CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2021

Contents

9:00 Classes

Curvature lies within (*Apurva Nakade*, $\boxed{\text{MTW}\Theta\text{F}}$)
Thanks to several high-res photos of outer space, we have experimentally verified one of the most counter-intuitive predictions of Einstein's theory of general relativity: that mass bends the space-time continuum. Gravity arises as the bending of this space-time continuum and is mathematically indistinguishable from curvature. But what does it mean for space-time to bend? How can we tell that our universe is curved without being able to look at it from the outside?

In this 2-chili class, we'll answer some of these questions for two-dimensional objects. We'll define curvature à la Gauss and use it to study the geometry of surfaces. We'll do some explicit computations and, time permitting, prove the Gauss–Bonnet theorem.

Class format: The class will be a mix of lectures and IBL. You should expect to spend a considerable part of the class time solving problems in breakout rooms.

Prerequisites: Differentiation (definition, interpretation, basic computations)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	🤔 👥 HW Optional	Squishy shapes

Finite fields and how to find them (Viv, $\boxed{\text{MTW}\Theta\text{F}}$)
You may be familiar with fields like $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. But what about fields that are a lot smaller? Like, a *lot* smaller?

The integers modulo p form a field (that is to say, a set where we can add, multiply, subtract, and divide according to rule 4) when (and only when) p is prime. However, there are many many more finite fields out there! In this class we'll see how to construct finite fields, and in particular see how constructing finite fields is like finding primes between N and $2N$. (If that sounds *completely unrelated*, come see why it is not.) We'll also explore the structure of finite fields, and, time permitting, show that we have found them all.

Class format: Lecture; I'll be sharing a tablet screen and writing.

Prerequisites: Group Theory; Ring theory, to the point where you're comfortable quotienting a ring by an ideal

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	HW Recommended	Algebraic structures Number theory

Quadratic forms (Gabrielle, $\boxed{\text{MTW}\Theta\text{F}}$)
A binary quadratic form is a function $f(x,y) = ax^2 + bxy + cy^2$, where $a, b, c \in \mathbb{Z}$, as you can read from Viv's class blurb last week. You can also read the definition of the discriminant, which is $b^2 - 4ac$, as well as the fact that the class number is finite and has a beautiful formula given by Dirichlet (which you proved if you took her class!).

Unfortunately, it is frowned upon for me to plagiarize Viv's class, so we will learn something else! Following Hatcher's Topology of Numbers, we are going to draw some pretty pictures and use them to derive some interesting facts about quadratic forms (including classifying quadratic forms by their discriminants, proving that the class number is finite, and figuring out which integers can be output by a quadratic form).

Class format: Interactive lecture!

Prerequisites: Familiarity with Legendre symbols may be helpful but not necessary.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Recommended	Number theory

The Schwarzschild solution (*Jon Tannenhauser*, $\boxed{\text{MTW}\Theta\text{F}}$)
Einstein's general theory of relativity says that matter tells spacetime how to curve and the curvature of spacetime tells matter how to move. Suppose our spacetime contains a single spherical, static mass such as the sun. The Schwarzschild solution of Einstein's equations describes the geometry of spacetime in the empty space surrounding the mass. If the mass is sufficiently concentrated, we have a black hole—a region of spacetime where gravity is so strong that nothing can escape.

In this course, we won't derive or solve the Einstein equations. Instead, we'll first spend a few days building up some needed formalism (coordinate transformations, tensors, metrics, and geodesics). Then we'll posit the form of Schwarzschild solution and see how its geometry leads to the bizarre physics of black holes. Finally, we'll discuss in grisly detail what will happen to an astronaut unlucky enough to fall into a black hole (spoiler alert: it's called "spaghettification").

Class format: Interactive lecture. Mostly I'll be talking, but students are welcome to ask questions anytime via chat, and sometimes I'll pose questions for students to answer via private chat. I plan to make skeletal PDF notes available, with blank spaces for calculations and proofs, to be filled in during class.

