Mathcamp 2009 Qualifying Quiz

Instructions

We call it a quiz, but it's really a challenge: a chance for you to show us how you approach new problems and new concepts in mathematics. What matters to us are not only your final results, but your reasoning. Correct answers on their own will count for very little: you have to justify all your assertions and ***prove*** to us that your solution is correct. (For some tips on writing proofs, see `www.mathcamp.org/proofs.php`.) Sometimes it may take a while to find the right way of approaching a problem. Be patient: there is no time limit on this quiz.

The problems start out easier and get harder. (At least we think so — but you may disagree.) None of the problems require a computer; you are welcome to use one if you'd like, but first read a word of warning at `www.mathcamp.org/computers.php`.

We don't expect every applicant to solve every problem: in the past, we have sometimes admitted people who could do only half of them, occasionally even fewer. However, you should definitely try all the problems and send us the results of your efforts: partial solutions, conjectures, methods — everything counts.

If you need clarification on a problem, please email `quiz09@mathcamp.org`. You may not consult or get help from anyone else. You can use books or the Web to look up definitions, formulas, or standard techniques, but any information obtained in this way must be clearly referenced in your solution. Please do not try to look for the problems themselves: we want to see how well you can do math, not how well you can use Google! Any deviation from these rules will be considered plagiarism and may disqualify you from attending Mathcamp 2009.

Good luck and have fun!

Problems 6 and 7

Problems

1. A group of Mathcampers went on a 3.5-hour hike on Mt. Rainier. In any continuous one-hour period during their hike, they covered exactly 2 miles. Does it follow that they hiked exactly 7 miles total? If not, what are the minimum and maximum distances they could have walked? *Prove your answer.*

2. (a) You wish to color each of the integers 0, 1, 2, ... , n red or blue, in such a way that

- both colors are used, and
- integers that differ by 7 or 11 have the same color.

Can you do this for all n? If not, what is the largest value of n for which it is possible? *Prove your answer.*

(b) How does the answer in (a) change if we replace the numbers 7 and 11 with arbitrary integers m and k? If you can't figure out the solution for the most general case, try it for some special cases — e.g., for a fixed value of k or for some specific relationship between m and k. Include your most interesting or general proofs in your solution.

3. Let S be a set of numbers. We'll call S *autonomous* if the number of elements in S is itself an element of S. For example, the set $\{2, 5\}$ is autonomous, as is the set $\{2, 3, 7\}$, but the set $\{3, 4\}$ is not. Find, with proof, the number of autonomous subsets of $\{1, 2, 3, \ldots, n\}$.

4. Suppose you start with the number 1 and go through a series of steps, where at each step you add a (positive integer) divisor of your current number to the number itself, to get a new number. For instance, the first step is forced: you have to take 1+1, so your new number is 2. Now you have two choices; the next number could be 2+1=3 or 2+2 = 4. If you choose 4, the next step after that can take you to 5, 6, or 8.

(a) Show that if N is less than or equal to 2^{n+1}, then you can always reach N using $2n$ steps or fewer.

(b) Show that if N is larger than 2^n and you can reach N in $n+1$ steps, then N is a sum of two powers of 2.

5. Three real numbers are chosen at random between 0 and 1. What is the probability that they form the side lengths of a triangle? *Prove your answer.*

6. Nathan and Abi are playing a game. Abi always goes first. The players take turns changing a positive integer to a smaller one and then passing the smaller number back to their opponent. On each move, a player may either subtract one from the integer or halve it, rounding down if necessary. Thus, from 28 the legal moves are to 27 or to 14; from 27, the legal moves are to 26 or to 13. The game ends when the integer reaches 0. The player who makes the last move wins. For example, if the starting integer is 15, Abi might move to 7, Nathan to 6, Abi to 3, Nathan to 2, Abi to 1, and now Nathan moves to 0 and wins. (However, in this sample game Abi could have played better!)

