CLASS DESCRIPTIONS—WEEK 1, MATHCAMP 2013

Contents

9:10 AM Classes

Combinatorial Game Theory. (🌶🌶, Alfonso, Tues–Sat)
We have a plate with blueberries, blackberries, and raspberries. We take turns eating them. On your turn, you may eat as many berries as you want (as long as you eat at least one), but they all have to be of the same type. Then it is my turn. Unless you are Marisa, the goal is not to eat a lot of berries: the winner is the person who eats the last one. Will you beat me?

The above is an example of an impartial combinatorial game. There are tons more where it came from, and you probably have encountered some. To solve them all, there are basically only two skills that you need to learn. If you want enlightenment, my young grasshopper, come to this class. I will motivate the two skills and I will guide you all in figuring them out. The beauty of this topic is as much in the journey as it is in the final results, and I do not want to deprive you of the pleasure of discovering it slowly. We will also attack many examples, from easy ones to actual open problems.

Homework: Recommended

Prerequisites: None

Required for: Partizan Game Theory

Cryptography. (🌶→🌶🌶, Matt, Tues–Sat)
Cryptography lets us send a message in such a way that anybody who lacks a secret (such as a password) cannot read it. But that's just the tip of the iceberg: cryptography can be used to do plenty of impossible-seeming things! We'll see how to prove to someone that you found Waldo without revealing where he is, how to communicate anonymously even when everyone can hear everything you say, and how to play poker over the phone without anyone being able to cheat!
(*Note that 🌶→ 🌶🌶 means that the class starts off 🌶 and will end the week as 🌶🌶.*)

Homework: Recommended

Prerequisites: None

Group Theory. (🌶🌶🌶, Mark, Tues–Sat)
How can you describe the symmetries of geometric figures, or the workings of a Rubik's cube? How do physicists predict the existence of certain elementary particles before setting up expensive experiments to test those predictions? Why can't fifth-degree polynomial equations, such as $x^5 - 3x + 2013 = 0$, be

solved using anything like the quadratic formula, although fourth-degree equations can? The answers to these questions are mostly beyond the actual scope of this class, but they all depend on group theory. Knowing some group theory is at least helpful, and often crucial, in other parts of mathematics. So come get your feet wet (we'll consider taking off socks and shoes, but you shouldn't take any of that too literally.) We'll move fairly quickly and with luck, after doing fundamental concepts (and examples,) we'll get to permutation groups, Lagrange's theorem, quotient groups, and the First Isomorphism Theorem.

Homework: Recommended

Prerequisites: None

Required for: Representation Theory, Symmetric Group Representations, Hydrogen Atom, Matrix Groups

Related to (but not required for): Ring Theory

Partially Ordered Sets. (, Kevin, Tues–Sat)
What do the principle of inclusion-exclusion, divisors, graph colorings, and polytopes have in common? We can study and prove results about these things using partially ordered sets! And the list goes on and on: subgroup structures, partitions, permutations... so many things come naturally with a partial ordering!

Unlike a total ordering, we can't always compare two elements in a partially ordered set. For example, it's natural to order sets based on containment, but then the sets $\{1, 2\}$ and $\{1, 3\}$ are incomparable—neither contains the other. Even though it might seem like a partial order doesn't provide much structure, we'll see that we can do quite a lot of magic with what seems to be very little.

Homework: Recommended

Prerequisites: None

Problem Solving: Polynomials. (, Pesto, Tues–Sat)
Polynomials are a frequent topic of Olympiad-style competitions, since there are many and interesting problems using them: for instance, find the product of the lengths of all the sides and diagonals of a regular n-gon of diameter 2. Don't see the polynomials? Come to the class and find them.

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to Olympiad-style problems you will have solved as homework the previous day.

Disjoint from last year's polynomials problem solving class.

Homework: Required

Prerequisites: None

10:10 AM Classes

Euler's Greatest Hits. (, *Jon Tannenhauser*, Tues–Fri)
Leonhard Euler (1707-1783) wrote 866 works in mathematics and science. In this course, we'll study four of his greatest hits in analysis and number theory.

(1) Euler's first big mathematical coup was summing the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$, which had been an open question for nearly 100 years. We'll look at several of Euler's methods for evaluating this sum.
(2) A pair of numbers is called *amicable* if the proper divisors of each add up to the other. Before Euler, three amicable pairs were known. Euler found 58 others—we'll see how.