Prerequisites: Physics: some basic mechanics vocabulary (energy, linear and angular momentum, wavelength, frequency), special relativity at the level of the week 2 course. Math: single-variable integral calculus and Taylor series, polar and spherical coordinates, partial derivatives, vectors.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️🌶️	HW Recommended	Math in real life

The calculus of variations (Ben, $\boxed{\text{MTW}\Theta\text{F}}$)
In the ordinary calculus, one asks questions about functions—things like "where does this function have a maximum?" where the answer is going to be some number in the function's domain. In the

calculus of variations, we study how to use the ideas of the calculus for questions like "which curve connecting these points has the smallest arc length?" where the answer is not some number, but some *function*.

We'll see a little bit of how these ideas relate to physics, and we'll prove and discuss one of the most important results for the study of modern physics. Noether's Theorem[1] says that the symmetries of the laws of physics correspond to conserved quantities. Before Noether, it was well-understood that some quantity that we call "energy" doesn't change over time, and most people implicitly assumed that the laws of physics were the same today as they were yesterday—they're symmetric over time, that is. Noether's Theorem, among other things, states that the latter fact *implies* the former—the fact that the laws of physics remain the same over time tells us that energy is conserved.

Class format: Interactive lecture

Prerequisites: Linear algebra, some familiarity with analysis, calculus (at least knowing what integrals are and knowing about Taylor series). We won't need any major results from multivariable calculus, but having seen it before might be useful, for context.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️🌶️	HW Recommended	Analysis

10:10 Classes

Factoring large prime numbers (Linus, M TWΘF)
My Python program tells me, in about 8 seconds, that the next prime after 10^{1000} is $10^{1000} + 453$. It also factors $2^{139} - 1$ into $5625767248687 \times 123876132205208335762278423601$ in about 3 seconds.

"Wait, I thought the fastest way to do this is to check up to $\sqrt{n}$?" That's only if you're taking an intro to programming class, where even including the $\sqrt{}$ is hailed as the Greatest Idea In Theoretical Computer Science.

This class is about better algorithms, in order of increasing fanciness, to tell whether a number is prime. Also, algorithms to factor large composite numbers. Also, how a mathematician factored $2^{93} + 1$ in 3 seconds... before the invention of computers, using parts from old bicycles.

(The title is a joke on a common misphrasing: if you Google it in quotes you can find it hundreds of times in the wild.)

Class format: Lecture. Expect plenty of pico-quizzes where I ask y'all to DM me answers in chat.

Prerequisites: Basic number theory: know about inverses mod N, Fermat's Little Theorem, and the Chinese Remainder Theorem.

Programming is NOT a prerequisite.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Recommended	CS & algorithms Number theory

Functions of a complex variable (2 of 2) (Mark, M TWΘF)
This is the continuation of the class with the same name from week 2; see the blurb for that class for more information. If you are thinking of taking this class in week 3 and you didn't take it in week 2, please consult with me to make sure you'll be OK and/or so I can help you catch up on whatever background you may be missing.

[1]Due to the remarkable German mathematician Amalie "Emmy" Noether. You may have encountered her in a ring theory class, as well.

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: The material from the week 2 class.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	HW Recommended	Analysis

Lights, camera, group actions! (Emily, M $\boxed{\text{TW}\Theta\text{F}}$)
One way of thinking of a group action is that it is a function which "applies" elements of a group to elements of a set to produce a new element of that set. We can have groups act on many different kinds of sets—from sets of numbers to sets of tic-tac-toe boards. They can also act on objects such as polygons or graphs. In fact, you have probably already seen a group action without realizing it: by definition, the dihedral group of size $2n$ acts on a regular polygon with n sides via rotations and reflections. Groups can even act on themselves, which we will find out to be quite important!

In general, the behavior of a group action give us useful information about the group and the set it acts on. We will explore how various theorems and algorithms hinge upon the theory of group actions, with my personal favorite being how to precisely represent any finite group as a permutation group.

