Mathcamp 2006 Qualifying Quiz

Instructions

We call it a quiz, but it's really a challenge: a chance for you to show us how you approach new problems and new concepts in mathematics. What matters to us are not only your final results, but your reasoning. Correct answers on their own will count for very little: you have to justify all your assertions and *prove* to us that your solution is correct. (For some tips on writing proofs, see `www.mathcamp.org/proofs.php`.) None of the problems require a computer; you are welcome to use one if you'd like, but first read a word of warning at `www.mathcamp.org/computers.php`. Sometimes it may take a while to find the right way of approaching a problem. Be patient: there is no time limit on this quiz.

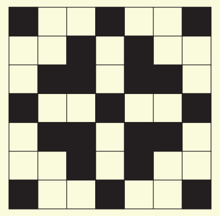
We don't expect every applicant to solve every problem: in the past, we have sometimes admitted people who could do only five, occasionally even fewer. However, you should definitely try all the problems and send us the results of your efforts: partial solutions, conjectures, methods -- everything counts.

If you need clarification on a problem, please email `quiz@mathcamp.org`. You may not consult or get help from anyone else. You can use books or the Web to look up definitions, formulas, or standard techniques, but any information obtained in this way must be clearly referenced in your solution. Please do not try to look for the problems themselves: we want to see how well you can do math, not how well you can use Google! Any deviation from these rules will be considered plagiarism and may disqualify you from attending Mathcamp.

Good luck and have fun!

Problems

1. In the following 7×7 grid, each square is colored black or white. This particular coloring has a lot of symmetries: the pattern of black and white squares is unchanged if the grid is rotated by 90°, 180°, or 270°, or if it is reflected in a horizontal, vertical, or diagonal line through the center. How many ways are there to color a 7×7 grid black and white with these same symmetries? How about an $n\times n$ grid?



2. (*Contributed by Khoa Lu Nguyen, Mathcamp 2005*) Is it possible to find 2006 distinct points in the plane so that for any two of the points P and Q, the circle with diameter PQ goes through at least one more point from the set?

3. Two sets A and B have the property that there are precisely 144 sets which are subsets of either A or B or both. How many elements are in the union $A\cup B$? (*Hint:* The set of subsets of a set S is called the power set of S. You may want to look it up if you've never encountered it before. Of course, if you do, you should reference your source in your solution.)

4. Arda and Cordelia are playing a game. Arda writes down a positive integer. Cordelia looks at it and decides whether she wants to go first or second. The players then take turns subtracting powers of 2 (i.e. 1, 2, 4, 8, 16, ...) from Arda's number. Negative numbers are not allowed, and the player who gets to 0 wins. For instance, a typical game might go as follows:

$$25 \to 17 \to 15 \to 11 \to 10 \to 6 \to 4 \to 0$$

(a) For which numbers written by Arda should Cordelia choose to go first, and what should her strategy be?
(b) What if the players must subtract powers of 3 instead of powers of 2?
(c) What if they subtract powers of a fixed positive integer k?

5. Start with four numbers arranged in a circle. Between each pair of adjacent numbers, write their average. Now erase the original numbers, so that once again you are left with four numbers in a circle. You can repeat this process as many times as you wish. (For example, if the original numbers were 1, 3, 7, 5 in that order, then after one iteration you get 2, 5, 6, 3; after two iterations, you get 7/2, 11/2, 9/2, 5/2, etc.)

Suppose after 20 interations, you obtain the numbers 1, 2, 3, 4, *not necessarily in that order*. Can you tell what numbers you had after one iteration? Can you tell what the original numbers were?

6. What is the maximum number of 90° angles that an n-sided polygon can have? The polygon need not be convex, but you have to distinguish between 90° and 270° angles. For instance, a hexagon in the shape of the letter L has five 90° angles and one 270° angle.

7. Order the numbers $1,2,\ldots,n$ any way you want. Add up the differences between successive numbers, with all differences counted positively: for example, if you start with the ordering 5,3,1,4,2, you get $2 + 2 + 3 + 2 = 9$. What is the largest number you can obtain in this way? Your answer, of course, will depend on n. (Be sure to prove that the number you get really is the largest possible!)

