Mathcamp 2012 Qualifying Quiz

Instructions

We call it a quiz, but it's really a challenge: a chance for you to show us how you approach new problems and new concepts in mathematics. What matters to us are not only your final results, but also your reasoning. Correct answers on their own will count for very little: you have to justify all of your assertions and prove to us that your solution is correct. (For some tips on writing proofs, see www.mathcamp.org/proofs.) Sometimes it may take a while to find the right way of approaching a problem. Be patient: there is no time limit.

The problems start out easier and get harder. (At least we think so - but you may disagree.) None of the problems require a computer; you are welcome to use one if you'd like, but first see www.mathcamp.org/computers.

We don't expect every applicant to solve every problem: in the past, we have sometimes admitted people who could do only half of them, occasionally even fewer. However, don't just try two or three problems and declare yourself done! The more problems you attempt, the better your chances. We strongly recommend that you try all the problems and send us the results of your efforts: partial solutions, conjectures, methods - everything counts.

If you need clarification on a problem, please email quiz12@mathcamp.org. You may not consult or get help from anyone else. You can use books or the Web to look up definitions, formulas, or standard techniques, but any information obtained in this way must be clearly referenced in your solution. Please do not try to look for the problems themselves: we want to see how well you can do math, not how well you can use Google! Any deviation from these rules is considered plagiarism and may disqualify you.

Have fun and good luck!

Contact Mathcamp

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Please do not send certified mail to Mathcamp; we will email you to verify that your application materials have arrived.

Problems

(1) A frog jumps along the number line. It starts at 0 and every second it jumps n units to the right (the same positive integer neach time). After one second, you decide that you want to catch the frog. It's dark, you can't see the frog, and you don't know what n is. (For all you know, it might be a super-frog, so *n* could be arbitrarily large.) However, at any given second, you are allowed to choose an integer and search there. If the frog is on that integer, you'll catch it; if not, you'll have to try again.

(a) Devise a strategy that will eventually catch the frog. (You'll need to explain which integer you plan to check at each second.)

(b) Now suppose the frog is allowed to start by going either to the left or to the right; once it chooses a direction, it always jumps *n* units in that direction. Can you devise a strategy for catching it if you don't know which way it's going, and don't know what *n* is?

(C) What if the conditions in part (b) hold, and you also don't know which integer point the frog started at? (After you have worked on this problem for a while, it may be useful to read the following article, particularly the section after the proof of Lemma 1: http://www.cut-the-knot.org/do_you_know/numbers.shtml.)

(2) Each integer on the number line is colored with exactly one of three possible colors-red, green or blue-according to the following rules:

+ the negative of a red number must be colored blue,

+ the sum of two blue numbers must be colored red.

(a) Show that the negative of a blue number must be colored red and the sum of two red numbers must be colored blue.

(b) Determine all possible colorings of the integers that satisfy these rules.

(3) Let p be an odd prime. A group of p campers sit around a circle, and are labeled with the integers 1, 2, ..., p in clockwise order. The camper with label 1 yells out the number 1. The camper sitting next to this camper in clockwise order yells out 2. The camper two spots in clockwise order from the camper who yelled out $\hat{2}$ yells out $\hat{3}$. This process continues: the camper seated n spots (in clockwise order) from the camper who yelled out *n* must yell out n+1. A camper gets a cookie anytime she or he yells out a number.

(a) Show that there is a camper who never gets a cookie.

(b) Of the campers who do get cookies, is there one who at some point has at least ten more cookies than the others?

(C) Of the campers who do get cookies, is there one who at some point has at least ten fewer cookies than the others?

(4) Let a be a rational number with 0 < a < 1. A lollipop in the xy-plane with *base* (a, 0) consists of a line segment from (a, 0) to some point (a, b) with b > 0, together with a filled in disc of radius

(6) An ocean has infinitely many islands. Every island is labeled by one of the integers $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$, with no two islands having the same label and every integer being the label of some island. Two islands are connected by a bridge if their labels differ by a power of two. For instance, there is a bridge connecting island 7 and island -25.

We define the *distance* between two islands k_1 and k_2 to be the minimum number of bridges needed to get from k_1 to k_2 . For instance, the distance between the islands 0 and 7 is 2. (You can move from island 0 to island 8, then to island 7; this is the m, since you can't go from 0 to 7 using just one bridge.)

