CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2026

Contents

9:10 Classes

Continuum Hypothesis (Week 1 of 2) (, Susan, TWΘFS)

In 1874, Georg Cantor published a revolutionary article called "Über eine Eigenschaft des Inbegriffes aller reelen algebraischen Zahlen," or "On a Property of the Collection of All Real Algebraic Numbers". In this article he proved, for the first time, that the real numbers, in a precise mathematical sense, are larger than the natural numbers.[1] This begs the question: How much bigger? Is it the next cardinality up, or does there exist something in between? An infinity between the infinities? Or perhaps more than one. Perhaps even an infinity of infinities between the infinities! The continuum hypothesis states that there is no set whose cardinality is strictly between that of the reals and that of the natural numbers. Cantor believed the continuum hypothesis to be true, but he never managed to prove it. In 1900, the problem remained open, and was listed as one of Hilbert's problems: does there exist a cardinality that is larger than countable, but smaller than the continuum? The question was resolved in 1963, almost a hundred years after the initial problem was posed, when a mathematician named Paul Cohen proved that the continuum hypothesis can neither be proved nor disproved from the standard axioms of set theory.[2] The methods used by Cohen in this paper are stunningly beautiful. And in this class, you'll get to see them. Come with me and explore this gorgeous proof of the lack of a proof!

Homework: Recommended

Class format: Interactive Lecture

Prerequisites: None, but be prepared to move fast!

Hasse–Minkowski Theorem, or, Why One Needs p-adic Numbers to Understand Rational Solutions of Quadratic Equations (→ , *Ivan Loseu*, TWΘFS)

The p-adic numbers form an interesting example of a number system. They are important for number theory, and their purpose can be informally described as dealing with one prime at a time. This class can be viewed as an introduction to p-adic numbers with a discussion of an application to a problem that a priori does not involve prime numbers: When does a homogeneous quadratic equation in several variables with rational coefficients have a rational solution?

[1] This article presents a lovely proof that is entirely different from the diagonalization argument that many of you have seen.

[2] It had already been established that it could not be disproved by Gödel in 1940. The new result in the 1963 paper was that it could not be proved.

We will cover both the basics of quadratic forms (the common mathematical term for homogeneous quadratic polynomials) and p-adic numbers, as well as quadratic forms over p-adic numbers. We will then have some discussion around the Hasse–Minkowski theorem that—roughly speaking—says that when you study homogeneous quadratic equations you can indeed deal with one prime at a time.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Introductory number theory.

Teichmüller Theory of the Torus (, Arya, TWΘFS)

Take a paper square, and glue opposite sides. If done correctly (i.e. in $\mathbb{R}^4$), you will get a torus which is flat—just like the paper you used to create it. In this class, we will study the geometry of this type of construction. We will look at the "space of all flat tori" (Teichmüller space) and study it using Lattices (in $\mathbb{R}^2$), Loops (on the torus), and Linear algebra. Along the way, we'll meet some beautiful critters like the curve graph, the Farey tessalation of the circle, and Möbius transformations in the upper-half plane. Be warned: This class *will* involve some divison by zero, under staff supervision.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: I will assume familiarity with linear algebra, specifically with 2×2 matrices and how they act on $\mathbb{R}^2$.

Weyl's Equidistribution Theorem: Randomness without Randomness (, Laithy, TWΘFS)

What does it mean for a deterministic sequence to behave randomly? If you rotate a point around a circle by a rational angle, it eventually repeats; if you rotate by an irrational angle, it never repeats. But irrational rotations do something even stronger: in the long run, they visit every arc of the circle in exactly the right proportion.

In this class, we will make this idea precise using equidistribution modulo 1. We will compare "dense" with "uniformly distributed," study examples like $\{n\alpha\}, \{\sqrt{n}\}, \{\log(n+1)\}$, and prove Weyl's criterion, which turns uniform distribution into a question about cancellation in exponential sums:

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k u_n}.$$

This is a beautiful bridge between geometry, Fourier analysis, and analytic number theory. Along the way, we will discuss primes, Benford's law, powers like $\{a^n\}$, and see that some surprisingly simple problems are still open!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Single-variable calculus; complex numbers (i.e. comfort with i, and $e^{2\pi i\theta}$ being the unit circle, but no more).

What They Teach You in a Game Theory Course (, Nikita, TWΘFS)

We will start from the basic question: How do we reason about situations where the outcome depends not only on what you do, but also on what other people do? We will look at games in normal and extensive forms, draw some trees, stare at some payoff matrices, and develop the standard tools for analyzing strategic behavior.