(a) Assuming both Nathan and Abi play according to the best possible strategy, who will win if the starting integer is 1000? 2000? *Prove your answer.*

(b) As you might expect, for some starting integers Abi will win and for others Nathan will win. If we pick a starting integer at random from all the integers from 1 to n inclusive, we can consider the probability of Nathan winning. This probability will fluctuate as n increases, but what is its limit as n tends to infinity? *Prove your answer.*

7. Consider the sequence of positive real numbers:

4, 1/3, 4/3, 4/9, 16/27, 64/243, ... ,

in which each term (after the first two) is the product of the two previous terms. Note that for this particular sequence, the first and third terms are greater than one while the second and fourth terms are less than one. However, after that the "alternating" pattern fails: the fifth and all subsequent terms are less than one. Do there exist sequences of positive real numbers in which each term is the product of the two previous ones and for which all odd-numbered terms are greater than one, while all even numbered terms are less than one? If so, find, with proof, *all* such sequences. If not, prove that no such sequences exist.

8. On the x, y coordinate plane, there is a finite set of line segments. Each line segment lies completely in the square bounded by (0, 0), (0, 1), (1, 0), and (1, 1), and the sum of the lengths of all the line segments is 18. Prove that there exists a straight line in the plane that crosses at least 10 of the segments.

TUITION AND FEES

Full camp fee: US$3,500

The camp fee includes tuition, room, board, and all extracurricular activities.

MATHCAMP STUDENT CARE POLICY

Dear parent,

Student safety and enjoyment are Mathcamp's first priorities. Students will be housed in secure campus dormitories, with male and female students in designated sections of the same building. In case of a medical problem, the hospital is minutes away. Students will have access to university athletic facilities and computers. Every effort will be made to enable students who so desire to attend weekly religious services of their faith. Mathcamp is committed to an atmosphere of mutual tolerance, responsibility, and respect, and is proud of its past record in helping to create such an atmosphere.

- Mira Bernstein, Executive Director, Mathcamp

SCHOLARSHIPS AVAILABLE!

Admission to Mathcamp is need-blind. We are deeply committed to enabling every qualified student to attend, regardless of financial circumstances.

Mathcamp awards over $100,000 in need-based scholarships every year. In the past four years, no admitted applicant has been unable to attend the camp for financial reasons. We give several full scholarships each year, and occasionally even help students with travel expenses. Please do not let financial considerations prevent you from applying! If you would like to be considered for a scholarship, simply complete the scholarship application at right.

Mathcamp 2009 Scholarship Application

Please provide the following information on a separate sheet of paper, signed by a parent or guardian:

- Annual family income (all sources)
- A list of all members of your household and their relationships to the student (for siblings, please also provide ages)
- The cost of schooling, if any, for all family members (private school, college, etc.)
- The estimated cost of round-trip travel to Mathcamp for the student
- The portion of the cost of Mathcamp (including both tuition and travel) that your family can afford to pay
- Any special circumstances that you would like us to consider

Ready to apply to Mathcamp?

We invite applications from every student aged 13 through 18 who is interested in mathematics, regardless of racial, ethnic, religious, or economic background.

Mathcamp accepts applications both on the web and by regular mail. We strongly encourage all students with Internet access to use the online application process. **The $20 application fee is waived for online applications.**

Online Application:
Go to `http://www.mathcamp.org/application` and follow the instructions there. You will still have the opportunity to submit your quiz or recommendation letters by postal mail.

Postal Application:
Please mail the following items in a single envelope:

1. A completed **application form** (see right) and your answers to the **qualifying quiz** (see above).

2. A brief **personal statement** about your interest in math and why you want to come to Mathcamp. Some things you could talk about: What would you like to gain from a summer at Mathcamp? What do you like about math? Which of the problems on our quiz did you enjoy most? Are there specific areas or kinds of math that you're especially interested in? If you've done any other math programs, projects, or independent reading, tell us about them!

3. A **list of math courses** that you've taken at the high-school level or above, with brief descriptions of what was covered. Also, if you have done any math competitions, please include scores and awards.

