CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2025

Contents

9:10 Classes

Calculus Without Limits (, Glenn, $\boxed{\mathrm{TW\Theta F}}$S)

I hate limits.

In school, you likely learned calculus defined using limits. “For every $\epsilon > 0$, there exists a $\delta > 0$, such that”—ugh, I’m tired just by writing it down! Limits are a powerful, rigorous tool, but at the same time, ϵ-δ definitions are often difficult to reason about, having so many nested quantifiers.

But did you know that limits were not invented until 1821 by Augustin-Louis Cauchy, more than 150 years after Isaac Newton introduced calculus in the 1660s? And in fact, the more intuitive methods (without limits) pioneered by Newton and Leibniz were commonly taught in schools until as late as the mid-1900s. It is entirely possible to do calculus without limits!

This course is about doing calculus in various ways, all avoiding limits. Each day, we’ll cover something different.

- Day 1: The historical Newton/Leibniz way, treating dx and dy as actual manipulable mathematical symbols.
- Day 2: A more modern way, borrowing big-O asymptotic notation from computer science.
- Day 3: A rigorous way to define “infinitesimal” numbers, called non-standard analysis.
- Day 4: How modern mathematicians understand the meaning of dx, at least in integrals.

Homework: Optional

Class format: Interactive lecture

Prerequisites: +1 chili if this is your first time doing calculus

Diophantine Approximation and Transcendental Number Theory (, *Sarah Peluse*, $\boxed{\mathrm{TW\Theta F}}$S)

This course will cover how the rational numbers sit in the real line, including how well various sorts of real numbers can be approximated by rationals with small denominators. We will also discuss the closely related area of transcendental number theory, and see why π and e are irrational numbers.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Some familiarity with elementary number theory (like modular arithmetic).

Diophantine Approximation and the Putnam (, Misha, TWΘF$\boxed{\mathrm{S}}$)

Here’s something that math competitions and real-world math problems have in common: occasionally, what it takes to make progress is to remember a key idea that you've seen before in a completely unrelated context. We'll see what happens when that key idea is related to rational approximations of irrational numbers.

I will focus on applications to the Putnam competition, but my goal is not to train you for an olympiad but to teach you how to take a proof technique you've seen, squint at it sideways, and use it to do something new.

Homework: None

Class format: A mix of lecture and problem-solving.

Prerequisites: It will help to have attended the Diophantine Approximation and Transcendental Number Theory class scheduled for the first four days of this week.

Magic of Harmonic Functions (🌶🌶🌶🌶, Alan Chang & Laithy, $\boxed{\text{TW}\Theta\text{FS}}$)
What do heat flow, gravitational potential, and soap films have in common? They're all governed by harmonic functions: functions whose value at every point equals the average of their values nearby. In this class, we will explore the consequences of this property, such as infinite differentiability, analyticity, the maximum principle and Liouville's theorem.

Harmonic functions serve as a deep bridge between physics, geometry and analysis, and we will explore why they show up naturally in these areas. If time permits, we may take a peek into the discrete world and see how they appear in random walks, or even glimpse the connection between harmonic and holomorphic functions in complex analysis

Homework: Recommended

Class format: Interactive lectures

Prerequisites: Multivariable calculus (partial differentiation and integration). Stokes theorem and the divergence theorem is recommended but not required.

Model Theory (🌶🌶🌶, *Aaron Anderson*, $\boxed{\text{TW}\Theta\text{FS}}$)
At Mathcamp, we encounter loads of different mathematical widgets. There are groups, graphs, posets, tosets, rings, fields, vector spaces, and more. That's a lot to keep track of, but with model theory, we can view all of these as examples of the same phenomenon.

We'll tie all these together with a nice logical framework. We'll give general definitions of "mathematical structures," "axiom systems," and "proofs".

Then we'll use those definitions to construct some Alice-in-Wonderlandishly weird examples. A theorem that makes structures big, a theorem that makes structures small, infinite natural numbers, infinitesimal reals, and tiny universes of set theory that can fit in your (countably infinite) pocket.

Homework: Required

Class format: Lecture. Homework will connect the lecture topic to other areas of math (required, but you can pick and choose which to do), and prove a big theorem I don't have time for in lecture (optional, but very fun IMO).

Prerequisites: Either graph theory, group theory, or ring theory (there are a lot of contexts where we can find model-theory examples, but it's important to have at least one you're comfortable with).

Oops All Algebra: An Introduction to Infinity Categories (🌶🌶🌶🌶, Purple, $\boxed{\text{TW}\Theta\text{FS}}$)
So you liked categories... what if I told you a category was secretly just a 1-category! And you can also study 2-categories or 16-categories or 140-categories! What if you keep going—it is possible to study ∞-categories?