8. The Rainbow Game is played by a team of seven. Each player gets a hat, which can be any one of the seven colors red, orange, yellow, green, blue, indigo, and violet. The colors of the hats are independent of each other and repetitions are allowed: for instance, it may happen that all the hats are green. Each player can see only the colors of the six hats worn by the rest of the team; no player can see the color of his or her own hat. The players are to guess the colors of their own hats, and if at least one player guesses correctly then the team as a whole wins.

The players may not communicate in any way during the game, and they must all announce their guesses simultaneously. They are, however, allowed to plan out a strategy in advance, and they hope to find a strategy which will guarantee them success for every possible arrangement of hats. Is there such a strategy? Either find one or show that it cannot exist. (As a warm-up, try two players and two colors, or three players and three colors.)

9. Let's define a curious sort of "distance" on the positive integers. We say that two positive integers a and b are *one step apart* if $ab+1$ is a perfect square; for instance, 2 and 24 are one step apart. If we can find a sequence of positive integers

$$a = c_0,\ c_1,\ \ldots,\ c_n = b,$$

such that c_i and c_{i+1} are one step apart for each i, we say that it is possible to go from a to b in n steps. If n is the smallest positive integer such that it is possible to go from a to b in n steps, then we say that a and b are *n steps apart*. For instance, it is possible to go from 2 to 7 in two steps using the sequence 2, 24, 7; moreover, $2\cdot7+1$ is not a square, so 2 and 7 are two steps apart.
(a) Show that any two distinct positive integers are a finite number of steps apart.
(b) How far apart are 1 and 4?
(c) How far apart are m and $m+1$?
Note: 0 is not a positive integer.

10. Is it possible to dissect an equilateral triangle whose sides have length 9 into 3000 convex quadrilaterals each of which has all side lengths greater than or equal to 1? Either show how to do it or show it can't be done.

TUITION AND FEES (all figures in US$)

The Mathcamp fee includes tuition, transportation to/from Portland airport, room, board, and all extracurricular activities.

Full camp fee: $3,195

Automatic tuition reduction: If family income is X (in US$): deduct $(100{,}000 - X)/50$ from full fee.

Maximum Reduction: $1200

Thus a student whose family makes $40,000 a year or less automatically qualifies for the maximum tuition reduction of $1200.

TRAVEL TO MATHCAMP: Students may either arrange their own transportation to Portland or make use of Mathcamp's discount rates on American Airlines, around $350 round trip for travel within the continental US and Canada. American Airlines has been a long-standing sponsor of Mathcamp, and we encourage students to take advantage of these low fares.

SCHOLARSHIPS AVAILABLE!

Admission to Mathcamp is need-blind. We are committed to enabling every student to attend, regardless of financial circumstances.

Beyond the automatic tuition reduction, Mathcamp gives out over $50,000 a year in additional scholarships. Awards are based on a combination of merit and need, but the emphasis is on making the camp affordable to every qualified student. In the past, scholarship amounts have ranged from $300 to the entire camp fee. If you would like to be considered for an additional scholarship, please complete the scholarship application (see right).

Mathcamp 2006 Scholarship Application

Please provide the following information on a separate sheet of paper, **signed by a parent or guardian**:

- Annual family income (all sources)
- A list of all members of your household and their relationships to the student. For siblings, please also provide ages.
- What portion of the Mathcamp fee can your family afford to pay?
- Please describe any special circumstances that you would like us to consider.

MATHCAMP STUDENT CARE POLICY

Dear parent,

Student safety and enjoyment are Mathcamp's first priorities. Students will be housed in secure campus dormitories, with male and female students in designated sections of the same building. In case of a medical problem, the hospital is minutes away. Students will have access to university athletic facilities and computers. Every effort will be made to enable students who so desire to attend weekly religious services of their faith. Mathcamp is committed to an atmosphere of mutual tolerance, responsibility, and respect, and is proud of its past record in helping to create such an atmosphere.