(a) Show that for any integer $r \ge 1$, you can find two islands in the ocean at distance r from each other

(b) An *infinite path* in our ocean consists of an infinite set of islands I and an infinite set of bridges \mathcal{B} , such that:

+ every island in I is connected to exactly two bridges in $\mathcal B$, \star for any two islands in I, you can get from one to the other using only bridges in \mathcal{B} .

Here is one example of an infinite path: let I be the set of all odd-numbered islands and let \mathcal{B} be the set of bridges between islands in I whose labels differ by 2. It is easy to see that I and \mathcal{B} satisfy both conditions for an infinite path:

+ Each island k in I is connected to exactly two bridges in ${\mathcal B}$ namely, the bridges leading to islands k-2 and k+2; + To get from any odd-numbered island to any other using bridges

in $\mathcal B$, you start at the smaller number and keep adding 2. As a warm-up, can you create an infinite path with the same I as in

the example above, but with a different \mathcal{B} ?

(c) Is it possible to construct an infinite path in our ocean such that, for any two islands k_1 , k_2 in I, the minimum number of bridges in $\mathcal B$ needed to get from k_1 to k_2 is exactly the distance between k_1 and k_2 ? For instance, the infinite path in our example does not have this property: it takes two bridges in \mathcal{B} to go from island 1 to island 5 (you have to go via island 3), even though the distance between these two islands is 1 (there is a single bridge not in \mathcal{B} that connects them to each other). If your answer is yes, give an example of sets I and $\mathcal B$ that work. If your answer is no, prove that it can't be done.

(d) Does there exist a set S of 9 islands such that:



additional bridges between any of the islands), and • the distance between any two islands in S equals the minimum number of bridges needed to get from one island to the other via islands in S?

What if, instead of a 3 \times 3 grid, we had an $n \times n$ grid: for which nis such a configuration possible?

(e) Suppose, in a sea far away, we have islands labeled in the same way, with two islands connected by a bridge if their labels differ by a power of 3. Is there a one-to-one correspondence between islands in the ocean and islands in the far-away sea, such that two islands in the ocean are connected by a bridge precisely when the corresponding two islands in the sea are connected by a bridge?

less than b, centered at (a, b). Determine whether or not it is possible to have a set of lollipops in the xy-plane satisfying both of the following conditi

+ for every rational number *a* with 0 < a < 1, there is a lollipop whose base is the point (a, 0),

+ no two lollipops touch or overlap each other.

If such a set of lollipops exists, explain how to construct it. If not, justify why not.

(5) A convex body in the plane is a region with positive area such that for any two points in this region, the entire line segment between them also lies within the region. Let P be the perimeter (i.e., boundary) of a convex body in the plane. We will assume throughout this problem that P is centrally symmetric: that is, if (a, b) is a point on P, then so is (-a, -b).

For any nonnegative real number k, we define kP to be the subset of the plane obtained by multiplying all the points of P by k in each coordinate. In other words, for each point (a, b) of P, the point (ka, kb) is in kP.

If (x_1, y_1) , (x_2, y_2) are two points in the plane, we define the *P*-distance between them to be the smallest nonnegative real number *k* such that when the set *kP* is translated by (x_1, y_1) (i.e., by x_1 units horizontally and by y_1 units vertically), the point (x_2, y_2) lies on it. For example, if P is the square with vertices (0, 1), (1, 0), (0, -1), (-1, 0), then the *P*-distance between (3, 5) and (4, 10) is 6.

(a) Let P be the perimeter of a disc of radius 1 centered at the origin. Find a formula for the P-distance between any two points (a, b) and (c, d) in the plane.

(b) Let *P* be the perimeter of a rhombus with vertices (2, 0), (-2, 0), (0, 3), (0, -3). Find a formula for the *P*-distance between any two points (a, b) and (c, d) in the plane.

(C) In part (a), we took it for granted that a filled-in disc of radius 1 is a convex body. Prove this rigorously, using the definition of convexity given above

(d) Suppose P is a convex quadrilateral. What are the possible P-distances between vertices of P? What about when P is a convex hexagon? (Remember: *P* must still be centrally symmetric!)