In a 1-category you have some objects and some morphisms between them. In a 2-category, you have objects, morphisms between them, and morphisms between morphisms. And the pattern continues—so in an ∞-category, this pattern continues forever.

Ok so this is a thing, but is it useful? Well not if you do it in the most direct way. It turns out that, for applications, laws like associativity ought to no longer hold directly; instead we want them to hold only "up to coherent equivalence," in which case this notion does have lots of applications to algebraic topology, algebraic number theory, (derived) algebraic geometry, representation theory, and more. But defining these coherent equivalences turns out to be quite difficult. In this class we will see one way to do so!

Homework: Recommended

Class format: Interactive lecture, possibly with some group work.

Prerequisites: The definitions of category, functor, and isomorphism

The Lost Art of Slide Rules (🌶️, Glenn, TWΘF$\boxed{\text{S}}$)
Before the invention of the calculator, there was the mighty slide rule: a pocket device that could be used to quickly perform multiplication, division, exponentiation, logs, trigonometric functions, and more. And they're not even really ancient history—your grandparents likely used these in school, and NASA's Apollo mission crews used slide rules to do computations in space! Significantly faster than an abacus, these fancy rulers were the device of choice for millions of engineers around the world up until the late 1900s. In this class, we'll learn how to use slide rules for various computations, as well as how to design a slide rule from scratch!

Homework: None

Class format: Hands-on worksheets and group work

Prerequisites: None.

10:10 Classes

Computing Past Infinity (🌶️🌶️🌶️, Della, $\boxed{\text{W}\Theta\text{FS}}$)
Usually when you run an algorithm, you expect it to finish at some point and tell you the answer. But what if instead it continued running forever? Could you still get information out if it by looking at the steps it took? Can it output an answer after running forever? If the algorithm still isn't done after running forever, can we run it some more?

We'll explore two algorithms that can take infinite time, and see how all of these questions can have affirmative answers. Along the way, we'll encounter fun characters including continued fractions and ordinal numbers.

Homework: Recommended

Class format: Lecture

Prerequisites: Basic graph theory

Cut That Out! (🌶️🌶️, Zach, $\boxed{\text{W}\Theta\text{FS}}$)
Let's cut shapes into other shapes and assemble them into even more shapes! We'll have a plethora of pretty pictures and a panoply of perplexing puzzles, including:

- Can you divide a square into (any number of) polygons that are similar to each other but all have different sizes? Can you do it with just 2 pieces? 3? Try it!
- Can you cut a square into (any number of) polygons and rearrange them exactly into an equilateral triangle? What about funkier polygons? If we allow curved cuts, can you rearrange

the square into a circle? What about a cube into a tetrahedron using 3D pieces? What if we allow infinitely many cuts? Or fractal cuts? or ...

Shaaaapes!

Homework: Optional

Class format: Lecture

Prerequisites: None

Functions of a Complex Variable (week 2 of 2) (🌶🌶🌶, Mark, $\boxed{\text{W}\Theta\text{FS}}$)

This is a continuation of the week 2 course. If you didn't take that course but you want to join now, please consult with Mark.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Week 1 of "Functions of a Complex Variable" (or equivalent knowledge)

Orthogonal Projections (🌶, Riley, $\boxed{\text{W}\Theta\text{FS}}$)

Are you a boring person? A square? Orthogonal even? Or perhaps I'm just projecting. If indeed you are a boring person like me, you probably enjoy seeing the same idea used over and over again in different contexts. Such an idea that we will focus on in this class is orthogonal projection. Minimizing distances is pretty cool. In this class, we will show the properties of orthogonal projections, find linear regressions using them, try to approximate complicated functions with polynomials, and go to the infinite-dimensional moon by looking at Fourier series. Be there, and be square!

Homework: Optional

Class format: Inquiry-based learning with some lecture portion,

Prerequisites: Knowing the concept of dimension and span may be helpful but is not required. Some of the later topics require an understanding of integrals and convergence of infinite series.

The Hales–Jewett Theorem (🌶🌶🌶🌶, Misha, $\boxed{\text{W}\Theta\text{FS}}$)

The Hales–Jewett theorem is a classic result in Ramsey theory that, informally, says that "high-dimensional tic-tac-toe can never end in a draw." It is known for (1) many applications to other problems, and (2) eeeeenormous upper bounds. We will see two proofs of this theorem, and also visit exciting locales such as hypergraphs, arithmetic progressions, and point constellations.