Topics will include combinatorial games, zero-sum games, mixed strategies, Nash equilibria, and many, many 2×2 examples, because apparently everything can be modeled by four numbers in a

square. We will learn why sometimes the "obvious" strategy is bad, why sometimes doing something randomly is optimal, and why two perfectly rational people can still end up in a situation that makes everyone sad.

There will be prisoners, battles of the sexes, matching pennies, chicken, and maybe some auctions. No prior knowledge of game theory is required, but you should be comfortable with basic algebra and willing to take tiny toy examples far too seriously.

Homework: Optional

Class format: Lecture + Handouts

Prerequisites: None.

10:10 Classes

Baire Necessities ( , Ben+Charloutte, TWΘFS)

If you spend enough time chatting with staff like Riley, Charloutte, Maya, Ben, or Susan, you might hear things like "most continuous functions are nowhere differentiable," "derivatives are mostly continuous," "most smooth functions aren't power series," and, "most continuous functions don't have convergent Fourier series."

These statements are often used by those staff to persuade us that terrible, terrible things exist, or (more rarely) that nice things happen. But it *is* worth asking a few questions here: Could these teachers be lying to us about horrible things as some strange form of art? Why are all of these statements related to each other? And—perhaps most importantly—what do these statements mean?

We won't see all of these applications this week—there's just not enough time. But we *will* see, and prove, the deep result that lies under all of them, which is the Baire Category Theorem. We'll build up the necessary terminology, discuss the proof, and then see a few of the explanations of the cryptic statements above.

Homework: Recommended

Class format: Mostly lecture

Prerequisites: You should have enough of an analysis course to have seen convergence of sequences done with εs and δs.

Fourier Analysis but Actually Algebra ( , Glenn, TWΘFS)

In the early 1800s, Joseph Fourier developed a new idea to study the propagation of heat, which was to represent functions as sums of trigonometric functions. However, his work was not rigorous, and mathematicians at the time did not even fully believe that he was correct. A full understanding of Fourier series through the lens of analysis did not come until the early 1900s. This ended up being a somewhat complicated endeavor, introducing new concepts like L^2 *functions* and Hilbert spaces.

However, essentially all of Fourier's ideas can be developed in finite spaces, by considering periodic functions with domain $\mathbb{Z}/n\mathbb{Z}$ (the integers modulo n) instead of periodic functions with domain $[0, 2\pi)$. In today's modern day, some (including me) would argue that this is the only important case, since any implementation of a Fourier series on a computer would necessarily have to use such a finite approximation. Fourier theory on finite spaces has all the same uses and theorems as the infinite theory, and can be developed rigorously much more quickly.

In this course, we'll rigorously develop the (finite) theory of Fourier series and transforms, and explore many applications. We'll also extend the Fourier transform to all finite Abelian groups, not just $\mathbb{Z}/n\mathbb{Z}$. A few examples of applications that we could show include:

- Writing a computer program that converts audio recordings to sheet music.
- Proving (a finite analog of) the Heisenberg uncertainty principle.

- Multiplying two n-digit numbers (or degree n polynomials) in about $n\log(n)$ operations, which takes about n^2 operations with standard "long multiplication."
- Giving another proof of Arrow's impossibility theorem: how it's impossible to have a "fair" voting system.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra (Week 1's intro class is enough; familiarity with matrices is required. We'll review the necessary concepts in eigenvectors and orthogonality.) Know the definition of a group.

Memorylessness ( , Misha, TWΘFS)

Bus schedules can often be random and unpredictable. However, usually, you can at least say that if you've been waiting for a while, you must be getting close to the time the bus should arrive, and it will be there any moment now. Actually, if you've been waiting for a while, you might be thinking that the bus is worse than you thought it was, and it may be a long wait indeed.

But what if there were a bus such that, no matter how long you wait for it, you still expect the same waiting time ahead of you as you did when you started waiting? In such a case, we might say that the arrival time is "memoryless": the random variable doesn't change its behavior over time, because it doesn't remember how long it's been.

We will learn about two memoryless distributions (a discrete and a continuous one) and use them to study topics ranging from physics to puzzles.

Homework: Recommended

Class format: Lecture

Prerequisites: Know what conditional probability is, or talk to Misha if you don't.