- *Mira Bernstein, Executive Director, Mathcamp*

Ready to apply to Mathcamp?

We invite applications from every student aged 13 through 18 who is interested in mathematics, regardless of racial, ethnic, religious, or economic background.

Mathcamp is now accepting applications on the web as well as by regular mail. We strongly encourage all students with Internet access to use the online application process. **The $20 application fee is waived for online applications.**

Online Application:
Go to `http://www.mathcamp.org/application` and follow the instructions there. You will still have the opportunity to submit your quiz or recommendation letters by postal mail.

Postal Application:
Please mail the following items in a single envelope:

1. A completed **application form** (see right) and your answers to the **qualifying quiz** (see above).

2. A brief **personal statement** about your interest in math and why you want to come to Mathcamp. Some things you could talk about: What would you like to gain from a summer at Mathcamp? What do you like about math? Which of the problems on our quiz did you enjoy most? Are there specific areas or kinds of math that you're especially interested in? If you've done any other math programs, projects, or independent reading, tell us about them!

3. A list of math courses that you've taken at the high-school level or above, with brief descriptions of what was covered. Also, if you have done any math competitions, please include scores and awards.

4. **Two recommendation letters.** The first letter should be from a teacher who knows you personally, preferably (but not necessarily) a mathematics teacher. The letter should comment on your creativity, initiative, and ability to work with others, as well as on your academic achievements. The second letter should be from another adult who knows you personally (e.g. employer, pastor, soccer coach, etc. - preferably someone outside of school and not a relative). This letter should address your maturity, independence, social and personal qualities. We are looking for students who are not only good at math, but who will thrive in the atmosphere of freedom and responsibility that characterizes Mathcamp, and who will make a positive contribution to the camp community. To ensure confidentiality, ask each recommender to seal their letter in an envelope and to sign across the seal. Include the sealed envelopes with your application.

5. A **US$20 application fee** (check or money order made out to **Mathematics Foundation of America**) or a note signed by your parent or guardian explaining that your family cannot afford it.

Please do not use certified mail; if you wish, you may e-mail or call us to check that your application has arrived. Send to:

Mathcamp 2006
129 Hancock Street
Cambridge, MA 02139

Applications received by April 26, 2006 will be given equal consideration.

Rolling admissions thereafter.

Mathcamp 2006 Application Form

Name: ______________________________

Address: ______________________________

City: ______________________________

State/Province/Country: ____________________

ZIP/Postal Code: ____________________

Telephone: (______) ____________________

E-mail: ______________________________

___ I check my e-mail at least twice a week.

Date of birth: ______________________________

Must be between August, 1987 and July, 1993.

Current grade in school: ________

Family income (if below US $100,000):________

Note: Admission to Mathcamp is need blind. Families earning less than $100,000 qualify for the automatic tuition reduction.

___ I would like to be considered for an additional scholarship. *Please submit a scholarship application on a separate page.*

How did you hear about Mathcamp? __________

____

"I declare that my solutions to the qualifying quiz are my own work. I did not receive any form of assistance from other people, and I have referenced every instance where I looked something up in a book or on the web."

Signature: ______________________________

Howarth Hall, University of Puget Sound, with Mt. Rainier in the distance

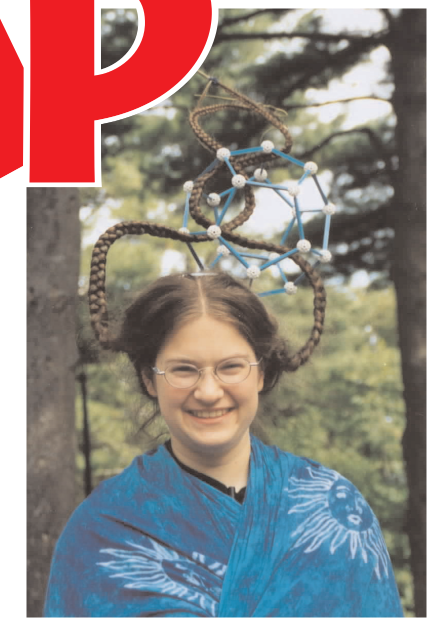
Discover MATHCAMP

Out of nothing I have created a strange new universe.