If time and your enthusiasm both allow it, we may also extend the theorem to the Graham–Rothschild parameter sets theorem, and learn the meaning behind Graham's number: a quantity that once held the record for the largest integer to be used in a mathematical proof.

Homework: Recommended

Class format: Lecture

Prerequisites: None

11:10 Classes

Dirichlet's Class Number Formula (🌶🌶🌶🌶, *Viv Kuperberg*, $\boxed{\text{TW}\Theta\text{FS}}$)

Here's a fun fact:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Some people think of this fact as a fact about the Taylor expansion of arctangent. But I think of it as a very deep fact about the function $x^2 + y^2$, or in other words, a specific example of Dirichlet's wonderful class number formula.

A *binary quadratic form* is a function

$$f(x,y) = ax^2 + bxy + cy^2,$$

with $a, b, c \in \mathbb{Z}$, which has *discriminant* $d = b^2 - 4ac$. The *class number* of a given discriminant d is the number of equivalence classes of binary quadratic forms with discriminant d under a certain group action.[1] Dirichlet's proof of his class number formula is a truly beautiful argument, with ideas ranging from group theory to clever averaging to, at one point, the area of an ellipse. Come explore one of my favorite proofs of all time!

Homework: Recommended

Class format: Lecture

Prerequisites: Group Theory; if you haven't seen group actions, talk to me! It is also helpful if you have seen the Chinese Remainder Theorem, and 2-by-2 matrices (how they multiply, and how they act on 2-dimensional vectors). You should either have seen quadratic reciprocity or be willing to believe it as a black box.

Geometric Group Theory (week 2 of 2) (, Arya, TWΘFS)
Last week, we learnt a bit about Cayley graphs of groups, and how their geometry tells you about algebraic properties of groups. This week, we shall talk about an assortment of topics that modern geometric group theorists care about. Roughly, these topics are in two flavours - "make stronger assumptions about the geometry of the Cayley graph to say more things about the group", and "find spaces other than the Cayley graph where the group acts on nicely". The pseudo-goal is that Arya wants to talk about his friends'/advisor's research. It'll be a party!

Homework: Optional

Class format: Interactive lecture

Prerequisites: Week 1 of GGT. If you didn't take that class, come find me!

Hat Puzzles (, Nikita, TWΘFS)
Disclaimer: We will discuss finitary hat puzzles, no axiom of choice stuff, though the puzzles will sometimes involve probability.

You all know some puzzles about wise men or prisoners wearing hats, looking at each other and guessing the color of their own hat. Nice puzzles spread through networks of mathematicians like a juicy bit of gossip, so I will spread it further. We will explore classic hat puzzles, prove optimal strategies, and discuss related problems. We will see Hall's and Tutte's theorems, optimal designs and Levine's notorious hat puzzle still waiting for a resolution.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

QR Factorization (, Kaia, TWΘFS)
Most matrices are dense. They have all or almost all nonzero entries.

They don't listen to what we say. They won't kindly choose to be triangular matrices instead, despite how clear we've made it to them that we prefer triangles to squares.

It's time to take control. We, as mathematicians, are endowed with the awesome power of rotations and reflections, which we shall use to make any matrix into a triangular matrix (up to orthogonal transformation), to defeat the evil dense matrices!

[1] Nic might tell you this has something to do with unique factorization in quadratic extensions of $\mathbb{Q}$, but we won't get into that.

Oh, yeah, and doing this efficiently and stably is also very useful for solving least squares problems, finding eigenvalues, and more.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Intro linear algebra: bases, orthogonality, eigenvalues and eigenvectors

Representation Theory of Finite Groups (week 1 of 2) (🌶🌶🌶🌶, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)
It turns out that you can learn a lot about a group by studying homomorphisms from it to groups of linear transformations (if you prefer, groups of matrices). Such a homomorphism is called a *representation* of the group; representations of groups have been used widely in areas ranging from quantum chemistry and particle physics to the famous classification of all finite simple groups. For example, Burnside, who was one of the pioneers in this area along with Frobenius and Schur, used representation theory to show that the order of any finite simple group that is not cyclic must have at least three distinct prime factors. (The smallest example of such a group, the alternating group A_5 of order $60 = 2^2 \cdot 3 \cdot 5$, is important in understanding the unsolvability of quintic equations by radicals.) We may not get that far, but you'll definitely see some unexpected, beautiful, and important facts about finite groups in this class, along with proofs of most or all of them. With any luck, the first week of the class will get you to the point of understanding character tables, which are relatively small, square tables of numbers that encode all the information about the representations of particular finite groups; these results are quite elegant and very worthwhile, even if you go no further. In the second week, the chili level may ramp up a bit (from about $\pi + 0.4$ to a true 4) as we start introducing concepts from elsewhere in algebra (such as algebraic integers, tensor products, and possibly modules) to get more sophisticated information.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra (including eigenvalues and eigenvectors, although I can try catching up people individually on some of that), group theory, and general comfort with abstraction.