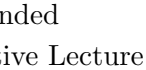
The Amazing Algebra of Braids! ( , Preston, TWΘFS)

Knots are twisty pieces of string that go in a loop. Links are several twisty looped pieces of string twisted together. Braids are several unlooped strings twisted together. The TWIST? Braids form a group, and questions about them are both topological and algebraic. This class will give a gentle introduction to braids and the algebra surrounding them, with an eye towards using braids to answer questions about knots. We'll explore the braid group and see how, like everything Preston does at camp this year, it is connected to left-distributivity. Prior knowledge of knots and braids may spark further inquiry, but the class will be friendly to all experience levels.

Homework: Recommended

Class format: Interactive Lecture

Prerequisites: Group theory.

Well-quasi-orders ( , Della, TWΘFS)

The problem with well-orders is that their Cartesian product isn't well-ordered (or even totally ordered). What if I want a similar property that is preserved by products? Well-founded is too weak: the product of two well-founded posets can fail to be well-founded. (Puzzle: how?)

It turns out a better answer is something called a "well-quasi-order"! We'll see that this property is preserved not only by product but by also much more powerful operations, talk about some connections to ordinal numbers and possibly reverse math, and understand why TREE(3) is so big.

Homework: Recommended

Class format: Lecture

Prerequisites: Some familiarity with ordinals.

1:10 Colloquia

Luck is a Thing That Comes in Many Forms (Misha, Tuesday)
And who can recognize her?

In this colloquium, we'll risk our money, our health, our lives, and the lives of others on mathematical games no sane and properly suspicious person would play. I will show you games where the worst player has the best luck; where the worst strategy is no worse than the most clever; where the best move is sometimes to sabotage yourself; where even the worst choice must sometimes be made; where the fast horse tends to lose to the slow horse. Please do not use what you learn here for evil purposes.

Future of Mathcamp (Staff, Wednesday)

Cohen–Lenstra Principle (August, Thursday)
As mathematicians we often study object that "naturally" appears and how frequently they do. The Cohen-Lenstra Principle governs a lot of this by a very simple heuristic of "a structure should appear with frequency inversely proportional to the number of automorphisms it carries". It is so powerful that often times when this heuristics fails it is because there is some uncovered structures that changes the number of automorphisms. We explore this principle in this colloquium in graph theory, group theory, and number theory.

Math Outreach (Campers, Friday)
(tba)

The Emperor Who Took "Mathcamp" Too Literally (Ce, Saturday)
Napoleon arrives at Mathcamp and decrees that whenever two campers meet, they must talk about exactly one of two things: math or camp. He is suspicious of overly single-minded groups, so he forbids both a large group in which every pair talks about math and a large group in which every pair talks about camp. His obvious next question is: how many mathematical conversations can still take place? Surprisingly, the answer undergoes a dramatic phase transition: allowing just one more camper into a forbidden all-math group changes the maximum possible number of mathematical conversations from surprisingly few to a substantial fraction of all conversations. We will see why such a tiny change in the rules leads to a completely different world—a phenomenon at the heart of Ramsey–Turán theory.

2:10 Classes

Fourier Analysis but Actually Analysis ( , Ben, TWΘFS)
In the early 1800s, Joseph Fourier developed a new idea to study the propagation of heat, which was to represent functions as sums of trigonometric functions. That might sound a little bit familiar.

The funny thing is, I will be proving some of the same results that Glenn is going to go over—the idea of using certain periodic functions as a nice orthonormal basis for a nice vector space, how to decompose functions into that basis, how to put it back together. We'll even see—or at least mention—some theorems in common, such as Plancherel's Theorem!

. . . But Glenn will be focused on *finite* sums, and *finite* sums—as we may have heard in Charlotte's analysis class last week—are much more nicely behaved than their infinite counterparts. So—for all of the excitement and difficulty that the students in Glenn's class will encounter—they will *not* encounter

the theoretical issues of convergence. For our part, these issues are *paramount*. Our *main question* is whether Fourier's idea *actually works* and—if so—what conditions are needed.

The story of the Fourier series is also historically important—the distinction between pointwise and uniform convergence was noticed *in part* due to issues appearing in the analysis of such series. Understanding the issues and strengths of the Fourier series was a major area of 19^{th} century analysis, and it motivates problems and fields of mathematics even to this day. For this reason, I think that it's important to understand the origin of the Fourier transform, and to study its properties—even if this does require a *lot* of analysis to understand the convergence.

Homework: Recommended

Class format: Lecture

Prerequisites: You should have enough of an analysis background to know the difference between uniform and pointwise convergence; Charlotte's Week 2 class should suffice.