Class format: Interactive lecture

Prerequisites: Group theory

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	(Optional: 🤔) HW Recommended	Algebraic structures Symmetries

Myth of the 13 Archimedean Solids (Lizka, M $\boxed{\text{TW}}$ΘF)
You know about the Platonic Solids: convex polyhedra whose faces are all the same kind of regular n-gon, like cubes and dodecahedra. You might have even seen a proof that there are exactly five of these. A slightly less well known set of polyhedra are the *Archimedean Solids*. In 1619, Kepler showed that there were exactly 13 Archimedean Solids. In 1997, Peter Cromwell included a full proof of this fact in his book, *Polyhedra*.

So people were surprised when Branko Grünbaum showed that there are actually *14* Archimedean Solids. (Moral 1. Beware the pseudo-rhombicuboctahedron.)

In this class, we will explore the Archimedean Solids and other cool sets of polyhedra to see how two lines of reasoning led to an identity crisis for the Archimedeans.

We will discuss the standard story told about this clash, in the process looking at Leonardo da Vinci's illustrations, Albrecht Dürer's experiments, and Kepler's astronomical theories. Then we will learn why this story is completely made up.

The short version of where the story goes astray is that it misses a distinction between discovery and creation in mathematics. For the long version, come to the class!

Class format: Interactive lecture

Prerequisites: None

Chilies	Class Actions	Themes (click for info)
🌶️	(Optional: 💬 👥 🚣) No HW	Rigid shapes

Lattices that make up the world (*Elizabeth Chang-Davidson*, MTW$\boxed{\Theta\mathrm{F}}$)
How many different ways are there to draw a lattice in three dimensions? In order to answer this question, we have to define what a lattice is, and what makes two lattices different from each other. The answer will take us through understanding a variety of symmetries in three dimensions, since we will be classifying these lattices through the ways that they can be left unchanged after various transformations. Many of the materials that surround us turn out to be made of one of these lattice types at a molecular level, and perhaps unsurprisingly, the molecular structure of materials governs many of their physical properties. Lattice structures help explain why diamonds are hard but graphite is soft, why some metals are brittle and others are ductile, and why salt crystals are cubes. We will finish by discussing some of these ways lattices are of interest to materials scientists, engineers, chemists, and crystallographers.

Class format: I will be screensharing me drawing/writing on a tablet for most of the time. Once or twice we will have a drawing exercise where you will be in a breakout room with other students drawing on a Jamboard together.

Prerequisites: Being able to add vectors in $\mathbb{R}^3$ and interpret it geometrically. (If you don't have this background, or want to solidify your understanding, feel free to Slack DM me or to find me during the 2nd half of TAU on Wednesday.)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	✏️ 👥 HW Recommended	Math in real life Rigid shapes Symmetries

Surreal numbers (Aaron, M$\boxed{\mathrm{TW}\Theta\mathrm{F}}$)
A few decades ago, John Conway and some of his colleagues spent several years hard at work playing games. They played classic games like Go, Nim and Dots-and-Boxes, and catalogued new ones like Hackenbush and Sprouts. Eventually a general theory emerged—not the probabilistic game theory of economists, but a deep combinatorial labyrinth of deterministic possibilities.

Out of all this complexity, we will carve a number system. In order to measure the unfairness of partisan games, we'll construct the *surreals*, a number system which contains not only the reals and the hyperreals, but the ordinals, and every infinite or infinitesimal element of any ordered field set theory can build.

Is this overkill to get a little better at the Go endgame? Of course, but as an algebraic structure, it can't be beat.

