CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2025

Contents

9:10 Classes

Embeddings, Universality, Hedgehogs, and Metrization (🌶🌶🌶, *Ben Dees*, TWΘFS)
Yes, hedgehogs[1].

In this class, we'll be exploring a variety of topological ideas, centered on discussing how to embed one topological space into another, and how to show that a topological space is homeomorphic to a metric space[2]. In these explorations, we'll talk about a charming result called Kowalsky's Hedgehog Theorem[3] and explore a few consequences.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: The basics of point-set topology (open sets, closed sets, continuity). Audrey's Week 1 class fulfills this prerequisite; I will also occasionally invoke Urysohn's Lemma, as Audrey is proving this result.

Geometric Group Theory (week 1 of 2) (🌶🌶, Arya, TWΘFS)
"Groups, like people, are defined by their actions." - Guillermo Moreno.

The goal of this class is to explore the interplay between the algebraic structure of groups, and the geometry of certain associated spaces. For example, if there is any justice in this world, you would want to think about the integers as "points unit distance away on a line", or a cyclic group as "points equidistant on a circle".

Given any group with a specified subset, you can similarly define a structure called the Cayley graph with respect to this subset. Certain groups also "act" nicely on some geometric spaces (for example, we can study the group of isometries of $\mathbb{R}^2$ by seeing how it acts on the plane). Understanding properties of such actions allows us to study both the algebraic properties of the group, and the geometric properties of the associated space. That's what geometric group theory is all about!

In this two week class, we'll introduce a lot of the building blocks of geometric group theory in Week 1. Week 2 will be an assorted collection of topics that people researching GGT care about. Homework is "required" for Week 1 - each homework set will have a small/reasonable subset of questions that you should try to attempt before next class for optimal understanding.

Homework: Required

[1] Spiny mammals in the subfamily Erinaceinae.
[2] The usual method is "embed the space into a metric space," which is the connection between these two questions.
[3] Named for the German mathematician Hans-Joachim Kowalsky.

Class format: Interactive lecture

Prerequisites: Intro group theory, and the definition of what a graph is. It's also recommended to either have seen what a basis of a vector space is, to know how to multiply matrices, or to talk to Arya before taking the class.

Required for: Geometric Group Theory (W2)

Hilbert Spaces (over $\mathbb{C}$) - What does 1 1/2 linear mean and why is it so helpful? (🌶🌶🌶, Audrey, TWΘFS)
We will figure out this 1 1/2 linear nonsense (and why it could maybe be argued, by me, that it is closer to 1 2/3 or 1 3/4 linear despite the naming convention). We will also look into how this type of structure on an infinite dimensional vector space gives us a ton of information on the functions from this space.

Homework: Recommended

Class format: Lecture

Prerequisites: Intro linear algebra- specifically vector spaces, vector subspaces, a basis. It could also be helpful if students have some notion of inner products, but this is not required.

Introduction to Graph Theory (🌶🌶, *Mira Bernstein*, TWΘFS)
A graph is a mathematical construct that consists of a bunch of objects (called "vertices"), some of which have connections between them (called "edges"). As you saw in Misha's opening colloquium, pretty much anything in math can be represented by a graph. This makes graph theory a powerful framework for approaching a very wide variety of problems.

In this class, we'll look at a sampling of famous elementary problems in graph theory:

- How can you tell if two different diagrams represent the same graph? (graph isomorphism)
- How can you tell by looking at a map of the trails in a park whether it's possible to take a hike in which you walk along each path exactly once? (Eulerian circuits)
- Draw three dots on a piece of paper, representing an electric utility, a water utility, and a gas utility. Draw three more dots, representing three houses. Is it possible to lay pipes/wires connecting each utility to each house, without any of the wires/pipes crossing? (Planarity)
- How many colors are needed to color a map if adjacent countries must have different colors? What if your map is on a torus? (Graph coloring)

The problems are fun in themselves, but our larger goal will be to build up a toolkit of fundamental graph theory concepts and techniques. Since graph theory is everywhere in math, having a basic knowledge of these graph theory tools is essential: you never know where they might come up.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None

Required for: Arithmetical Structures on Graphs (W2); Model Theory (W3); Computing past infinity (W3); Hat puzzles (W3)