How to Pronounce "Lucas" ( , Misha, TWΘFS)

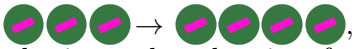
If you see the name Lucas attached to a theorem, odds are that it's the 19^{th} century French mathematician Édouard Lucas. (Pronounce "Lu" to rhyme with "déja vu" and "cas" to rhyme with "faux pas".) This class is a sequence of mostly-unrelated topics to which Lucas made a significant contribution.

We will certainly investigate patterns in the Fibonacci numbers, the Tower of Hanoï, and Pascal's triangle. Time permitting, we may also investigate Lucas's contributions to graph theory, geometry, or the search for Mersenne primes.

Homework: Recommended

Class format: Lecture

Prerequisites: Modular arithmetic up to and including Fermat's little theorem.

Pre-Galois Theory ( , August, TWΘFS)

Yep, finally. I am an algebraic number theorist after all.

Galois theory is the study of field extensions and polynomial solutions. Its most famous result of all is the "unsolvability" of quintics, but it is just overall the most beautiful undergraduate subject of maths—in my opinion. (A close second might be complex analysis.) We might not be able to get as far as quintics because the AOs only gave me 3 chilis, but we will see—I might go 4 chili on the last day to race ahead and get to it. (Most of the class will be a careful and manageable treatment still.)

Hence, this class is called Pre-Galois theory. This does not mean, 'Historically before Galois decided to duel someone.' It is the beginning of a class that may lead into Galois theory eventually. We will study many, many abstract fields that you might not have encountered or considered in the past, and some beautiful results we may prove along the way. Topics include:

- Tower law, minimal polynomials, finite and algebraic extensions;
- Ruler and compass constructions, and the impossibility of constructing certain things with a ruler and a compass;
- The existence of finite fields of all possible sizes;
- More.

Homework: Recommended

Class format: Lecture

Prerequisites: Know what rings and fields are; Understand what a vector space is, its dimension, and what a basis is. (-1 chili for day 1 if you already know about ideals, irreducible/prime elements, ED implies PID implies UFD, quotient rings, etc.)

Geometry through Matrices (, Ce, TWΘFS)
Welcome to a geometry club with absurdly strict rules. In one room, every pair of points must be an odd integer distance apart. In another, the entire configuration may use only two distances. Elsewhere, every pair of lines make exactly the same angle with one another. How many points or lines can fit before the rules become impossible? Although these sound like problems for rulers and compasses, the real answers come from matrices. We will encode distances and angles using linear algebra, then use rank, eigenvalues, and dimension to uncover hidden constraints. Along the way, we will meet odd-distance sets, two-distance sets, equilateral sets, and equiangular lines, and discover how surprisingly rigid geometry becomes in higher dimensions.

Homework: Recommended

Class format: Lecture with interactive discussions

Prerequisites: Linear algebra, especially familiarity with matrices

The Secret Life of Hanoï (, Neelam, TWΘFS)
Think the Towers of Hanoï is just a childhood game? Think again! Beneath its simple rules lie a playground of infinite trees, beautiful fractal graphs, and rigid symmetries. This course bridges the gap between a classic puzzle and cutting-edge geometric group theory. We will use automata to dissect the puzzle's state space, dive deep into the automorphisms of rooted trees, and map out the exact structural terrain we need to conquer the legendary Grigorchuk group. Don't miss the ultimate setup for one of math's most fascinating monsters.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Introductory group theory.

3:10 Classes

Congruent Numbers and Elliptic Curves (, *Ruthi Hortsch*, TWΘFS)
Let n be a positive integer. We call n a *congruent number* provided that there is a right triangle that has all rational sides and area n. Can we find an easy way of determining whether n is a congruent number? Tunnell's theorem gives us a way, but it uses hard and modern mathematics: the theory of elliptic curves and modular forms, which are fundamental objects of study in advanced number theory. It is also dependent on a weak form of an unsolved conjecture called the Birch and Swinnerton-Dyer Conjecture. (A proof—or counterexample—of this conjecture is worth a million dollars!)

In this class we will discuss congruent numbers and Tunnell's theorem, define elliptic curves and their basic properties, explain why answering questions about elliptic curves will tell us about congruent numbers, and get some idea about how this leads us to Tunnell's result.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic modular arithmetic. Group theory is not required but is useful context.