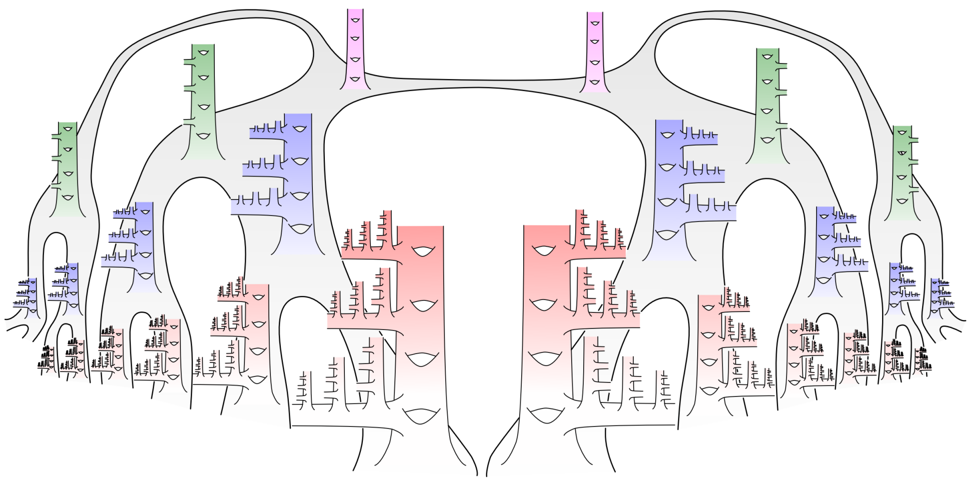
Class format: This class will be mostly problem-solving, while we play some games and start to figure out how to build a number system out of them. Once we have a handle on how the surreals work, I'll tell you a few more results about their big-picture structure.

Prerequisites: Some set theory, including ordinal arithmetic

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	📖 🤔 👥 HW Recommended	Algebraic structures Games

12:10 Classes

Classifying infinite-type surfaces (Assaf, MTWΘF)
An infinite-type surface is a surface which is not finite type. Think about infinite tori stuck together, or a never-ending flute, or whatever the heck this thing is[2]:



You may have heard of the classification of (orientable) surfaces using genus and boundary components, but for infinite-type surfaces, the classification is much much more interesting, involving Cantor sets, and some really fun topological spaces. In this class, we will develop the theory of infinite-type surfaces and state the classification theorem, which was proven in 1963 by Ian Richards. Along the way, we will meet some famous infinite-type surfaces with some great names.

So... imagine Jacob's Ladder, leaning on a Budding Cantor Tree over a Shark Tank. If you would like to learn how to visualize this not as an OSHA violation, but as infinite-type surfaces, come to this class!

Class format: Lecture, with long-ish periods of collaboration

Prerequisites: Topology (know the meaning of a "homeomorphism between compact Hausdorff topological spaces")

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	💬 👥 (Optional: 📖) HW Recommended	Squishy shapes

Graph colorings (Mia, MTWΘF)
A new island has been discovered in the Arctic Ocean! While the geographers are arguing over how to divide the island, the cartographers begin to wonder about the map: how many colors are needed to color the countries so that any two countries that share a border get different colors? The Four Color Theorem says just four. We won't prove this – it took over 100 years and a computer program that checked 1,936 different cases to prove this theorem – but we will use this question as a springboard to others.

Suppose the countries decide that they have *non-negotiable* color preferences. For instance, the country Zudral demands to be cyan or magenta. And the country Scaecia refuses to be anything but

[2]Image taken from *Homeomorphic subsurfaces and the omnipresent arcs* by Federica Fanoni, Tyrone Ghaswala, Alan McLeay

light blue, sky blue, or cornflower blue. Given that each country now has a list of allowable colors, how does that change the cartographer's ability to color the map?

Or what if we are allowed countries to be shaded in with several colors? In this case, Zudral could be indecisive and be one half cyan and one half magenta. Or what if we changed up the objective entirely and instead of focusing on the total number of colors used, we tried to minimize the number of colors "seen" on the neighboring countries?

Secretly, the questions above can be changed into questions about graph colorings, specifically, list coloring, fractional coloring, and "local coloring" respectively. With each new coloring, there arises a new chromatic number and we return to our central questions:

(1) What are the bounds for this chromatic number?
(2) Can we construct a family of graphs that forces this chromatic number arbitrarily far from its bounds?