(e) Let *P* be the perimeter of some centrally symmetric convex body, and let (a, b) be a point on P. What is the largest possible P-distance from (a, b) to another point on P? Will (a, b) be at this P-distance from just one other point on P or from multiple other points? (If any of your answers depend on the geometry of P and/or on the choice of (a, b), explain how.)

(f) In principle, we could define *P*-distance even when *P* doesn't come from a convex body and/or is not centrally symmetric. But it turns out that in both of these cases, the definition is problematic: the resulting quantity doesn't behave in the ways we expect a "distance" to behave. Can you determine what problematic issues arise?



(7) Last summer, the graduate students teaching at Mathcamp call them "mentors") arranged themselves into a pyramid with four layers, as shown in the picture above.

Now suppose we generalize this to a pyramid with n layers (n mentors in the bottom row, n-1 mentors in the row above, etc).Assume that all mentors have weight 1 and that each mentor supports his/her own weight plus half the weight supported by the one or two mentors leaning on him/her. For instance, in our fourlayer mentor pyramid, the weights supported by the mentors are:



(a) Find the weight supported by the mentor at the bottom left corner of a pyramid with n layers.

(b) Now suppose the pyramid has infinitely many layers. Let W(k,m) be the weight supported by the (k+1)-th mentor from the left in the (m+1)-th layer. For instance, W(0,0)=1, W(0,1)=W(1,1)=3/2, and so on. Determine a recursive formula satisfied by the function W(k, m), for $k, m \ge 0$.

(c) Find an explicit formula for W(k, 2k) in terms of k, for $k \ge 0$. (d) Extra credit: What can you say about W(k,m) in general?

(Note: If you're not sure what we mean by "explicit" and "recursive" formulas, take a look at: http://www.regentsprep.org/Regents/ math/algtrig/ATP3/Recursive.htm)



We invite applications from every student aged 13 through 18 who is interested in *mathematics, regardless of racial,* ethnic, religious, or economic background.

Internet access to use the online application process.

Online Application:

Postal Application: print out the application packet.

Student Care Policy

Dear parent: Student safety and enjoyment are Mathcamp's first priorities. Students will be housed in secure campus dormitories, with male and female students in designated sections of the same building. In case of a medical problem, we have a camp nurse on call, and the hospital is minutes away. Students will have access to university athletic facilities and computers. Every effort will be made to enable students who so desire to attend weekly religious services of their faith. Mathcamp is committed to an atmosphere of mutual tolerance, responsibility, and respect, and is proud of its past record in helping to create such an atmosphere.

The Mathematics Foundation of America invites you to apply to the twentieth annual

Canada/USA

Applications are due April 25, 2012 + Visit www.mathcamp.org for more information

Applying to Mathcamp 2012

Ready to Apply to Mathcamp?

Mathcamp accepts applications both on the web and by regular mail. We strongly encourage all students with

The \$20 application fee is waived for online applications.

Go to http://www.mathcamp.org/apply/ and follow the instructions. You'll still have the opportunity to submit our quiz or recommendation letters by postal mail.

Go to http://www.mathcamp.org/applybymail/ and

An application to Mathcamp consists of the following:

1) Some basic information about yourself and your math background. We will ask you to describe the math courses that you've taken at the high-school level or above, along with scores and awards from any math competitions you've done.

2) A brief personal statement about your interest in math and why you want to come to Mathcamp.

3) Your solutions to the 2012 Qualifying Quiz (see below).

4) Two recommendation letters, academic and personal.

- + The first letter should be from a teacher who knows you well, preferably a math teacher. The letter should comment on your creativity, initiative, and ability to work with others, as well as on your academic achievements.
- · The second letter should be from another adult who knows you personally (e.g. an employer, pastor, soccer coach, etc. preferably outside of school and not a relative). This letter

- Mira Bernstein, Executive Director, Mathcamp

Cost and Scholarships

Full Camp Fee: \$4000 (This includes tuition, room, board, and extracurriculars.)

Admission to Mathcamp is need-blind. We are deeply committed to enabling every qualified student to attend, regardless of financial circumstances.

