questions; you will spent a big chunk of the time working, alone or in groups, sometimes with our help, on all the steps of the construction. Some of the class time will be spent on presentation and discussion of your proofs.

This course is time-consuming, but all the work (homework included) is contained in the two to three daily hours.

Homework: Required.

Prerequisites: Basic group theory, linear algebra (matrix multiplication, and understand how a matrix represents a linear transformation).

1:10 Classes

Divergent Series. (🌶🌶, Sachi, Tuesday–Saturday)
Shhh! We're going to sum series that the establishment doesn't want you to sum.

Have you ever wanted $1 - 1 + 1 - 1 + 1 - \cdots = \frac{1}{2}$ to be true?

What should the value of $1 - 2 + 3 - 4 + 5 - \cdots$ be?

Abel said that "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever." Clearly he was trying to hide something.

Homework: Recommended.

Prerequisites: Calculus, knowledge of complex numbers and $e^{i\theta}$ form.

Cluster: Summing Series.

Geometric Group Theory. (🌶🌶🌶🌶, Susan, Tuesday–Saturday)
Given a group with a finite set of generators, we can build a visualization called the Cayley graph. Unfortunately, the Cayley graph is not a group invariant—different choices of generators result in different graphs. But these different graphs have a certain same-ey-ness about them. For example, all Cayley graphs of the integers look long and thin. So what's going on here?

In this class we'll define the notion of a quasi-isometry, the formalization of this idea of same-ey-ness. We'll explore this idea, and show that each group is associated to a particular metric space, and that this metric space is unique up to quasi-isometry. We'll also play with group actions, and show that if a group has a particular kind of action on a particular kind of metric space, we can use that action to find a finite set of generators for the group.

Homework: Recommended.

Prerequisites: Group Theory.

Cluster: Groups.

Group Actions. (🌶🌶🌶, Don, Friday–Saturday)
Groups are great—they've got operations, they've got identities, they even have inverses. And let me tell you, I have the best groups. But you know what the groups of today don't have enough of? Applications! We need to put our groups back to work, and in this class, that's exactly what we're going to do.

The problem with everybody else's group actions is that they don't have enough freedom. That's not a problem for me—my actions are fantastic. They have the most freedom. If you want to make group theory great again by learning about affine spaces and applications to mathematical physics, come to this class!

Homework: Recommended.

Prerequisites: Group Theory.

The Chip-Firing Game. (🌶🌶🌶, *Sam Payne*, Tuesday–Thursday)

Tropical geometry provides powerful combinatorial techniques for studying curves and surfaces and higher dimensional spaces defined by polynomial equations, by looking at piecewise linear solution sets over nonarchimedean fields, such as the p-adic numbers and Laurent series.

In the special case of curves, the piecewise linear shadows are graphs with finitely many vertices and finitely many edges. Algebraic geometers study curves in space by looking at configurations of points obtained by intersecting with a hyperplane, and examining how these configurations of points move as the hyperplane moves. When tropical geometers study how the shadows of these configurations of points move, we see that they move according to the combinatorial rules of the chip-firing game.

My three day class will be an introduction to the combinatorics of the chip-firing game, highlighting relations to classical combinatorics (such as the matrix-tree theorem and Dhar's burning algorithm), and also with a view toward applications in algebraic geometry—we will go over most or all of the combinatorial arguments needed to prove the Brill–Noether theorem, which determines when every smooth projective curve of genus g admits a degree d embedding in projective space of dimension r—the statement goes back to the days of Riemann, but filling in the details of the proof took another hundred years.

Homework: Recommended.

Prerequisites: None.

Colloquia

There Are Eight Flavors of Three-Dimensional Geometry. (*Moon Duchin*, Wednesday)

Picture it: circa 1977. Hippies, disco, Elvis dies, Star Wars comes out… and some guy comes along and works out that for all the possible ways to build a three-dimensional shape, you only need EIGHT building blocks. Seven won't do, and there's no ninth one. I'll tour you through the eight 3D geometries and put this amazing fact in context.

Tropical Geometry. (*Sam Payne*, Thursday)

Tropical geometry provides powerful combinatorial techniques for studying curves and surfaces and higher dimensional spaces defined by polynomial equations, by looking at piecewise linear solution sets over nonarchimedean fields, such as the p-adic numbers and Laurent series.

In the special case of curves, the piecewise linear shadows are graphs with finitely many vertices and finitely many edges. Algebraic geometers study curves in space by looking at configurations of points obtained by intersecting with a hyperplane, and examining how these configurations of points move as the hyperplane moves. When tropical geometers study how the shadows of these configurations of points move, we see that they move according to the combinatorial rules of the chip-firing game.

In colloquium, I will try to give an overview of the basic notions of tropical geometry, and explain how the combinatorial chip-firing rules arise from the Poincaré–Lelong formula, governing solutions to a certain partial differential equation over nonarchimedean fields.

Oops! I Ran Out of Axioms. (*Steve Schweber*, Friday)

Is there a question which the axioms we use for math can't solve? It's reasonable to think that the answer is no; or at least, that if we do find such a question, that we can just add a couple new axioms and then *those* will be enough to answer everything. However, in 1931, a mathematician named Kurt Gödel surprised everyone by proving that we will never have a complete list of axioms: given any "reasonable" set of axioms, there will be true statements which are not provable from those axioms. Moreover, these statements aren't weird or vague—they are perfectly concrete statements about natural numbers, and over the next few decades Gödel's result was improved to the following: given any "reasonable" set of axioms, there is a *polynomial* p with integer coefficients, which has no integer solutions, such that the axioms cannot prove that p has no integer solutions.

I'll talk about what exactly this means, and how it turns out to be true.

Visitor Bios

Moon Duchin. Moon works in geometric topology and geometric group theory. She particularly looks at the large-scale geometric structure of groups and unusual metric spaces. One recurring theme is taming the geometric infinite by either attaching a "boundary at infinity" to a space you want to study, or else approaching it dynamically by understanding what happens after you flow or jump around in your space for a really long time. She's also actively interested in history, philosophy, and cultural studies of science.

Sam Payne. Sam studies algebraic varieties with techniques from nonarchimedean geometry. He has nothing against archimedean geometry (not really, anyway), and holds a deep and enduring admiration for Archimedes, who single-handedly defended the city of Syracuse against Marcellus and the forces of the Roman Empire. For a while. That ended badly, but the point is that Archimedes discovered the laws of hydrostatics and the principle of the lever, invented devices like The Claw and The Screw and The Spiral and The Method. The last of which is more a mathematical technique than a device—it's basically the Fundamental Theorem of Calculus, but 2000 years before Newton and Leibniz. And he used it to compute the volume of the sphere(!). And then he wrapped it in bacon and shot it into space. What this is all leading up to is that nonarchimedean geometry, although not discovered by Archimedes, really owes its (co)identity to this great man, and if he were here at Mathcamp, he would get along with Sam just fine, and they would take a little while to talk about what's been happening in math for the last 2300 years, and then they would prove something awesome about algebraic curves and surfaces using p-adic analysis.

Steve Schweber. Steve studies mathematical logic, with a focus on computability and set theory. In his spare time, he juggles large objects with his feet and unicycles. His birthday is July 16.