(3) We'll survey some of Euler's work on integer partitions, including the pentagonal number theorem

$$(1-x)(1-x^2)(1-x^3)\cdots = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)}\,.$$

(4) Which numbers are both square and triangular? We'll derive Euler's formula for the k^{th} square triangular number,

$$N_k = \frac{1}{32}\left((3+2\sqrt{2})^k - (3-2\sqrt{2})^k\right)^2\,.$$

(*Note that the days are self-contained; you can come to later classes without attending earlier ones.*)

Homework: Recommended

Prerequisites: None

Related to (but not required for): Combinatorics of Partitions

Finding the Center. (🌶🌶, Pesto, Sat)
Given n points in the plane, what's the best way to find the center of the smallest circle containing them if:

- Your computer's infinitely powerful, but you're not;
- You're powerful, but your computer's not;
- You and your computer are powerful, but unlucky?

Homework: None

Prerequisites: None

Inside Convexity. (🌶🌶🌶, Alex, Tues–Sat)
Intuitively, a convex set is a set with no holes or dimples. (Of course, the first thing we'll do in class is provide a more formal definition.) For example, a filled-in circle is convex, while a torus (i.e. doughnut) is not. Convex sets have lots of nice properties that arbitrary sets don't have. For instance, given a collection of convex sets in d-dimensional space, if every $d+1$ convex sets have a common element, then all of the convex sets contain a common element (We will prove this in class.) A particularly interesting class of convex sets is polytopes. In this class, we'll talk about properties of convex sets in general and then focus on polytopes and how they are (or aren't) different from polyhedra.

Homework: Recommended

Prerequisites: None

Introduction to Number Theory. (🌶🌶, Ruthi, Tues–Sat)
When is n of the form x^2+y^2? Writing out some examples, maybe we can find some patterns and make hypotheses about this, but how does one find a complete answer? And how can we prove it?

A fundamental part of a mathematician's toolbox is the notion of modular arithmetic. If you don't know what people mean when they say "mod n" or "mod 7", this is the class for you! We'll learn a whole lot about the structure this gives us, and how it relates to questions like the sum of two squares.

Homework: Required

Prerequisites: None

Required for: You don't have to have taken this specific class, but the concepts from this class will be required for Algebraic Number Theory and PRIMES is in P.

Linear Algebra. (🌶🌶, Nic, Tues–Sat)
Linear algebra is the area of math that deals with vectors and matrices. It is one of the most useful methods in mathematics, both within pure math (as you can see from the number of Mathcamp classes that require this one as a prerequisite) and in its applications to the real world. One could argue that most of what mathematicians (and physicists, and engineers, and economists) do with their time is try to reduce hopelessly complicated non-linear problems to linear ones that can actually be solved. Thus for many applied fields, the most important math to know is not calculus, but linear algebra. Obviously we can't cover all of linear algebra in one week, but this class will give you a basic background, as well as a preview of some of the most important results. We're going to start out on the plane, where linear algebra springs out of geometry. We'll define linear maps and give an intuitive preview of the central themes of linear algebra. Then we'll leave our two-dimensional pictures behind and introduce the more general concepts of vector spaces, linear independence, dimension, kernels, images, eigenvectors, eigenvalues, and diagonalization. (If you don't know what any of these words mean, that's great: come to the class! If you know all of them, then you probably don't need this class.)

Homework: Recommended

Prerequisites: None

Required for: Representation Theory, Matrix Groups, Using Linear Algebra, Flag Varieties, Multilinear Algebra, Symmetric Group Representations, Symmetric Functions and Schubert Calculus, Topology of Surfaces, Algebraic Number Theory

How Not to Prove the Continuum Hypothesis. (🌶🌶🌶🌶, Susan, Tues–Sat)
$\mathbb{N}$ is countable and $\mathbb{R}$ is uncountable, but what happens in between? What does it mean for a subset of $\mathbb{R}$ to be small? Is a "small" set necessarily countable? In this class we will explore different notions of smallness and largeness, touching on ideas from topology, measure theory, and logic. Especially logic! We'll learn the rules to several games played on infinite sets that can give us valuable information about how the sets behave. One of these games will allow us to almost-but-not-quite prove the continuum hypothesis.

Homework: Recommended

Prerequisites: None

Related to (but not required for): Set Theory, Ultrafilters and Hindman's Theorem

11:10 AM CLASSES

4.99 Color Theorem. (🌶🌶, Noah, Tues–Sat)
The goal of this class is to state an open problem which I have tried to work on in my research but got stuck on.