- Janos Bolyai, discoverer of hyperbolic geometry

Mathcamp is a chance to...

- Live and breathe mathematics: fascinating, deep, difficult, fun, mysterious, abstract, interconnected (and sometimes useful).
- Gain mathematical knowledge, skills and confidence--whether for a possible career in math or science, for math competitions, or just for yourself.
- Set and pursue your own educational goals: choose your classes, do a project, learn what **you** want to learn.
- Study with mathematicians who are passionate about their subject, from internationally known researchers to graduate students at the start of their careers, all eager to share their knowledge and enthusiasm.
- Make friends with students from around the world, and discover how much fun it is to be around people who think math is cool.



During the five weeks of Mathcamp I learned more than in all my classes last year. I met people who are as interested in math as I am, and who know much more math than I do. I got to call Ph.D.'s by their first names. I got to interrupt teachers in class and be appreciated for it. I got to argue with famous professors, and they didn't assume they were right. Vin [one of the mentors] still tries to teach me to solve the Rubik's cube, all the way from England! Camp gave me the kind of freedom I had never felt before. I will never forget the experience and I will do everything possible (and impossible) to come back next year.

Yuliya Gorlina (Skokie, IL, USA)

Academics and Activities

A Variety of Choices...

The Mathcamp schedule is full of activities at every level, from elementary to the most advanced:

- courses lasting anywhere from a few days to five weeks.
- lectures and seminars by distinguished visitors.
- math contests and problem-solving sessions with experienced trainers.
- hands on workshops and individual projects.

You can see schedules from past years at www.mathcamp.org/academics.php.

The Freedom to Choose

Mathcamp does not have a set curriculum or a list of requirements. We encourage the faculty to teach what they are most passionate about, and let the students choose what they are interested in learning. With the help of an academic advisor, you will design a program of study that reflects your own interests and goals. You can take any classes you want, and even the number of classes that you attend each day is up to you: you can use the rest of the time to review what you've learned, talk to one of your professors, work on problems, or just take a break. For many students, the freedom to take charge of their own education is one of the aspects of Mathcamp that they value most.

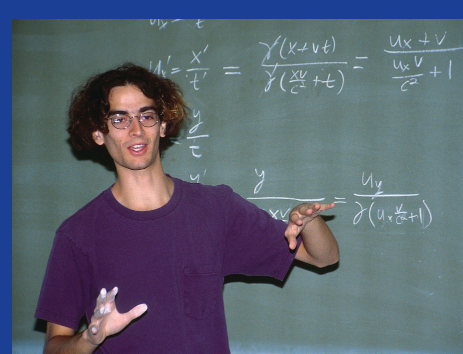
"There's no other place where you can be taught this kind of math"

-Henry Wong (West Palm Beach, FL, USA)

Classes

Course offerings vary from year to year, depending on the interests of the students and faculty. Some of the topics taught in previous years have included:

Geometry and Topology Euclidean and non-Euclidean (hyperbolic, spherical, projective) geometries • Geometric transformations • Combinatorial topology • Algebraic geometry • Knot theory • The Brouwer Fixed-Point Theorem **Calculus and Analysis** Topics in calculus • Fourier analysis • Complex analysis • Real analysis • Dynamical systems • Non-standard analysis **Algebra and Number Theory** Primes and factorization algorithms • Congruences and quadratic reciprocity • Linear algebra • Groups, rings, and fields • Galois theory • Representation theory • *p*-adic numbers • Numbers and games **Computer Science** Cryptography • Algorithms • Complexity theory • Computer graphics • P vs. NP **Set Theory and Logic** Cardinals and ordinals • Gödel's Incompleteness Theorem • The Banach-Tarski Paradox **Discrete Mathematics** Combinatorics • Generating functions • Graph theory • Ramsey theory • Partitions • Probability • Finite geometries **Connections to Science** Relativity and quantum mechanics • Dimensional physics • Neural networks • Biological systems **Discussions** Philosophy of Mathematics • Ethnomathematics • Women in mathematics • "How to survive being smart" **Problem Solving** Proof techniques • Elementary and advanced methods • Contest problems of various levels of difficulty • Relays and team competitions