Mathcamp awards over \$100,000 in need-based scholarships every year. In the past seven years, no admitted applicant has been unable to attend the camp for financial reasons. We give several full scholarships each year, and occasionally even help students with travel expenses. Please do not let financial considerations prevent you from applying! If you'd like to be considered for a scholarship, just complete the short application at right.

should address your maturity, independence, social and personal qualities. We are looking for students who are not only good at math, but who will thrive with the freedom and responsibility that characterize Mathcamp, and who will make a positive contribution to the camp community.

5) If you would like to be considered for financial assistance, please include the scholarship application (see instructions below). Note that admission to Mathcamp is need-blind.

6) For postal applications only: A US \$20 application fee (check or money order made out to Mathematics Foundation of America) or a note signed by your parent or guardian explaining that your family cannot afford it.

All applications received by April 25, 2012 will be given equal consideration.

Scholarship Application

Please have a parent or guardian provide the following information, along with her or his email address:

- + 2011 family income (all sources).
- Expected family income for 2012. (If significantly different from 2011, please explain.)
- + A list of all members of your household (supported by the above income) and their relationships to the applicant. For siblings, please provide ages.
- The cost of schooling, if any, for household members (private school, college, etc).
- The estimated cost of round-trip travel to Mathcamp for the applicant.
- The portion of the cost of Mathcamp (including both
- tuition and travel) that your family can afford to pay.
- + Any special circumstances you want us to consider.



"Out of nothing I have created a strange new universe." – Janos Bolyai, co-discoverer of hyperbolic geometry

Mathcamp is a chance to...

+ Live and breathe mathematics: fascinating, deep, + Study with mathematicians who are passionate about difficult, fun, mysterious, abstract, interconnected (and their subject, from internationally known researchers to sometimes useful)

• Gain mathematical knowledge, skills and confidence – whether for a possible career in math or science, for math competitions, or just for yourself.

• Set and pursue your own goals: choose your classes, do a project, learn what you want to learn.

graduate students at the start of their careers, all eager to share their knowledge and enthusiasm. + Make friends with students from around the world

think math is cool.

"Mathcamp was the first place where I really understood the beauty and intricacies of abstract mathematics." - Paul Hlebowitsch (Iowa City, IA, USA)

Academics

A Variety of Choices

The Mathcamp schedule is full of activities at every level, from introductory to the most advanced:

• Courses lasting anywhere from a few days to five weeks

+ Lectures and seminars by distinguished visitors

• Math contests and problem-solving sessions + Hands-on workshops and individual projects

You can learn more at:

http://www.mathcamp.org/academics

Classes

Course offerings vary from year to year, depending on the interests of the students and faculty. Some of the topics taught in previous years have included:

Discrete Mathematics: Combinatorics + Generating functions • Partitions • Graph theory • Ramsey theory + Finite geometries + Polytopes and Polyhedra + Combinatorial Game Theory + Probability

Algebra and Number Theory: Primes and factorization algorithms + Congruences and quadratic reciprocity + Linear algebra • Groups, rings, and fields • Galois theory • Representation theory + p-adic numbers

Geometry and Topology: Euclidean and non-Euclidean (hyperbolic, spherical, projective) geometries + Geometric transformations • Combinatorial topology • Algebraic geometry + Knot theory + Brouwer Fixed-Point Theorem

Calculus and Analysis: Fourier analysis + Complex analysis • Real analysis • Measure theory • Dynamical systems • Non-standard analysis

Computer Science: Cryptography + Algorithms + Complexity • Information Theory • P vs. NP

Logic and Foundations: Cardinals and ordinals + Gödel's Incompleteness Theorem • The Banach-Tarski Paradox • Model theory • Category theory

Connections to Science: Relativity and quantum mechanics + Dimensional physics + Voting Theory + Bayesian Statistics + Neural networks + Mathematical biology + Cognitive Science

Discussions: History and philosophy of mathematics • Math Education + "How to Give a Math Talk" + College, Graduate School and Beyond

Problem Solving: Proof techniques • Elementary and advanced methods + Contest problems of various levels of difficulty • Weekly "Math Relays" and team competitions