In 1852, Guthrie asked whether the countries on any map can by colored with four colors such that no adjacent countries have the same color. In 1976, Appel and Haken showed that this was possible. Their proof is somewhat controversial because it uses a computer in a serious way. By contrast the proof of the 5 color theorem is much much shorter and easier. This suggests an obvious question: What about numbers between 4 and 5? Is the 4.99 color theorem easy like the 5 color theorem or hard like the 4 color theorem? Do things get harder gradually as you move past 5 or is there a jump in difficulty? Of course, at first glance it doesn't make sense to try to color a map with 4.99 colors, but it turns out that we'll be able to make this question precise. We'll prove the x color theorem for any $x \geq 5$ and look at how some of the techniques used in the proof of the 4 color theorem might be used for other numbers below 5.

Homework: Optional

Prerequisites: None

Related to (but not required for): Introduction to Graph Theory, Finding the Perfect Match, Flows, Traveling Salesman Problem

Crash Course on Mathematical Abstraction. (🌶, Mira, Tues–Sat)
This course covers fundamental mathematical concepts and tools that all other Mathcamp courses assume you already know: basic logic, basic set theory, proofs by induction and contradiction, and a few other tidbits. If you are new to advanced mathematics or just want to make sure that you have a firm foundation for the rest of your Mathcamp courses, then this class is highly recommended.

Here are some problems to help you test your knowledge:

(1) Negate the following sentence without using any negative words ("no", "not", etc.): *"If a book in my library has a page with fewer than 30 words, then every page in that book contains a word that starts with a vowel."*
(2) Prove that there are infinitely many prime numbers.
(3) Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
(4) Prove that an equivalence relation on a set S partitions S into a set of disjoint equivalence classes.
(5) Prove that addition modulo 2013 is well-defined.
(6) What is wrong with the following argument (aside from the fact that the claim is false)?
  Claim: On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.
  Proof: We proceed by induction on n. The claim is clearly true for $n = 2$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k+1$, we add another city to this island. By assumption, the new city is connected by a road to at least one of the old cities, and you can get from that old city to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k+1$, so by induction it holds for all n. QED.
(7) Explain what it means to say that the real numbers are uncountable. Then prove it.

If you are not 100% comfortable with most of these questions, then you can probably benefit from this crash course. If you found this list of questions intimidating, then you should *definitely* take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Homework: Required

Prerequisites: None

Required for: Everything!

Extreme Probability! (week 1 of 2). (🌶🌶, Aaron, Tues–Sat)
If you digitize a year's worth of *New York Times* articles, how much disk space will you need? If you put a cast iron skillet in an oven, how much heat will it soak up? If you drop a bacterium in a hostile ocean, will its descendants live to rule the waves? These questions are all studies in extreme probability: the weird stuff that happens when the theory of a single coin toss is scaled up to a million or a trillion.

The first week of this course will take us from the basics of probability theory—no experience required!—to the earliest glimmer of extreme probability, the "law of large numbers." In the second

week, we'll set out into the wilds of extreme probability, applying deep ideas like "entropy" and "percolation" to problems in information theory, physics, and biology.
Homework: Recommended
Prerequisites: None
Required for: Week 1 of this class is required for Machine Learning.
Related to (but not required for): The Probabilistic Method

The Forehead Game: Multiparty Communication Complexity. (, Tim!, Tues–Sat)
You and your friends are each dealt a playing card from a regular deck of cards. But you're not allowed to look at it. Instead, you must stick your card to your forehead so that everybody else can see it, but you can't. Suppose I come by and see you all sitting there with cards on your foreheads, and I ask "Is the sum of your cards odd?". Naturally, being busy people, you want to answer while talking to each other as little as possible. You could have your friend tell you what card you have, and then you could answer my question. Since there are 52 cards in the deck, this takes $\log_2 52 \approx 6$ bits of communication. If you instead get your friend to just tell you whether your card is even or odd, you will still be able answer my question, and this only takes 1 bit of communication.

What if I had asked a different question? It turns out there are some questions that you could only answer by saying a lot to each other: that is, questions with high "multiparty communication complexity".