1:10 Classes

Continuous But Nowhere Differentiable Functions Are Everywhere (🌶🌶, Charlotte, $\boxed{\text{TW}\Theta\text{FS}}$)
The "Monsters of Analysis," first discovered by Weierstrass, inspired this lovely quote from Hermite: "I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives." This was the reaction to the existence of just one continuous but nowhere differentiable function – imagine how Hermite would have reacted upon discovering that in fact, *"most" functions that are continuous are nowhere differentiable.*

In this class we'll figure out what we mean by "most" and prove that yes, your calculus teachers have been lying to you – in some topological sense, most continuous functions are horrible rather than nice. This class is also my apology for not running an Introduction to Real Analysis Class at Mathcamp while possessing the power of the AC. (Blame Arya.) As such, we'll actually define and prove some things that may have been glossed over in other analysis classes and colloquiums (blame Ben) – with particular focus on uniform convergence and completeness.

Homework: Recommended

Class format: Interactive lecture plus group work

Prerequisites: Be comfortable with the definition of a metric space, and with the ϵ-δ definition of continuity. You should also be comfortable with the definitions of open and closed sets, and the closure of a set.

Einstein's Theory of Gravity 1: Special Relativity (🌶🌶, Laithy, $\boxed{\text{TW}\Theta\text{FS}}$)
What if I told you that time passes differently for your head than for your feet? That a moving train is literally shorter than a stationary one? That "now" means different things to different observers? Welcome to Einstein's universe.

In 1905, a 26-year-old patent clerk named Albert Einstein challenged our intuitions about reality with two simple postulates:

- The laws of physics looks the same to every inertial observer.
- Light always travels at the same speed whether you're chasing after it or running away.

Through the postulates, which are supported by numerous experiments, Einstein revealed that space and time are not the fixed stage Newton imagined, but rather a unified, flexible, geometric "4-dimensional manifold" called spacetime.

In this class, we'll begin by exploring why Newton's perfectly sensible picture of absolute time and space falls apart when things move really fast. We will realize that speeds don't add up as Newton thought, and we will derive the Lorentz transformation (the correct transformations describing a change of reference) from the postulates of relativity. We will derive the non-Euclidean geometry of spacetime, called Minkowski spacetime, and study its causal structure and counterintuitive consequences — such as time dilation, length contraction, and the twin paradox. We will rebuild mechanics and dynamics by defining the 4-velocity, relativistic momentum, relativistic energy of moving particles. We will also derive the famous $E = mc^2$ as one of the most powerful (and most dangerous) predictions of Minkowski spacetime.

Homework: Recommended

Class format: Linear algebra (linear independence, basis, matrices). Multivariable calculus (partial differentiation)

Prerequisites: None.

Infinite Trees (week 1 of 2) (🌶🌶🌶, Susan, $\boxed{\text{TW}\Theta\text{FS}}$)
König's infinity lemma states that a tree of infinite height with finite levels must have an infinite branch. So let's ask the obvious[2] followup question: what happens when you have a tree of uncountable height with countable levels?

Surprisingly,[3] no it doesn't! In this class we'll explore what it means for a tree to have uncountable height, and show that it's possible for such a tree to have only countable branches and levels. And this isn't even the weirdest object we'll see. We'll explore an extra-set-theoretic axiom called the Diamond Lemma, and show how it can be used to construct trees whose very existence is independent of our standard collection of set theory axioms.

Come to this class if you're excited about exploring infinite objects, want to meet a mathematical fortune teller that gets the right answers for the wrong reasons, or are dying to know how to pronounce the name "Aronszajn."