Introduction to Ring Theory (🌶🌶🌶, Mark, TWΘFS)
Many, if not most, people would agree that the three most important kinds of structures introduced in abstract algebra are groups, rings, and fields. (Some people would include vector spaces - whose "home" is in linear algebra - and others would include "algebras" - which combine the structures of rings and vector spaces.) Rings are sets with two operations, addition and multiplication, of which the second need not be commutative. Important rings include matrix rings, polynomial rings, and rings of integers (either the ordinary integers, or the analogue of $\mathbb{Z}$ in a larger field than $\mathbb{Q}$). We'll look

at general properties of rings, at examples, and at specific extra conditions that one can impose to make rings "behave better" (which often means "behave more like $\mathbb{Z}$"). Presumably, the examples will include rings that occur naturally in algebraic number theory and in algebraic geometry. In particular, we probably won't spend much time on non-commutative rings.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None, although having studied some sort of algebraic structure (the most likely ones are groups, vector spaces, and fields) would probably help.

10:10 Classes

Badly Behaved Sets (🌶🌶, Sam, WΘFS)
Have you ever wondered how we measure the size of a set? One may consider the number of elements of a set. However, in this case, we get into all sorts of problems with having to define all sorts of infinities. Another way you may think about doing this for intervals, is by taking the length of the interval, or in higher dimensions, the area/volume of n-dimensional boxes. But how do you extend this idea to sets that aren't intervals or aren't even a finite union of intervals like $\mathbb{Q}$? To do this, we will define and study the Lebesgue measure, which allows to extend this idea of length/area/volume to more complicated sets. But beware, you will soon see that not all is happy in Lebesgue land, as you will come to find that there are indeed sets that we can't measure in this way. In order to find such a set, we will use the mythical and often questioned, axiom of choice. There will be a lot of other cool things on the way.

Homework: Recommended

Class format: Interactive lecture and worksheets

Prerequisites: Basic set notation, convergence and properties of sequences and series, open and closed sets.

Finite Geometries (🌶, Misha, WΘFS)
In 1899, Hilbert wrote a set of 16 axioms that uniquely characterize Euclidean geometry: there is only one object[4] that obeys them, and it is the Euclidean plane.

That's a lot of axioms (in binary, it would be 10000 axioms) and so we'll stop reading after axiom 3 and take what we can get. What we can get still includes the Euclidean plane; however, it also includes, for example, the set of cards in a SET deck.

Serious people study finite geometries to design multi-factor experiments, or to buy tickets that guarantee them a profit in lotteries. They also give us a zoo of examples small enough to understand completely, and yet rich enough to shed light on more ordinary geometry.

Homework: Recommended

Class format: Interactive lecture, with some group activities that might happen during class or might be left for homework, I haven't decided yet.

Prerequisites: None.

Introduction to Descriptive Set Theory (🌶🌶🌶🌶, Maya, WΘFS)
The full title of this course is: Finding the Cantor Space Inside Polish Spaces, or, an Introduction to Descriptive Set Theory

This course will introduce some fundamental spaces and concepts from descriptive set theory, heading towards Cantor-Bendixson derivatives and ranks and the Cantor-Bendixson theorem, which is

[4]Up to isomorphism, to be technical—more technical than we will get this week.

very cool for multiple reasons: it is tied with the origins of set theory (as Ben mentioned in his Week 1 colloquium, though if you didn't attend it that's perfectly alright), and it settles the continuum hypothesis for a whole class of spaces.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: To dive into this course, you need to be familiar with topological spaces, bases for topological spaces, the subspace topology, basic properties of open and closed sets, Cauchy sequences, the relationship between closed sets and limits of sequences. If you are familiar with, say, all but a couple of these concepts, you will still be fine. If you are missing more than a couple, talk to me! Thorough experience with the topology of the real line will also suffice.

Singular Value Decomposition (🌶🌶🌶, Kaia, WΘFS)

How can we change from one basis to another? When do the eigenvectors of a matrix form a basis? Why is a basis of eigenvectors so special anyways?

One version of the Spectral Theorem tells us that the eigenvectors of a symmetric matrix form a basis, and allow us to diagonalize a matrix. But what can we do if our matrix isn't symmetric? What if it isn't even square?

And what does any of this have to do with denoising and compressing images?

Learn the answer to these questions and more as we build up to the Singular Value Decomposition, and think about the best way to approximate a matrix by another matrix of lower rank.