Spotlight on a Class

Set Theory as a Foundation for Mathematics (2011) + What is a number? Stop and think. Do you know? Do they even exist (as a mathematical concept)? Fortunately for all of mathematics, the answer is "yes", and in this class we'll see why. We'll build the numbers you know and love from the natural numbers up through the reals. We'll see how even the most basic properties that you never think about can be proved (such as the fact that addition is commutative: x + y = y + x). You've probably seen proof by induction, but I bet you've never seen a real proof that this common technique works. We'll show that it is possible to rigorously prove it, once you've stated it

The Freedom to Choose

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Mathcamp does not have a set curriculum or a list of requirements. We encourage the faculty to teach what they are most passionate about, and we let the students choose what they are interested in learning. With the help of an academic advisor, you will design a program of study that reflects your own interests and goals. You can take any classes you want, and even the number of classes that you attend each day is up to you: you can use your time to review what vou've learned, talk to one of your professors, work on problems, do a project, or just take a break. For many students, the freedom to take charge of their own education is one of the aspects of Mathcamp that they value most.

Projects

Every student at Mathcamp is encouraged to do a project, supervised by one of the mentors or faculty. Projects range in scope from creative applications of simple techniques to advanced problems connected to faculty research. Project topics in previous years have included:

- Periodicity of Fibonacci numbers mod *n*
- Information theory and psychology
- Knight tours on an *m*-by-*n* chessboard • Cellular automata
- Cops and robbers on a graph
- Constructing the regular 17-gon + Admissible covers of algebraic curves
- Mathematical Finance
- Algorithmic composition of music • Intelligent ways of searching the web
- Probability in sports
- The elasticity equation of string
- + Digital signal processing
- Light paths in universes with alternate physics
- + Playing 20 Questions with a Liar
- + Dirichlet's Theorem on Arithmetic Progressions
- Non-Orientable Knitting

"One cannot compare my ideas of what `I'm interested in math' meant before and after Mathcamp.'

- Asaf Reich (Vancouver, BC, Canada)

"There was no pressure: the incentive to learn came from within." - Keigo Kawaji (Toronto, ON, Canada)

"Mathcamp took every limitation I thought I had—social, academic, and personal—and shattered it."

– Andrew Kim (Dover, MA, USA)

carefully: if you have a statement P(...) and you want to show that P is true for each natural number n, then it's sufficient to prove that P(0) and that P(k) implies P(k+1). However, we can't even think about proving that induction works without a very precise and rigorous definition of what numbers are in the first place! So, how can we define numbers? What is there more basic than numbers that we could build numbers out of? The answer is sets. You might know sets as "collections of objects", but for us, those "objects" will themselves be sets. In fact, we'll be able to start from just the empty set and build up everything you know about numbers (and more!), using only the logical concept of sets. It turns out that everything we study in mathematics can be expressed using sets, something I hope to convince you of by the end of the week.



"Mathcamp was definitely the most fun I've ever had." – Avichal Garg (Cincinatti, OH, USA)



"Mathcamp isn't really a camp. It's more of a five-week long festival - a congregation of people who celebrate math, enjoy math, learn math and essentially live math. Through it al I've discovered cool theorems that I wouldn't have understood before and cool people I didn't know existed. I've learnt that I actually know close to nothing about the weird and wonderful subject that is mathematics, and that I will probably pursue it for the rest of my life. Math on, Mathcamp!' - Yongquan Lu (Singapore)

Site of Mathcamp 2012

"It's not often that you find a place that is exciting to the mind and liberating to the spirit. Mathcamp is both."

"I've changed so much in my two years here. I think about math in a new, deeper way. I approach problems differently. I've gained perseverence and learned to ask for help without shame and give it with joy."

- Hallie Glickman-Hoch (Brooklyn, NY, USA)





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and discover how much fun it is to be around people who



University of Puget Sound



- Greg Burnham (Memphis, TN, USA)





People and More

Regular Faculty

Mira Bernstein (Executive Director, Mathcamp) Interests: Algebraic Geometry, Mathematical Biology, Information Theory

Mark Krusemeyer (Carleton College) Interests: Abstract Algebra, Combinatorics, Number Theory, Problem Solving

David Savitt (University of Arizona) *Interests:* Number Theory, Arithmetic Geometry

Mohamed Omar (Caltech) Interests: Combinatorics, Applied Algebraic Geometry, Discrete Optimization

Visiting Faculty

John H. Conway (Princeton) + One of the most creative thinkers of our time, John Conway is known for his groundbreaking contributions to such diverse fields as knot theory, geometry of high dimensions, group theory, transfinite arithmetic, and the theory of mathematical games. Outside the mathematical community, he is perhaps best known as the inventor of the "Game of Life."