There was no pressure at all — the incentive towards learning came from within.
– Keigo Kawaji (Toronto, ON, Canada)

If you want to learn 'cool' math with awesome people and mentors this is *the* best place!
– Shinyoung Hwang (Northfield, MA, USA)

Thanks so much for showing me how math is truly the most beautiful form of art.
– Qian Yang (Pembroke Pines, FL, USA)

Spotlight on a class:

Theoretical Computer Science (Summer 2005)

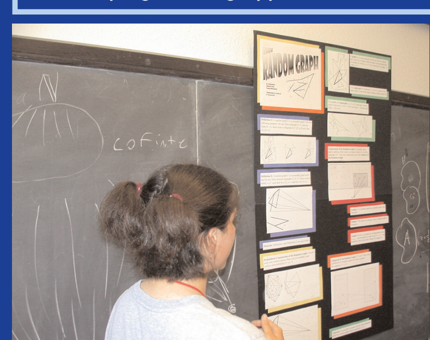
Theoretical computer science is a mathematical framework for analyzing what computers can and cannot do. In this class, we formulated a model of computation and used it to prove that there are problems which no computer, no matter how fast, will ever be able to solve. We then analyzed how long problems take to solve: primality testing is fast, whereas factoring large numbers is (probably) very slow (the basis of modern cryptography). This led us to define the complexity classes P and NP, and to ask the famous question: is P = NP? This is not just one of the biggest open problems in all of mathematics, but worth a million dollars to anyone who solves it. (Unfortunately, we didn't.)

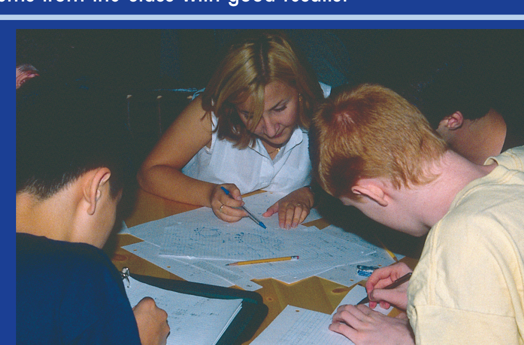
After P and NP, we examined space complexity (PSPACE and NPSPACE), BPP (computation using randomization), and other topics. By the end, we had constructed a "roadmap" to computational problems and their difficulty. The class was pure mathematics; no computer was harmed or touched in its teaching. However, several spin-off projects had teams of students programming applications of theorems from the class with good results.

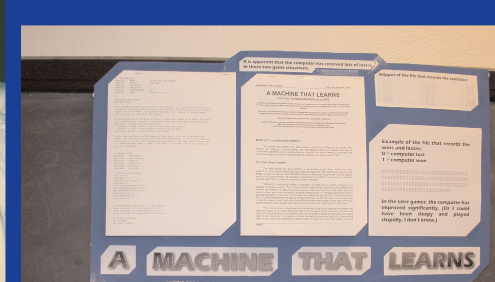
Projects

Every student at Mathcamp is encouraged to do a project, supervised by one of the mentors or faculty. Projects range in scope from creative applications of simple techniques to advanced problems connected to faculty research. Some previous years' projects have been on:

- Billiard-ball geometry
- Information theory and psychology
- Periodicity of Fibonacci numbers mod *n*
- Knight tours on an *m*-by-*n* chessboard
- Non-convex polyhedra
- Cellular automata
- Cops and robbers on a graph
- Constructing the regular 17-gon
- Admissible covers of algebraic curves
- Mathematical Finance
- Algorithmic composition of music
- Intelligent ways of searching the web
- Probability in sports
- The elasticity equation of string
- Digital signal processing
- Light paths in universes with alternate physics