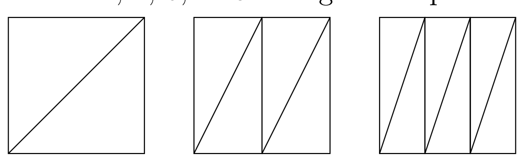
Homework: Recommended

Class format: Interactive lecture

Prerequisites: Intro linear algebra or equivalent

Triangles in a Square: How hard can it be? (🌶🌶, Glenn, WΘFS)

You can very easily cut a square into 2, 4, 6, or 8 triangles of equal area, or any even number.



But how about an odd number of triangles? It's hard to find explicit lengths for to cut a square into, say 5 pieces, but it almost feels like you ought to be able to nudge the points around until they just all land in the right spots. However, it turns out: the answer is no! There is no way to cut a square into an odd number of triangles of equal area.

Even more surprising than the answer being no, this fact is about pure Euclidean geometry, and yet the proof involves:

- Sperner's lemma, a result in **graph theory** and **combinatorial topology**,
- p-adic numbers, a topic in **algebraic number theory** and **analysis**,
- **axiom of choice**, and
- a little bit of determinants, from **linear algebra**.

It really is all the branches of math coming together to solve one problem! No expectation that you've seen any of these before. We'll explain each part and put them together to prove this theorem of Monsky.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

11:10 Classes

Arithmetical Structures on Graphs (🌶🌶, *Joel Louwsma*, TWΘFS)
Given a graph (a set of vertices, some of which are connected by edges), label the vertices by positive integers such that, at each vertex, the number there divides the sum of the numbers at adjacent vertices. If I give you a graph, can you always find such integers? How many different ways can this be done? What if we require that the greatest common divisor of the integers used is 1? These labelings are called arithmetical structures, and we'll see how to formulate questions about arithmetical structures in terms of linear algebra by using matrices associated to the graph.

Every arithmetical structure gives a different version of the chip firing game (discussed in Wednesday's colloquium). We'll also explore how answers to questions about whether all chip configurations are equivalent (or whether all chip configurations are equivalent to a multiple of a certain configuration) depend on the graph and how they depend on the arithmetical structure. Although these types of questions are not difficult to state or start exploring, they lead quickly to topics of current research interest.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None

Breaking the Axiom of Choice (🌶🌶🌶🌶, Steve, TWΘFS)
The axiom of choice (or rather, one of the many equivalent versions of the axiom of choice) states that whenever I have a collection of nonempty sets $(X_i)_{i \in I}$, there is a way to "pick" an element of each set - precisely, there is a function f with domain I such that $f(i) \in X_i$ for each $i \in I$. This is an extremely powerful axiom used to prove many important things. We won't see that.

The big problem with AC, in my opinion, is that it's *too* compelling; it's not clear how it could possibly be false! This unfortunately obscures lots of interesting math around the axiom of choice.

The purpose of this class is to see how to create "toy universes" of mathematics in which the axiom of choice *fails.* Do you want an infinite set that can't be split into two infinite pieces? What about an uncountable group that has only countably many subgroups? Come to this class!

Homework: Recommended

Class format: Lecture

Prerequisites: Group theory, familiarity with naive set theory and the statement of the axiom of choice

Functions of a Complex Variable (week 1 of 2) (🌶🌶🌶, Mark, TWΘFS)
Spectacular (and unexpected) things happen in calculus when you allow the variable (now to be called $z = x + iy$ instead of x) to take on complex values. For example, functions that are "differentiable" in a region of the complex plane now automatically have power series expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both in and outside math. For example, functions of a complex variable were used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if a and b are positive integers with $\gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, \ldots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann Hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet's theorem is certainly beyond the scope of this class and heat conduction probably is too, but we'll prove an important theorem due to Liouville that 1) leads to a proof of the so-called "Fundamental Theorem

of Algebra", which states that any nonconstant polynomial (with real or even complex coefficients) has a root in the complex numbers and 2) is vital for the study of "elliptic functions", which have two independent complex periods, and which may be the topic of a week 5 class. Meanwhile, we should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and venturing boldly forth into the complex plane! This class runs for two weeks, but it should be worth it. (If you can take only the first week, you'll still get to see a good bit of interesting material, including one or two of the things mentioned above.)