4. **Two recommendation letters.** The first letter should be from a teacher who knows you well, preferably (but not necessarily) a math teacher. The letter should comment on your creativity, initiative, and ability to work with others, as well as on your academic achievements. The second letter should be from another adult who knows you personally (e.g. an employer, pastor, soccer coach, etc. – preferably someone outside of school and not a relative). This letter should address your maturity, independence, social and personal qualities. We are looking for students who are not only good at math, but who will thrive in the atmosphere of freedom and responsibility that characterizes Mathcamp, and who will make a positive contribution to the camp community. To ensure confidentiality, ask each recommender to seal their letter in an envelope and to sign across the seal. Include the sealed envelopes with your application.

5. A **US$20 application fee** (check or money order made out to **Mathematics Foundation of America**) or a note signed by your parent or guardian explaining that your family cannot afford it.

Please do not use certified mail; if you wish, you may e-mail or call us to check that your application has arrived. Send to:

Mathcamp 2009
129 Hancock Street
Cambridge, MA 02139

All applications received by April 25, 2009 will be given equal consideration.

Rolling admissions thereafter.

Mathcamp 2009 Application Form

Name: _____

Address: _____

City: _____

State/Province/Country: _____

ZIP/Postal Code: _____

Telephone: (_____) _____

E-mail: _____

☐ I check my e-mail at least twice a week.

Date of birth: _____
Must be between August 1990 and June 1996.

Current grade in school: _____

Please check all those that will apply if you are accepted to Mathcamp:

☐ I authorize Mathcamp to give out my email address to other campers (e.g. to future roommates, or to those who live near me for coordinating travel to camp).

☐ I will need a visa to enter the US.

☐ I will be able to attend the full five weeks. (If not, please attach explanation.)

Note: all planned absences from the camp must be discussed with us in advance. Ordinarily, we do not want students to miss more than a few days of camp.

☐ I would like to be considered for a scholarship.

Please submit a scholarship application on a separate page. Note that admission to Mathcamp is need-blind.

How did you hear about Mathcamp? _____

_____

"I declare that my solutions to the qualifying quiz are my own work. I did not receive any form of assistance from other people, and I have referenced every instance where I looked something up in a book or on the web."

Signature: _____

The Mathematics Foundation of America invites you to apply to the seventeenth annual

CANADA/USA MATHCAMP

July 5 - August 9, 2009

For Mathematically Talented High-School Students From Around The World

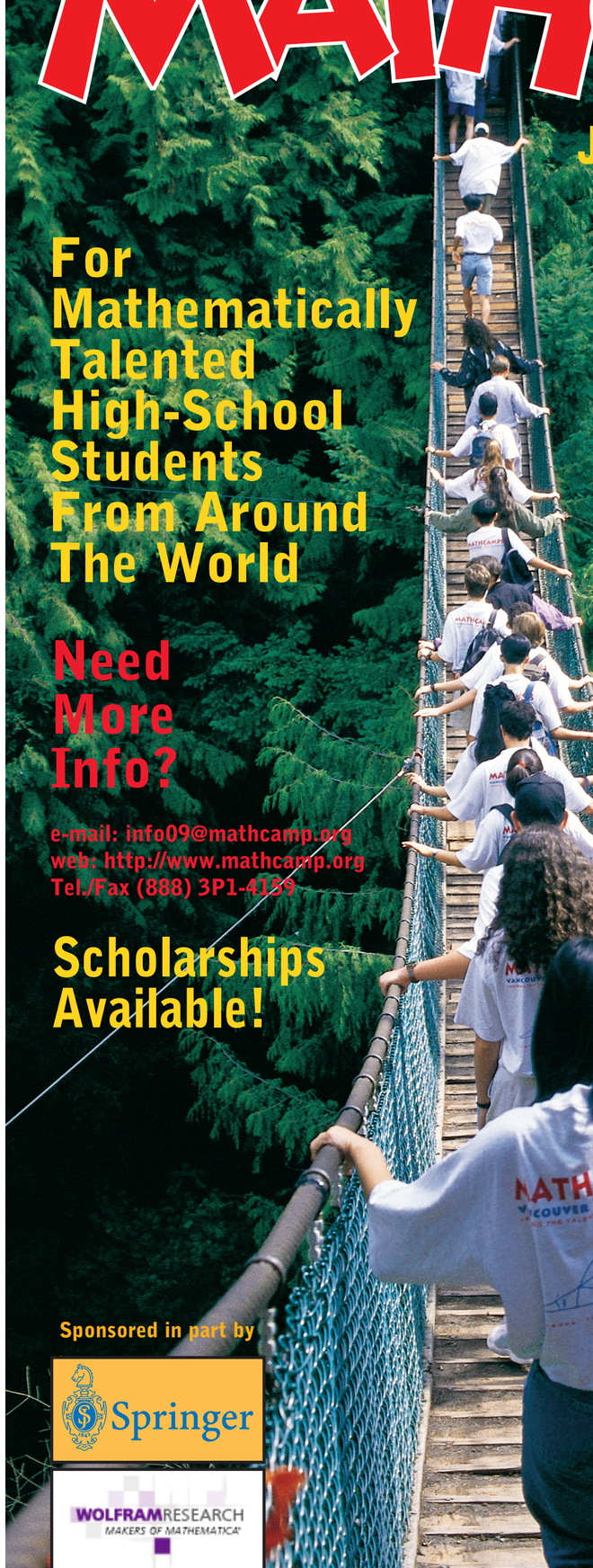
At the University of Puget Sound, Tacoma, Washington, USA