Homework: Recommended

Class format: Interactive Lecture

Prerequisites: None

Problem Solving: Cheating in Geometry (🌶🌶🌶 → 🌶🌶🌶🌶, *Zack Chroman*, $\boxed{\text{TW}\Theta\text{FS}}$)
Geometry is hard. Sometimes you can bash geometry problems with algebra, but algebra is hard too. Everything would really be a lot nicer if geometry were easy, like if every pair of lines intersected or if every circle passed through the same two points. Helpfully, projective geometry (motto: "what if geometry were better") exists! In projective space, everything is great, lines and curves behave how

[2] for sufficiently weird values of "obvious"

[3] for sufficiently small values of "surprise"

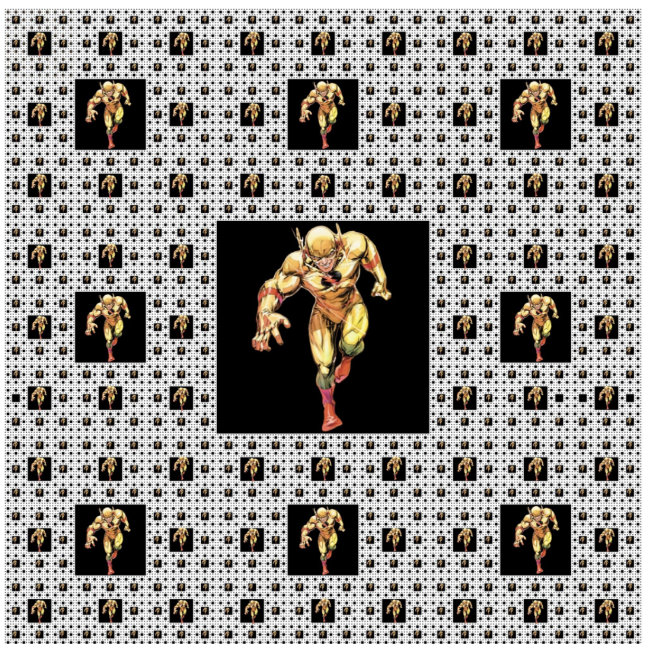
they should, and geometry is easy.[4] We'll build some intuition for projective space through examples, and discover some geometric and algebraic tools which will sometimes allow us to solve hard geometry problems quickly and easily, in particular the somewhat infamous "method of moving points." Side effects may include, but are not limited to: an inability to return to thinking about angles and lengths, a tendency to write solutions that will make your graders sad, and following every sentence with "you know, this would really be a lot nicer over CP2"

Homework: Recommended

Class format: Interactive lecture + group work

Prerequisites: None—in particular, no experience with olympiad geometry will be assumed.

Reverse Flash: Fractal Geometry (🌶🌶🌶🌶, Narmada, $\boxed{\text{TW}\Theta\text{FS}}$)
The Reverse-Flash was a meta-human speedster from the twenty-second century, a time criminal, descendant of Eddie Thawne, and the archenemy of Barry Allen/The Flash. We will also commit time crimes in this class by learning fractal dimensions in reverse: first we will look at examples of what we want the dimension to be, then we will list properties that the dimension should satisfy, and finally we will discover a definition of dimension that works.



Each day, I will give you some tools to discover for yourself the definition of one of the standard fractal dimensions. At the end of the class, we'll switch to a more lecture-based style to see effective tools for computing dimensions and possibly to discuss the Kakeya conjecture. Even if you've seen some definitions of fractal dimensions before, there will still be interesting problems for you to prove their properties and compute some examples, so talk to me to find out if this class will be a good fit for you!

Homework: Recommended

Class format: IBL and group work

Prerequisites: Comfortable manipulating epsilon-delta limits; Know what open and closed balls are; Comfortable manipulating logs; Know what a continuous function is; Know what a limsup and liminf are.

[4] OK, maybe not easy, but at least there aren't any angles.

Colloquia

Sparsest Cut and Metric Spaces (*Alan Chang*, Tuesday)
How do you divide a graph into two fairly balanced parts without cutting too many edges? This is the sparsest cut problem. In this colloquium talk, we will discuss how the geometry of metric spaces helps us find good solutions.

Chebyshev's Bias (*Viv Kuperberg*, Wednesday)
Here's a programming exercise:

Let $f_1(N)$ denote the number of primes $p \leq N$ such that p is congruent to 1 mod 3, and let $f_2(N)$ denote the number of primes $p \leq N$ such that p is congruent to 2 mod 3. What is the smallest integer N such that $f_2(N) < f_1(N)$?

The answer is bigger than you might expect! (Try it!) In this colloquium, we will talk about what's going on here. No prior knowledge of number theory needed beyond knowing what a prime number is and knowing what it means for a number to be congruent to a mod q.

Tic-tac-toe and Additive Combinatorics (*Sarah Peluse*, Thursday)
I'll introduce some of the most important problems and results in additive combinatorics. We will begin by discussing the Hales–Jewett theorem, which has implications for the game of tic-tac-toe, and hopefully get to the famous Green–Tao theorem on arithmetic progressions in the primes.