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Multivariable calculus (the week 1 crash course will be enough)

Homomorphic Encryption (🌶🌶, Eric, TWΘFS)
Normally you might think about encryption as a closed box: encrypting something is like putting it in a box and locking it. You can send the box around to lots of people safely, but they can't really do anything with the stuff inside the box without unlocking it. *Homomorphic* encryption is the idea that you can encrypt data in ways that let you perform operations on it while keeping it totally encrypted. In this class we'll learn about a system for performing fully homomorphic encryption that is just based on addition and multiplication of integers! We'll show how to take a simple system that kind of works and extend it using a self-referential *bootstrapping* technique so that we can safely perform arbitrary computations on encrypted data!

In terms of the actual math content of this class there will be lots of thinking carefully about integer and modular arithmetic; some discussion about boolean logic and boolean circuits as our model of computation; and little tidbits of algebra, combinatorics, and probability scattered throughout.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Comfort with modular arithmetic and writing integers with binary rather than decimal expansions.

The Polynomial Method in Combinatorics (🌶, Charlotte & Narmada, TWΘFS)
The polynomial method in combinatorics has existed for tens of years, but it really popped off in 2010 with Dvir's proof of the famous Kakeya conjecture over finite fields. Now, the Kakeya conjecture isn't even the most interesting application of the polynomial method! We'll look at two powerful tools from algebra—the Schwartz-Zippel Lemma and the Combinatorial Nullstellensatz—and use them to find surprisingly effective solutions to counting problems. If you like working with polynomials and solving counting problems, come to our class!

Homework: Recommended

Class format: Mostly group work

Prerequisites: None

1:10 Classes

Catalan Structures (🌶🌶, Riley, TWΘFS)
When is a binary tree less than another binary tree? Why is (ab)(c(de)) like the least common multiple of ((ab)c)(de) and ((ab)(cd)e)? Why are these related? In this class, we learn about combinatorial bijections between different families of objects that the Catalan numbers count. Additionally, we look at the most usual partial order on Catalan objects, the Tamari lattice, and aim to prove that it is

indeed a lattice. If you are fond of drawing, you may particularly like this class as there will be lots and lots of it!

Homework: Recommended

Class format: IBL with occasional lecture portions

Prerequisites: Know what a combinatorial bijection is

Category Theory from Scratch (🌶🌶🌶, Della & Purple, TWΘFS)

When we study the cartesian product of sets, there are some key properties we prove about it, like its associativity and commutativity.

When we study the direct product of groups, there are some key properties we prove about it, like its associativity and commutativity.

When we study the direct sum of vector spaces, there are some key properties we prove about it, like its associativity and commutativity.

When we study the product of topological spaces, there are some key properties we prove about it, like its associativity and commutativity.

When we study the smash product of pointed connective spectra, there are some key properties we prove about it, like its associativity and commutativity.

Wait. Can we do all these proofs at once?

Homework: Recommended

Class format: Mostly exploratory group work

Prerequisites: Standard constructions on sets (e.g. cartesian product, disjoint union, bijections), and at least one of the other examples mentioned in the blurb (or similar)

Problem-Solving: Induction (🌶, Zach, TWΘFS)

Take your inductive problem-solving and proof-writing skills to the next step (and the next, and the next…). This class targets:

- Confident inductioneers interested in strengthening their induction ~~hypotheses~~ skills by solving challenging problems and seeing induction used in new and surprising ways.
- Newcomers to this most prized proof technique, hoping to build a solid base ~~case~~ with practice, writing feedback, and discussions of common pitfalls.
- Everyone in between!

You'll work together to select and solve problems with a wide range of difficulties, on topics including: alternate induction formulations (strong induction, well-ordering-principle); invariants; hypothesis strengthening; batching and nesting; Cauchy induction and other reordering; common mistakes; infinite constructions; well ordered sets; and more. Be prepared to present your favorite solutions to the rest of the class as well!

Homework: Recommended

Class format: Lecture, group problem solving, and student presentation

Prerequisites: Has written a proof by induction before

Solving Sudokus Fractionally: Linear Programming (🌶🌶, Nikita, TWΘFS)

What if a Sudoku cell could be 30% six and 70% nine? In the linear relaxation of Sudoku, each empty cell is assigned a fractional amount of each digit, adding up to 1. We then ask: when is the solution uniquely determined?