Jim Gates (UMD) + Jim Gates uses mathematical models involving supersymmetry, supergravity, and superstring theory to explore nature. One of his current focus areas includes Adinkras, a new mathematical concept, linking computer codes like those in browsers to the equations of fundamental physics as if our physical reality resides in the science fiction movie, "The Matrix."

Craig Sutton (Dartmouth) + Craig Sutton works on problems in inverse spectral geometry, where one explores the extent to which the geometry of a manifold is encoded in the spectrum of its associated Laplace operator. Of particular interest to him is understanding whether we can "hear" the geometry of spheres and other symmetric spaces

Allan Adams (MIT) + Allan Adams works on quantum versions of algebraic and differential geometry, and uses black holes in 5 spacetime dimensions to study high-temperature superconductors in the usual 4.

Kirsten Wickelgren (Harvard) + Kirsten Wickelgren would like to solve polynomial equations with loops on associated surfaces. This was conjectured by one of the founders of modern algebraic geometry, Alexander Grothendieck. He has all of our admiration.

Joe Rabinoff (Harvard) + Joe Rabinoff studies geometric objects over fields like the rational numbers, but equipped with a strange absolute value that would have made Archimedes' head explode. These objects turn out to be useful for doing things like solving Diophantine equations.

Josh Tenenbaum (MIT) + Josh Tenenbaum is a professor of Cognitive Science and a member of the MIT Computer Science and Artificial Intelligence Lab. In his research, he builds mathematical models of human and machine learning, reasoning, and perception. He thinks about neural networks, information theory, and statistical inference.

Students

We never cease to be amazed at what a varied and interesting bunch of young men and women our students are! While everyone at camp shares a love of mathematics, their other interests run the gamut. Each year's camp is a collection of 115 students who are musicians and writers, jugglers, dancers, athletes and actors, artists, board game players, hikers, computer programmers, students of science and philosophy - all sharing their interests and experiences with each other.

Most of the students at camp come from North America, but many come from overseas. Students have come to camp from Bulgaria, India, Japan, Lithuania, Macedonia, Mexico, Poland, Romania, Saudi Arabia, Serbia, South Korea, Tanzania, Turkey, and many other places around the globe.

It is a testament to our students' maturity and independence that they can be serious about doing math, while still finding so many different ways to have fun. Many camp activities are organized entirely by campers, and students routinely cite each others' company as one of the best aspects of camp.

Mentors and **Junior Counselors**

The residential staff at camp is made up of Mentors and Junior Counselors ("JCs"). Mentors are graduate students in mathematics and computer science; they teach most of the classes at camp, picking the course topics freely from among their favorite kinds of math. JCs, all of them camp alumni, are undergraduates who run the non-academic side of camp (from field trips to first aid to frisbee games). Each student is assigned a Mentor or JC as their residential advisor; RAs live on the same hall as their advisees and look out for them on a day-to-day basis. Like campers, the Mentors and JCs often return year after year to Mathcamp.

Beyond Math

Mathematical activities are scheduled for five days a week; whatever math happens on the other two days is purely informal. The weekend is reserved for relaxation and the incredible number of activities that quickly fill the schedule. All of these activities are optional, and students can choose simply to relax with friends.

Field trips in the past have included hiking, sea kayaking, whitewater rafting, amusement parks, and museums. Lots of activities happen on-campus, too: there are rehearsals for the choir and the contemporary a cappella group, juggling and salsa dancing lessons, improv, and even making berry pies. There is an annual team "puzzle hunt" competition, a talent show, and ice cream made with liquid nitrogen. Campers also organize many events themselves-from sports and music to chess and bridge tournaments-and each year, a group of students creates the camp yearbook.

"Coming to Mathcamp has given me a community with which to interact, not just five weeks a year, but all year round." – Eric Wofsey (St Louis, MO, USA)