Even though this started as just a game, the fact that such questions exist has a number of unexpected consequences. One consequence is that a computer (or technically, a "Turing Machine"—we'll define what this is) with k different heads on its hard drive is more powerful than one with $k-1$ heads. Another consequence is that there is a way of generating "pseudorandom" numbers that computers can't tell apart from truly random numbers.
Homework: Optional
Prerequisites: None
Related to (but not required for): Simple Models of Computation, Communication Complexity, Markov Algorithms, Rewrite Systems

Ultrafilters and Hindman's Theorem. (, Steve, Tues–Sat)
Combinatorics is full of results saying that functions on an infinite set are well-behaved infinitely much of the time. An easy example of this is the Pigeonhole Principle: given a function $f : \mathbb{N} \to X$, for a finite set X, no matter how crazy f is there is always some infinite set $S \subseteq \mathbb{N}$ on which f is constant – that is, some hole winds up with infinitely many pigeons. A slightly trickier instance of this is Infinite Ramsey's Theorem for pairs: if f is a function which assigns 0 or 1 to each pair of distinct natural numbers, there is some infinite set H such that any pair from H gets assigned the same color as any other pair. (If you haven't seen Ramsey's Theorem, don't worry – we'll prove it in class.)

However, what if "infinite" just isn't big enough? What if we want our function to be nicely behaved on a **really** big set? For example, for a function $f : \mathbb{N} \to X$ with X finite, maybe we want f to be constant on a set which is not only infinite, but closed under finite sums. Can we always find such a set – and if so, what's the most ridiculous way we can prove it?

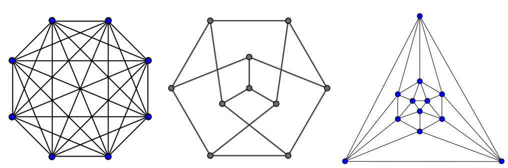
In this class we'll do combinatorics using ultrafilters — bizarre, beautiful objects from the mysterious land of set theory! Ultrafilters cannot even be proved to exist without the axiom of choice, but that won't stop us from using them to build big homogeneous sets. Oh, and we'll also need to say the words "compact space," "topological semigroup," and "idempotent" a few times.
Homework: Recommended
Prerequisites: This class is completely self-contained, but assumes experience with mathematical rigor and abstract definitions.
Related to (but not required for): How Not to Prove the Continuum Hypothesis, Set Theory, Hyperreals

1:10 PM Classes

Geometry is Awesome, Dude! (, Asi & Zach, Tues–Sat)
If someone asked you for the center of a triangle, what would you tell them? Perhaps the point where a cardboard cutout could balance on the tip of a pen? Or maybe the point equidistant from the vertices? Perhaps, even, the place where the three altitudes meet? These are all reasonable answers to the question, but in general, these are very different locations that seem entirely unrelated. So why are these three points always collinear?!

This is just one of the many beautiful and counterintuitive "coincidences" that pervade Euclidean geometry. Beginning with the "Euler line" described above, this course will explore many amazing and fundamental properties of triangles and circles that you didn't know you didn't know.
Homework: Recommended
Prerequisites: None

Introduction to Graph Theory. (, Marisa, Tues–Sat)
Graphs are a tool we use to model relationships, and are beautiful objects all by themselves. In this introductory class, we'll meet some combinatorial problems that are solved with graphs, and we'll explore some properties of graph structure: forest, trees, caterpillars; walks, trails, and paths; cliques; eccentricity; and some other things that have less funny names. And I'll tell you why these are three of my favorite graphs:



Homework: Recommended
Prerequisites: None
Required for: You don't have to have taken this specific class, but the concepts from this class will be required for Traveling Salesman Problem, Finding the Perfect Match, Flows, and Graph Minors

Multivariable Calculus Crash Course. (, Mark, Tues–Sat)
In real life, interesting quantities usually depend on several variables (such as the coordinates of a point, the time, the temperature, the number of campers in the room, the real and imaginary parts of a complex number, ...). Because of this, "ordinary" (single-variable) calculus often isn't enough to solve practical problems. In this class, we'll quickly go through the basics of calculus for functions of several variables. As time permits, we'll look at some cool applications, such as:

- If you're in the desert and you want to cool off as quickly as possible, how do you decide what direction to go in?
- What is the total area under a bell curve?
- What force fields are consistent with conservation of energy?

Homework: Recommended
Prerequisites: Single-variable calculus (differentiation and integration)
Required for: Functions of a Complex Variable

Spacetime Supersymmetries. (🌶🌶🌶, *Jim Gates*, Tues–Sat)
This class will begin with an introduction to the concept of symmetry groups in the context of the rotation group in three dimensions. This is followed by an introduction to spacetime supersymmetry (SUSY) in the context of non-relativistic quantum mechanics. Following this, supernumbers, superfunctions, etc. are discussed. After this, SUSY in four dimensional simple models is presented and a set of fundamental problems is identified. A new set of tools called 'Adinkras' will be discussed and open problems will be presented as a possible research direction.