This class is an introduction to linear programming, which, despite its name, is not computer programming, but a beautiful branch of pure math about optimizing linear constraints. We'll see how classic Sudoku-solving techniques like X-wing and Swordfish are secretly applications of Hall's Marriage Theorem, and use LP to analyze and solve puzzles in a new way.

Homework: Recommended

Class format: Handouts + interactive lecture

Prerequisites: Being able to solve sudoku

The Other Other Analytic Number Theory (Modular Forms) (week 2 of 2) (🌶🌶🌶🌶, *Dave Savitt*, TWΘFS)

(This is the same blurb as Week 1.)

It is sometimes said that there are five elementary operations in arithmetic: addition, subtraction, multiplication, division, and modular forms.

This two-week class will be a hands-on introduction to modular forms. We'll start the first week with some vignettes about infinite products from the work of Euler: his formula for the sine function

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

and the pentagonal number theorem

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{k \in \mathbb{Z}} (-1)^k q^{k(3k-1)/2}.$$

(Think for a moment about how amazing this formula is, how much unexpected cancellation has to happen for the right-hand side to have terms only in degrees $k(3k-1)/2$.). We'll also introduce the Bernoulli numbers and some of their basic properties.

Then we'll fast forward to the 19th century to see how these are all tied together by the theory of modular forms, with some amazing applications to arithmetic. For example, defining

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

we'll prove the congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$, where $\sigma_{11}(n)$ denotes the sum of the 11th powers of the divisors of n. If time permits we'll prove Jacobi's four-square theorem, that the number of ways of writing a positive integer n as a sum of four squares is 8 times the sum of the divisors of n that are not divisible by 4. (If time doesn't permit, the proof will be in the lecture notes.)

Homework: Recommended

Class format: Interactive lecture

Prerequisites: The Other Other Analytic Number Theory week 1

The Shape and Soul of a Surface: the Gauss Bonnet Theorem (🌶🌶🌶, Laithy, TWΘFS)
What do a sphere, a donut, and a pretzel have in common? Each has a different shape — but even more interestingly, a different soul. In this class, we'll explore one of the most beautiful and surprising theorems in mathematics: the Gauss–Bonnet Theorem, which reveals that the total curvature of a surface encodes deep information about its topology — that is, its fundamental shape.

We'll start by understanding two key characters in this story: the Gaussian curvature, which captures how a surface bends, and the Euler characteristic, a topological invariant tied to the number of vertices, edges, and faces in a triangulation. Then we'll bring them together in the Gauss–Bonnet Theorem, a stunning formula that links geometry and topology in a single elegant equation.

As you can imagine, the Gauss Bonnet theorem has many powerful applications. If time permits, we will study some of its applications, including holonomy and parallel transport, the map-colouring problem in graph theory and why every country has at most five neighbours, and how geometry can detect "holes" on a surface.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Differential Geometry of Surfaces class. (It will be good to know what a suface is and its Gaussian curvature is).

COLLOQUIA

Chip Firing on Graphs (*Joel Louwsma*, Wednesday)
Suppose we have a graph (a set of vertices, some of which are connected by edges) and that at each vertex there is some number of chips. Each vertex can give a chip to *each* of its neighbors, and each vertex can take a chip from *each* of its neighbors. Using these moves, can we get from any configuration of chips to any other configuration of chips? If not, how many inequivalent chip configurations are there? What if we allow vertices to have a negative number of chips? And what do the answers to these questions tell us about the structure of the graph we started with?

What if Algebra but Equality is Vibes? (Purple, Thursday)
In math we often encounter operations—like the multiplication of real numbers—with nice properties, like the associative property $(xy)z = x(yz)$. We then define algebraic gadgets via listing certain desirable properties, and algebra is in some sense the study of these abstract algebraic gadgets. But what if these equalities held not literally "on the nose," but only up to some weaker notion of equivalence? And then what if that notion of equivalence, itself, has some properties of its own, each of which also only hold up to some second notion of equivalence? And then what if this process repeats forever?

This seemingly far-fetched situation in fact occurs in many places in modern mathematics. The study of situations like this—sometimes called homotopy theory or higher algebra—is an extremely active area of research. This talk will give a zero-prerequisite geometric example of such a situation, and discuss some of the abstract tools we use to get a handle on it.

On Colors (Nikita, Friday)
I'll tell you all I know about colors. We will explore why the space of colors is three-dimensional and learn how to perceive impossible colors.