Applications received by April 25, 2009 are given full consideration.
Late applications will be considered as long as space is available.

Need More Info?

e-mail: info09@mathcamp.org
web: http://www.mathcamp.org
Tel./Fax (888) 3P1-4159

Scholarships Available!



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WOLFRAM RESEARCH
MAKERS OF MATHEMATICA


The University of Puget Sound

Discover MATHCAMP

Out of nothing I have created a strange new universe.
- Janos Bolyai, co-discoverer of hyperbolic geometry

Mathcamp is a chance to...

- Live and breathe mathematics: fascinating, deep, difficult, fun, mysterious, abstract, interconnected (and sometimes useful).
- Gain mathematical knowledge, skills and confidence – whether for a possible career in math or science, for math competitions, or just for yourself.
- Set and pursue your own goals: choose your classes, do a project, learn what ***you*** want to learn.
- Study with mathematicians who are passionate about their subject, from internationally known researchers to graduate students at the start of their careers, all eager to share their knowledge and enthusiasm.
- Make friends with students from around the world, and discover how much fun it is to be around people who think math is cool.

Mathcamp isn't really a camp. It's more of a five-week long festival - a congregation of people who celebrate math, enjoy math, learn math and essentially live math. Through it all I've discovered cool theorems that I wouldn't have understood before and cool people I didn't know existed. I've learnt that I actually know close to nothing about the weird and wonderful subject that is mathematics, and that I will probably pursue it for the rest of my life. Math on, Mathcamp!
– YQ Lu (Singapore)

It's not often that you find a place that is exciting to the mind and liberating to the spirit. Mathcamp is both.
– Greg Burnham (Memphis, TN, USA)

Academics and Activities

A Variety of Choices...

The Mathcamp schedule is full of activities at every level, from elementary to the most advanced:

- courses lasting anywhere from a few days to five weeks
- lectures and seminars by distinguished visitors
- math contests and problem-solving sessions
- hands on workshops and individual projects

You can see schedules from past years at
www.mathcamp.org/academics

The Freedom to Choose

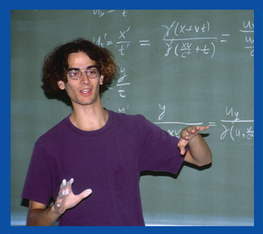
Mathcamp does not have a set curriculum or a list of requirements. We encourage the faculty to teach what they are most passionate about, and we let the students choose what they are interested in learning. With the help of an academic advisor, you will design a program of study that reflects your own interests and goals. You can take any classes you want, and even the number of classes that you attend each day is up to you: you can use the rest of the time to review what you've learned, talk to one of your professors, work on problems, do a project, or just take a break. For many students, the freedom to take charge of their own education is one of the aspects of Mathcamp that they value most.