Homework: Optional

Prerequisites: Linear Algebra is a corequisite

The Category of Sets. (🌶🌶🌶, Waffle, Tues–Sat)
Category theory is sometimes thought of as a horribly abstract and general part of math that you can't wrap your mind around without knowing about homological algebra and algebraic topology and other scary-sounding things. Rest assured, we will meet no such monsters in this class! Instead, we will study some basic concepts in category theory in a concrete and familiar setting, namely that of sets and functions between sets. If we use numbers to count how many elements a set has, this setting becomes even more familiar: it's just ordinary arithmetic.

However, the way in which we will do arithmetic will be far from ordinary! Instead of assuming we already know what "numbers" are and how to add and multiply them, we will be able to *define* these notions using nothing but the concept of a function, and we will also prove the ordinary rules of arithmetic using only functions. The advantage of this "categorical" approach is that almost everything we prove will apply not only to ordinary sets and arithmetic, but also to any other kind of mathematical object you might someday encounter. Plus, it's really cool.

Homework: Required

Prerequisites: None

Required for: Category Theory

Related to (but not required for): Programming in Haskell

Colloquia

Permutation Groups, Coxeter Algebras & SUSY. (Tuesday, *Jim Gates*)
This presentation relates the property of supersymmetry to the mathematical structures of Permutation Groups, and Coxeter Algebas.

On the Controversy Between Messrs. Leibniz and Bernoulli Concerning Logarithms of Negative and Imaginary Numbers. (Wednesday, *Jon Tannenhauser*)
In the early 18th century, a debate raged among the pioneers of calculus as to how to extend the logarithm function to negative and imaginary numbers. Johann Bernoulli said that the logarithm of a negative number should equal the logarithm of its absolute value. Gottfried Wilhelm von Leibniz disagreed, maintaining that the logarithm of a negative number cannot be any real number. After reviewing the arguments and counterarguments on both sides, we'll see how the resolution, given by Euler in 1749, turned out to be more interesting than either antagonist suspected. Along the way, we'll touch on highlights in the history of calculus and complex variables.

Vanning Trees. (Thursday, Zach)
As a criminal mathtermind on the run from the law (of large numbers), you have constructed a complicated network of hideouts to navigate via underground tunnels. To throw the authorities off

your (not-necessarily acyclic) path, you sporadically move from each hideout to one of its randomly chosen neighbors in your trusty van. As you do this, every time you visit a new hideout for the first (I mean zeroth) time, you mark the corridor you just traversed as that hideout's "escape route" to use in case of a surprise raid. Once you have visited all hideouts, your choices of escape routes form a smaller network for quickly driving to freedom—an emergency vanning tree spanning your hideouts. What is this tree likely to look like? Which trees are more likely to arise? Do answers to these questions in any way facilitate your daring escape? This last question is irrelevant to my talk but is still fun to think about.

Bear Catch-a-Berry Theorem. (Friday, Asi)
A hungry bear lives on a flat plane. Since all of you always sign in on time, the bear can't eat you, and so it must resort to eating berries. The bear plays a game with a berry, who tries to avoid getting eaten. One by one, the bear selects non-empty subsets of the plane. Each time, the berry must move to a point that lies in *all* the previously picked subsets. However, the berry must always stay inside the berry field, which is just another non-empty subset of the plane.

Can the bear always catch the berry? You can check that without any constraints on the game, the bear can easily catch the berry. But that's too easy for the bear. Can we find some constraints to impose on the game to make it more interesting? Turns out that if we force the bear to choose *open dense* subsets of a *metric space*, the game is not so easy anymore. In this colloquium we'll find out what these words mean, and also whether we can set up things so that the berry never gets caught.

Visitor Bios

Jim Gates uses mathematical models involving supersymmetry, supergravity, and superstring theory to explore nature. One of his current focus areas includes Adinkras, a new mathematical concept, linking computer codes like those in browsers to the equations of fundamental physics as if our physical reality resides in the science fiction movie, "The Matrix."

Jonathan Tannenhauser teaches math at Wellesley College. In the past he has worked in theoretical physics, finance, and the genomics of birdsong, and more recently he has become interested in the history of mathematics. This is his fifth visit to Mathcamp.