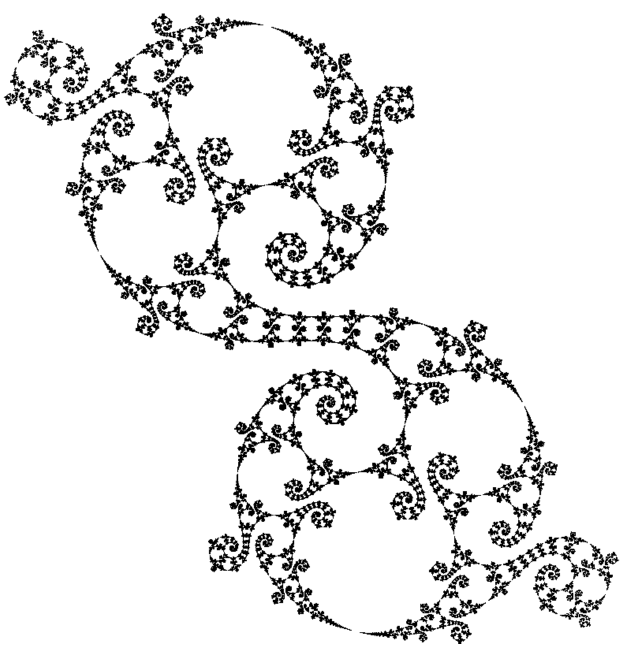
Note: Although maps are an excellent motivating example, we will be focusing on general graphs, not just planar graphs!

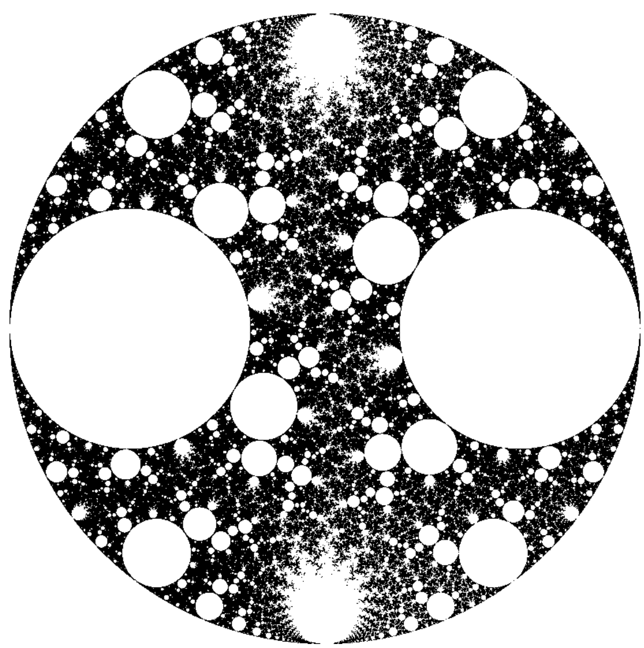
Class format: Interactive lecture

Prerequisites: Graph theory

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	(Optional: 💬) HW Recommended	Graph Theory

Kleinian groups and fractals (*Dan Gulotta*, $\boxed{\text{MTW}\Theta\text{F}}$)





The fractals above are very symmetric. But the symmetries are not the familiar, Euclidean ones; instead they are of the form $z \mapsto \frac{az+b}{cz+d}$, where z is a complex coordinate. Such symmetries are called Möbius transformations. You can generate more fractals like this here: https://guests.mpim-bonn.mpg.de/gulotta/mc21.html.

So how do we construct a group of Möbius transformations that produces fractals like these? Surprisingly, the answer involves something called uniformization. Uniformization is the process of expressing a complicated surface as a quotient of a simpler surface by a group action. This allows us to use group theory to understand geometry and topology, and vice versa.

Our exploration of uniformization will lead to a debate about the proper way to gift-wrap a donut (Fuchs prefers wrapping paper with no holes while Schottky prefers paper with no corners). We will manage to satisfy both sides by taking a bite out of the donut, and we'll get pretty pictures as a reward.

Class format: Most of the time will be spent on lecture, with some opportunities for discussion and exploration. I will use a virtual whiteboard.