"Mathcamp was the first place where I really understood the beauty and intricacies of abstract mathematics."
– Paul Hlebowitsch (Iowa City, IA, USA)

Classes

Course offerings vary from year to year, depending on the interests of the students and faculty. Some of the topics taught in previous years have included:

Discrete Mathematics Combinatorics • Generating functions • Graph theory • Ramsey theory • Finite geometries • Probability • Partitions • Combinatorial Game Theory
Algebra and Number Theory Primes and factorization algorithms • Congruences and quadratic reciprocity • Linear algebra • Groups, rings, and fields • Galois theory • Representation theory • *p*-adic numbers • Numbers and Games
Geometry and Topology Euclidean and non-Euclidean (hyperbolic, spherical, projective) geometries • Geometric transformations • Combinatorial topology • Algebraic geometry • Knot theory • The Brouwer Fixed-Point Theorem
Calculus and Analysis Fourier analysis • Complex analysis • Real analysis • Dynamical systems • Non-standard analysis
Computer Science Cryptography • Algorithms • Complexity • Information Theory • P vs. NP
Logic and Foundations Cardinals and ordinals • Gödel's Incompleteness Theorem • The Banach-Tarski Paradox • Model theory • Category theory
Connections to Science Relativity and quantum mechanics • Dimensional physics • Neural networks • Mathematical biology • Bayesian statistics • Cognitive Science
Discussions History and philosophy of mathematics • Math Education • "How to Give a Math Talk" • College and Beyond
Problem Solving Proof techniques • Elementary and advanced methods • Contest problems of various levels of difficulty • Weekly "Math Relays" and team competitions



There was no pressure at all — the incentive to learn came from within.
– Keigo Kawaji (Toronto, ON, Canada)

This summer, I made a number of cool connections between various subjects — hyperbolic geometry, physics, etc... I love seeing how different ideas connect with different applications.
– Anna de Bakker (Winnipeg, MB, Canada)

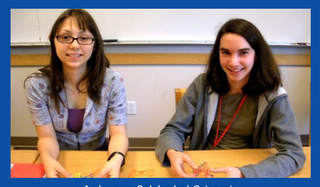
Thanks so much for showing me how math is truly the most beautiful form of art.
– Qian Yang (Pembroke Pines, FL, USA)

Spotlight on a Class

Computability and Complexity (Summer 2008)

Theoretical computer science is a mathematical framework for analyzing what computers can and cannot do. In this class, we formulated a model of computation and used it to prove that there are problems which no computer, no matter how fast, will ever be able to solve. We then analyzed how long problems take to solve: primality testing is fast, whereas factoring large numbers is (probably) very slow (the basis of modern cryptography). This led us to define the complexity classes P and NP, and to ask the famous question: is P = NP? This is not just one of the biggest open problems in all of mathematics, but worth a million dollars to anyone who solves it. (Unfortunately, we didn't.)

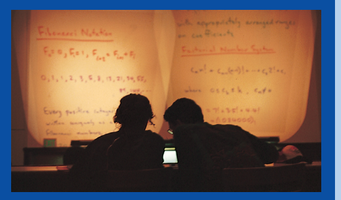
After P and NP, we examined space complexity (PSPACE and NPSPACE), BPP (computation using randomization), and other topics. By the end, we had constructed a "roadmap" to computational problems and their difficulty. The class was pure mathematics: no computer was harmed (or even touched) in its teaching. However, several spin-off projects had teams of students programming applications of theorems from the class with good results.