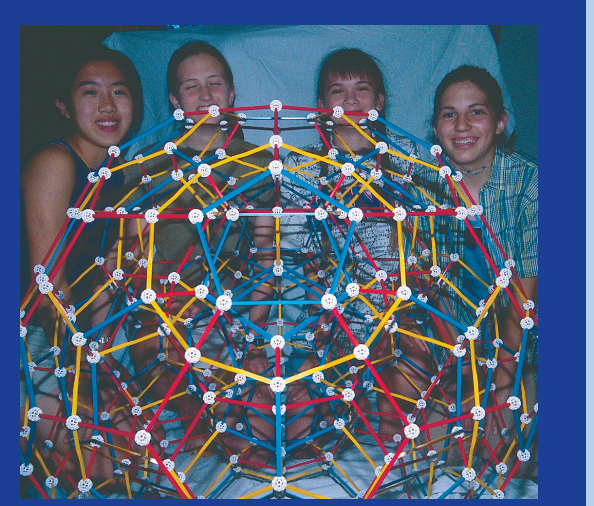
"Mathcamp was definitely the most fun I've ever had"

– Avichal Garg (Cincinnati, OH, USA)

Beyond Math

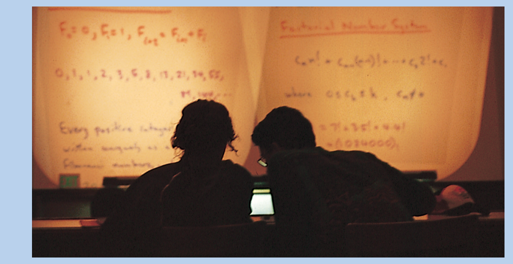
Ordinarily, mathematical activities are scheduled for five days a week; whatever math happens on the other two days is purely informal. Otherwise, the weekend is reserved for relaxation and the incredible number of activities that quickly fill the schedule. All of the activities are optional, and students can choose simply to relax with friends.

Field trips in the past have included hiking trips to the mountains, whitewater rafting, sea kayaking, amusement parks, and museums. Additionally, many activities are done on-campus, such as a team "puzzle hunt" competition, the Contrapositones (our own a capella singing group), a talent show, juggling and swing dancing lessons, and even improvisational theater. Campers also organize many events themselves, from sports and music to chess and bridge tournaments, to the production of the camp yearbook.



People

"Everyone should go to Mathcamp. [...] Interesting people. Math people. People both serious and silly. Mathcamp is the most amazing summer paradise for five weeks. Certainly it's been great for me. It changes the world. Just go."
– Anna de Bakker (Winnipeg, MB, Canada)



Regular Faculty

Mira Bernstein (Wellesley College)
Interests: Algebraic geometry, mathematical biology, information theory

Bogdan Enescu (Deputy Leader of the Romanian IMO Team since 1995)
Interests: Problem solving, problem creation

Mark Krusemeyer (Carleton College)
Interests: Algebra (abstract and linear), combinatorics, number theory, problem solving

David Savitt (University of Arizona)
Interests: Number theory, arithmetic geometry

Visitors

John H. Conway (Princeton)
One of the most creative thinkers of our time, John Conway (pictured below, at lunch with Mathcamp students) is known for his ground-breaking contributions to such diverse fields as knot theory, geometry of high dimensions, group theory, transfinite arithmetic, and the theory of mathematical games. Outside the mathematical community, he is perhaps best known as the inventor of the "Game of Life".

Allan Adams (Harvard)
Allan Adams studies quantum versions of algebraic and differential geometry that play a fundamental role in string theory, and uses these tools to explore the fate of tachyons, moose diagrams, and other puzzles involving black holes and spacetime singularities. Dr. Adams believes that everyone should understand quantum mechanics, which is as beautiful and strange as it is true, and looks forward to discussing it at Mathcamp.

Marina Meila (University of Washington)
Marina Meila works at the interface of statistics and computer science. She is interested in machine learning by probabilistic methods and in reasoning under uncertainty. In her research, she develops new mathematical and computational methods for inferring structure from data, with applications ranging from computer vision and robotics to bioinformatics and the social sciences.