Prerequisites: I will assume that you are familiar with complex numbers, the definition of a group, and 2×2 matrices.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	👥 (Optional: 💬) HW Recommended	Symmetries

Noncommutative ring theory (1 of 2) (Susan, $\boxed{\text{MTW}\Theta\text{F}}$)

A ring R is called "simple" if its only ideals are $\{0\}$ and R itself. A commutative ring is simple if and only if it is a field. A good first guess would be that a noncommutative ring is simple if and only if it is a division ring. This would also be a wrong first guess. Unfortunately, in noncommutative rings simplicity gets complicated.

In this class we'll prove the Wedderburn-Artin's structure theorem for semi simple rings, and see how simplicity and division rings are related. We'll start this week with a quick jaunt through module theory. We'll define modules, prove Zassenhaus's lemma and the Schreier refinement theorem, and use these results to prove the Jordan–Hölder's theorem, which essentially says that every sufficiently nice module has a unique decomposition into its simplest components.

Class format: Interactive lecture

Prerequisites: Ring theory, or a solid understanding of the definition of rings, domains, division rings, and ideals.

Required for: Noncommutative ring theory (2 of 2) (W4)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️🌶️	(Optional: 💬) HW Recommended	Algebraic structures

What are your numbers worth? or, the part of algebraic number theory we can actually do (Eric, $\boxed{\text{MTW}\Theta\text{F}}$)

In this class we will figure out what numbers are worth. Some numbers will be worth a lot, other numbers will be worth negative amounts, yet still others will be worth fractional amounts. We will learn the difference between knowing a number (very extremely mind-bogglingly hard) and knowing what a number is worth (surprisingly incredibly magically easy). We will gain the power of being able to figure out the worth of numbers (and many cool corollaries of this power) by making line doodles. In actuality this is a course about the local part of algebraic number theory, but those words won't show up much until the last day. (If you want more technical words: we'll be learning about the p-adic valuation(s) on $\mathbb{Q}$ and finite extensions thereof, through the lens of Newton polygons. But don't worry if you don't know any of these words yet!)

Class format: Here is the tentative plan. Day 1 will be an interactive lecture. Day 2 will be a very short lecture followed by you solving problems on a worksheet. Day 3 will be entirely worksheet.

Day 4 will be back to interactive lecture. Day 5 will be split between lecture and worksheet. All the lecturing will be done through screensharing handwritten slides from an ipad.

Prerequisites: You'll need to be comfortable with modular arithmetic, at the level of being able to easily translate between statements in modular arithmetic and statements in divisibility of integers; Misha's Topics in number theory course in week 1 will set you up well. On day 4 we will do a bit of linear algebra with coefficients in $\mathbb{Z}/p\mathbb{Z}$, but at a 🌶️ pace so don't be scared. There will be a few very optional homework problems that use background in ring theory and linear algebra.

Chilies	Class Actions	Themes (click for info)
🌶️	📖 ✏️ 🤔 (Optional: 👥) HW Recommended	Number theory

Class action descriptions

Different classes will have you engage with the content in different ways. The following actions are meant to help you get a sense for what you might expect to do in each class.

If an action is selected, this means that the teacher is intentionally building this activity into your learning experience. An action may be listed as optional if the teacher is planning to offer space for this activity, but will not expect everyone to participate in it.

- **Active Listening:** You should be ready to do this for every class. May include listening to someone present information, reading things on slides, taking notes, and having opportunities to ask or respond to questions.
- 📖 **Reading:** Reading and interpreting a document for yourself (problem statements, paragraphs, etc.). This does not include anything that will be explained out loud at the same time, such as lecture notes or slides.
- 💬 **Speaking:** Sharing things out loud with others in the class. Ideas, problem solutions, etc.
- ✏️ **Writing:** Drawing or typing your thoughts during the class. Proofs, responses to questions, shared whiteboard, etc.
- 🤔 **Problem solving:** Thinking about how to solve problems that have been given to you.
- 👥 **Collaborating:** Working with others in a small group to accomplish a task.
- 🚣 **Other activitying:** Anything not included in the previous categories: playing a game, using a piece of software (besides Zoom), building a craft, etc.