Elite Ball Knowledge (, Maya+Riley, TWΘFS)
This class aims to teach you some elite ball knowledge. That is, theorems about metric spaces that prominently use sets of points that are less than some fixed distance away from a central point, also known as metric balls. Some ball knowledge you will gain:

- You will see a variety of metric spaces and ask what the balls are in these metric spaces.
- You will see some more exotic spaces.
- You will learn about compactness.

- You will get some practice proving things in metric spaces.

In short, you will come away from the class having learned knowledge about metric balls, knowledge one may call elite (when compared to a high school class). Chat might js become certified ball knowledge havers fr fr.

Homework: Recommended

Class format: Inquiry-based learning (IBL)

Prerequisites: Everyone WILL Louve Analysis, or the equivalent.

Graph Eigenstuff (, Bowen, TWΘFS)

Graphs can be studied by drawing them, coloring them, or counting paths in them. In this class, we will study graphs by turning them into matrices and asking what their eigenvalues know.

It turns out that eigenvalues know a lot. They can detect whether a graph is connected, whether it is bipartite, how random walks behave, and how well-connected the graph is. We will start with familiar examples: complete graphs, cycles, paths, trees, and regular graphs. By computing their eigenvalues, we will build intuition for what adjacency matrices are really measuring.

A major theme of the class will be random walks. Multiplying by the adjacency matrix can be interpreted as moving around the graph, and the eigenvalues tell us how quickly this walk forgets where it started. This leads naturally to *expander graphs*: sparse graphs that behave, in many ways, like random graphs. We will work toward results such as the eigenvalue interlacing theorem, the expander mixing lemma, spectral gap estimates, and maybe a Cheeger-type inequality relating eigenvalues to expansion.

The class will be inquiry-based: Students will compute examples, find patterns, and then prove theorems from those patterns. The goal is to see how much graph theory can be done with linear algebra.

Homework: Required

Class format: Inquiry-based learning (IBL), with worksheets, examples, and guided proofs

Prerequisites: Linear algebra is required: students should be comfortable with matrices, eigenvalues, and eigenvectors. Basic graph theory is recommended but not required; knowing vertices, edges, paths, cycles, connectedness, and bipartiteness will make the class easier to appreciate.

Making Algorithms Continuous ( , Narmada, TWΘFS)

How do you run an algorithm on a graph? In the *global* model, you feed your n-vertex graph to a computer, it runs a (hopefully efficient) process on the vertices and edges, and outputs your n-vertex graph with labels on it. But what if your graph is modelling a real-life network like the internet, so it has one gajillion vertices? It turns out that our computers can't even get past the first step of eating all one gajllion vertices, so you would just end up with a busted computer.

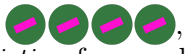
The *LOCAL* model of computation—which is how the internet actually works—puts a little computer at each vertex, and each computer runs an algorithm based on its *local* information. In this class, we'll look at how these differ from global algorithms and what it means for them to be 'efficient'.

But wait! Why did I put the word 'continuous' in the title? It turns out that the existence of efficient LOCAL algorithms on finite graphs corresponds exactly to the existence of continuous algorithms on strange infinite graphs. We'll see why at least one direction of this correspondence is intuitive, and if there's time, I will try to convince you of the other direction. This is a very new and still active area of mathematics, so the proofs will not all be rigorous, but the ideas themselves will be challenging!

Homework: Optional

Class format: Interactive lecture

Prerequisites: Graph theory (know the definitions of proper colorings, trees, and bipartite graphs); Know what open and closed sets in $\mathbb{R}^n$ are (specifically, can you find two ways to argue that $\mathbb{Q}$ is not an open set?); Know the definition of a continuous function.

The Many Faces of the Euler Characteristic ( , Purple, TWΘFS)

Euler defined what we now call the *Euler characteristic* of a graph: the alternating sum $v - e + f$, where v, e, and f are the number of vertices, edges, and faces, respectively. This Euler characteristic turns out to be the first example of a very general phenomenon in which we can systematically build algebraic invariants of objects which are glued together from fundamental pieces. Such objects include sets (and "negative sets"), graphs, polyhedra (and, more generally, polytopes in arbitrary geometries), manifolds, cell complexes, chain complexes, vector spaces, and modules. For each such family of object, we can define a kind of generalized Euler characteristic; the theory of algebraic invariants produced in this way is called cut-and-paste K-theory. In this class, we'll work through a number of these examples, with the goal of building towards that theory.

(This class won't assume familiarity with many of the objects mentioned above—only the ones listed in the prerequisites.)

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Comfort with the definitions of group, ring, vector space, group homomorphism, ring homomorphism, and linear map.