A class on Polyhedral Origami


A class on Hyperbolic Geometry

Projects

Every student at Mathcamp is encouraged to do a project, supervised by one of the mentors or faculty. Projects range in scope from creative applications of simple techniques to advanced problems connected to faculty research. Project topics in previous years have included:



- Periodicity of Fibonacci numbers mod *n*
- Information theory and psychology
- Knight tours on an *m*-by-*n* chessboard
- Cellular automata
- Cops and robbers on a graph
- Constructing the regular 17-gon
- Admissible covers of algebraic curves
- Mathematical Finance
- Algorithmic composition of music
- Intelligent ways of searching the web
- Probability in sports
- The elasticity equation of string
- Digital signal processing
- Light paths in universes with alternate physics
- Playing 20 Questions with a Liar
- Dirichlet's Theorem on Arithmetic Progressions
- Non-Orientable Knitting

"Mathcamp was definitely the most fun I've ever had."
– Avichal Garg (Cincinatti, OH, USA)

Beyond Math

Mathematical activities are scheduled for five days a week; whatever math happens on the other two days is purely informal. The weekend is reserved for relaxation and the incredible number of activities that quickly fill the schedule. All of the activities are optional, and students can choose simply to relax with friends.

Field trips in the past have included hiking trips to the mountains, whitewater rafting, sea kayaking, amusement parks, and museums. Lots of activities happen on-campus, too, such as a team "puzzle hunt" competition, the Contrapositones (our chorale), the Kernel (our contemporary a cappella singing group), a talent show, juggling and contra dancing lessons, and even improvisational theater. Campers also organize many events themselves — from sports and music to chess and bridge tournaments — including the production of the annual camp yearbook.

When 100 campers all come back saying they just had the best summer of their life, you know there's got to be something special about this place.
– Tristan Brand (Whitefield, ME, USA)

People

Regular Faculty

Mira Bernstein
(Executive Director, Mathcamp)
Interests: Algebraic Geometry, Mathematical Biology, Information Theory

Mark Krusemeyer
(Carleton College)
Interests: Abstract Algebra, Combinatorics, Number Theory, Problem Solving

David Savitt (University of Arizona)
Interests: Number Theory, Arithmetic Geometry

Noah Snyder (UC Berkeley)
Interests: Quantum Topology, Representation Theory, Elementary Number Theory

Visitors

John H. Conway (Princeton)
One of the most creative thinkers of our time, John Conway (pictured below, at lunch with Mathcamp students) is known for his ground-breaking contributions to such diverse fields as knot theory, geometry of high dimensions, group theory, transfinite arithmetic, and the theory of mathematical games. Outside the mathematical community, he is perhaps best known as the inventor of the "Game of Life."

James Tanton
(St Mark's Institute of Mathematics)
James likes taking elementary ideas in mathematics and pushing them in strange and unusual directions. He works with students of all ages and their teachers to look for really cool ways to prove advanced theorems with very basic and elementary tools (light bulbs, triangular palaces, bits of felt, dots and boxes) thereby proving to the world that mathematics — real, creative and truly exciting mathematics — is absolutely accessible to all. He has written a book or two, has worked at all levels of mathematics education, founded an outreach Institute, and has conducted research in algebraic topology and a splash of number theory. But his major goal and pleasure is to advance true joy in thinking about mathematics, much like Mathcamp!

Megumi Harada (McMaster Univ.)
Megumi Harada works on symplectic geometry, which provides a mathematical framework for classical physics and also ties in to quantum physics, combinatorics, and graph theory. Megumi loves to draw pictures and to get people excited about math. In 2005, she was the only mathematician among the 10 finalists in a televised competition for "Ontario's Best Lecturer" (billed as "reality TV with a high IQ"). She has also appeared regularly on TV Ontario's *More 2 Life* program to discuss mathematics in everyday life.

Dave Patrick (Art of Problem Solving)
Dave Patrick is a textbook writer and instructor at Art of Problem Solving (AoPS). He is the author of two of AoPS's textbooks (and is currently working on a third), and has taught problem-solving courses for AoPS at all levels, from MATHCOUNTS to the Putnam, and most everything in between. In his past life doing mathematics research (at MIT and the University of Washington), Dave studied noncommutative algebraic geometry, in particular the classification and structure of noncommutative ruled surfaces.