George Hart (SUNY Stony Brook)
George Hart is both a professor of computer science and a mathematical sculptor; his work can be seen at www.georgehart.com. At Mathcamp, he leads hands-on workshops in which participants explore the geometry of three- (and four-) dimensional space using the mathematical construction set Zometool.

Megumi Harada (University of Toronto)
Megumi Harada works on symplectic geometry, which is the mathematical framework for classical physics. Symplectic geometry also connects to many other fields, such as quantum physics, combinatorics, and graph theory. Dr. Harada was the only mathematician among the 10 finalists in a televised competition for Ontario's Best Lecturer (billed as "reality TV with a high IQ"). She appears regularly on TV Ontario's More 2 Life program to discuss mathematics in everyday life.

Josh Tenenbaum (MIT)

Moon Duchin (University of California at Davis)

Abigail Thompson (University of California at Davis)

... and others!

Mathcamp has provided me with bits of information that keep showing up in the strangest places. I'm in college now, doing a double major in mathematics and electrical engineering. The funny thing is how I usually have experience with the mathematics involved, even though you wouldn't realize it by looking at the courses I've taken. For instance, the big thing that we do in electrical engineering right now is using phasors... well, I inadvertently got my first taste of that from a problem I did at Mathcamp '98. Usually I ace the tests that the EE students fail, especially when the questions include the words "show that..." or "prove"—which aren't as uncommon as one might expect in an engineering course.
– Sandor Swartz (Rolla, MO, USA)

Mentors and Junior Counselors

Mathcamp's graduate student "mentors" are recruited from some of the best universities on the continent (such as Harvard, MIT, Stanford, McGill, etc.). They teach classes on their favorite subjects, and act as an important link between the students and the older faculty. The mentors, assisted by camp alumni known as junior counselors, are also in charge of running the non-academic aspects of the camp and making sure that no one is ever bored — a duty which they perform diligently and with gusto! As one student put it, *"Our mentors are the coolest camp staff ever."*

"Imagine a place where you can discuss a perfect world that exists only in the mind and where you can play and have fun all day, even make up your own rules, and where everyone you meet loves playing in and learning about this world. That's Mathcamp."
– Curtis Fonger (North Palm Beach, FL, USA)

Hikers break for an August snowball fight.

"Go, just go! Trust me!"

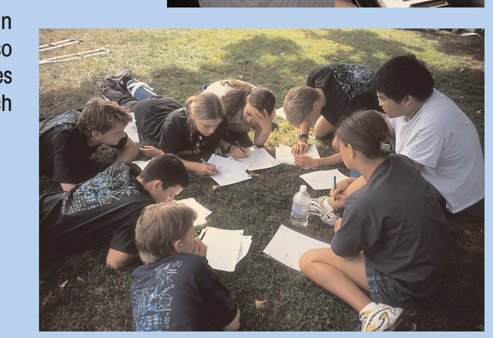
-Jian Xu (Toronto, ON, Canada)

Students

We never cease to be amazed at what a varied and interesting bunch of young men and women our students are! While everyone at camp shares a love of mathematics, their other interests and accomplishments run the gamut. Each year's camp is a collection of musicians and writers, athletes and actors, hikers, artists, jugglers, board game players, dancers, computer programmers, students of science and philosophy - all sharing their interests and experiences with each other.

Most of the students at camp come from Canada or the US, but many come from overseas. Students have come to camp from Japan, Poland, England, Taiwan, Turkey, Romania, Tanzania, and many other places from around the globe.

It is a testament to our students' maturity and independence that they can be serious about learning and doing mathematics while still finding so many different ways to have fun in the evenings. Many camp activities are organized entirely by campers, and students routinely cite each others' company as one of the best aspects of the camp.



"Mathcamp (for those of you who like math) is seriously a paradise. You have great people to meet, games to play, places to go, and things to do. When 100 campers all come back saying they just had the best summer of their life, you know there's got to be something special about this place."
– Tristan Brand (Whitefield, ME, USA)

"I can't even begin to describe the wonderful mentors, counselors, and campers that make Mathcamp the absolute best way to spend a summer."
– Gloria Lee (Bayside, NY, USA)