Josh Tenenbaum (MIT)
Josh Tenenbaum is a professor of cognitive science at MIT. In his research, he builds mathematical models of human and machine learning, reasoning, and perception. His interests also include neural networks, information theory, and statistical inference.

... and others!

The atmosphere of Mathcamp made it possible to be myself totally and completely, without fear of being judged. Coming to Mathcamp gave me the courage to explore new ideas and take on new challenges, not only at camp itself, but later, in the real world as well.
– Erika Parks (Colorado Springs, CO, USA)

Mathcamp has provided me with bits of information that keep showing up in the strangest places. I'm in college now, doing a double major in mathematics and electrical engineering. The funny thing is how I usually have experience with the mathematics involved, even though you wouldn't realize it by looking at the courses I've taken. For instance, the big thing that we do in electrical engineering right now is using phasors... well, I inadvertently got my first taste of that from a problem I did at Mathcamp. Usually I ace the tests that the EE students fail, especially when the questions include the words "show that..." or "prove"—which aren't as uncommon as one might expect in an engineering course.
– Sandor Swartz (Rolla, MO, USA)

Mentors and Junior Counselors

The residential staff at camp is made up of Mentors and Junior Counselors ("JCs"). The Mentors are graduate students in mathematics and computer science; they teach most of the classes at camp, picking the course topics freely from among their favorite kinds of math. JCs, all of them camp alumni, are undergraduates who make sure that the non-academic side of camp runs smoothly. From planning field trips to organizing afternoon frisbee games, they make sure that no one is ever bored. Each student is assigned a Mentor or JC as their residential advisor; RAs live on the same hall as their advisees and look out for them on a day-to-day basis. Like campers, the Mentors and JCs often return year after year to Mathcamp.


An unusual way to deliver the solution to a math problem


An impromptu gathering of the Contrapositones

I've changed so much in my two years here. I think about math in a new, deeper way. I approach problems differently. I've gained perserverence and learned to ask for help without shame and give it with joy.
– Hallie Glickman-Hoch (Brooklyn, NY, USA)

Imagine the thrill of setting out in a boat and stumbling across an island that you never knew existed. At Mathcamp you get that thrill every day. By the end, it feels as if a whole continent has opened before you, waiting to be explored.
– Alex Kandell (Catonsville, MD, USA)


Hikers break for an August snow ball fight.

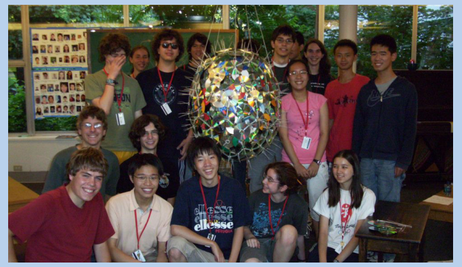
Students

We never cease to be amazed at what a varied and interesting bunch of young men and women our students are! While everyone at camp shares a love of mathematics, their other interests and accomplishments run the gamut. Each year's camp is a collection of musicians and writers, athletes and actors, hikers, artists, jugglers, board game players, dancers, computer programmers, students of science and philosophy - all sharing their interests and experiences with each other.

Most of the students at camp come from Canada or the US, but many come from overseas. Students have come to camp from Japan, Poland, England, Taiwan, Tanzania, Turkey, Romania, Lithuania and many other places around the globe.

It is a testament to our students' maturity and independence that they can be serious about doing math, while still finding so many different ways to have fun in the evenings. Many camp activities are organized entirely by campers, and students routinely cite each others' company as one of the best aspects of camp.


Mathcamp staff making ice cream with liquid nitrogen


A truncation of the rhombic triacontahedron made out of 200 CDs

At Mathcamp, you do things you never thought possible and you learn things you never dreamed of. If you've ever wanted reality to include things it didn't contain for you, come to Mathcamp.
– Vicky Howse (Saskatoon, SK, Canada)

"Go, just go! Trust me!"
– Jian Xu (Toronto, ON, Canada)