CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2013

Contents

9:10 AM Classes

Extreme Probability! (week 2 of 2). (🌶🌶🌶→ 🌶🌶, Aaron, Tue–Sat)
In the first week of this course, we built up the basic elements of probability theory, starting from scratch. Now, it's time for an adventure! Armed only with the tools we built in Week 1 (or brought from home), we'll set out into the wilds of extreme probability, applying deep ideas like "entropy" and "percolation" to problems in information theory, physics, and biology. If you want to learn about data compression, temperature, magnetization, species extinction, or polymers, you should come along for the ride.

Week 1 is sufficient background for this part of the course, but not necessary. Anyone with probability experience is welcome to join! If you want to jump in, talk to me about your background so I can help you get up to speed if necessary. Don't be scared off by the chili rating—the first day or two, when we prove our main result about entropy, may drift into the three-chili range, but you should be able to take the rest of the course at either two or three chilis, depending on your background and motivation.
Homework: Recommended
Prerequisites: Week 1, or equivalent. There will be extra fun for students who know about derivatives.

Linear Algebra and Group Theory to the Hydrogen Atom. (🌶🌶🌶, David Jordan, Tue–Sat)
In this class, we'll survey some of the beautiful mathematical structures underlying the orbit of an electron about a proton, which together comprise the hydrogen atom.

We'll begin by discussing rigid symmetries of three dimensional space. We'll exploit a crucial such symmetry in the electric potential, translate this into the language of group theory and linear algebra, and... voila! We'll discover the s, p, d, ... shells from chemistry class, and we'll be able to predict the discrete spectrum of radiation from atomic particles—one of the great discoveries of 19th century physics. The best part is that we need almost no input from chemistry or physics—this is really a story about the mathematics of symmetry!

There will be pictures—theoretical and experimental—of actual electron orbitals, and they will be beautiful.

This class can be considered a pre-introduction to representation theory: you won't need to know what representation theory is before the class, but my hope is that you'll be itching to know what it is by the end.

Homework: Recommended
Prerequisites: Familiarity with linear algebra is a prerequisite. Familiarity with some group theory is helpful, but not necessary. Familiarity with physics or chemistry is beside the point ☺. Several copies of the first chapter from Stephanie Frank Singer's book *Linearity, Symmetry, and Prediction in the Hydrogen Atom* have been printed and left in the Mathcamp library. It is requested that you peruse (not in detail) the first 13 pages before coming to the first class. Don't worry if there are things you don't understand—that is expected!—read it to get an idea of the bigger questions and get excited about the class.
Related to: Matrix Groups.

Ring Theory. (🌶🌶🌶, Mark, Tue–Sat)
Rings come up throughout mathematics almost as often as groups; ring theory and group theory are both parts of "abstract algebra", and the two have a somewhat similar flavor. Actually, a ring is a special kind of Abelian group, but with a second "multiplication" operation that interacts well with addition. Examples include the ring of integers and its generalizations in algebraic number theory, rings of polynomials, rings of continuous functions, rings linked to curves, surfaces, and higher-dimensional objects in algebraic geometry, rings of matrices, ... In this class we'll cover some of the fundamental notions of ring theory. This includes the idea of an "ideal" (originally short for "ideal number"), a concept first introduced in the context of trying to rescue a failed proof of Fermat's Last Theorem, in order to get unique factorization to work in a number system where it didn't. (It later turned out that the concept of an ideal was far more fundamental than the somewhat specialized problem that led to it.) If all goes well, near the end of the class you'll see a very elegant and rigorous way to define the complex numbers based on the reals that removes any doubt you may have had about "the existence of imaginaries", and perhaps also how a very similar construction allows us to make a number system (field) with exactly four elements, or exactly 121 elements, in which you can add, subtract, multiply, and divide, except by zero. (Staff supervision doesn't help in this case, alas. Note: The integers modulo 4 or modulo 121 will *not* do; you can't divide by 11 in the integers modulo 121, for example.)
Homework: Recommended
Prerequisites: Knowing what an Abelian group is. More generally, some experience with group theory would be helpful.

Simple Models of Computation. (🌶🌶, Pesto, Tue–Sat)
Almost all programming languages are equally powerful—anything one of them can do, they all can. We'll talk about two *less* powerful models of computation—ones that can't even, say, tell whether two numbers are equal. They'll nevertheless save the day if you have to search through 200MB of emails looking for something formatted like an address.[1]

This is a math class, not a programming one—we'll talk about clever proofs for what those models of computation can and can't do.
Homework: Recommended
Prerequisites: None.

[1] xkcd.com/208

Surfaces and Symmetries (week 1 of 2). (🌶🌶, Susan, Tue–Sat)
There are thousands of beautiful symmetry patterns in the world, but only seventeen of them can tile the plane. What a strange number—where does it come from? In this class, we will discover a topological proof of this strange and beautiful theorem. We will be drawing, cutting, pasting, folding, and smershing our way to an understanding of how symmetries work, and how we can magically calculate exactly how many there are!

Homework: Required

Prerequisites: None.

10:10 AM Classes

Epsilon the Enemy. (🌶🌶, Ruthi, Tue–Sat)
Do you stay up late wondering why the real numbers are special? Do you love calculus but worry that saying "closer and closer" doesn't really explain how close you mean? In this class we will learn how to make calculus less wishy-washy. We'll talk about what exactly the real numbers are, the rigorous definition of a limit and (if we have time) derivatives and integrals.

Homework: Recommended

Prerequisites: calculus

Related to: Probability, any kind of Calculus, Harmonic Analysis, Metric Spaces.

Matrix Groups. (🌶🌶🌶, Asi, Tue–Sat)
Mathematicians are like alchemists: both kinds of people like to combine random objects and hope that some magic happens. This is exactly what we'll do in this class. In linear algebra, we multiply matrices. In group theory, we multiply group elements. Why not combine the two? Poof! We get matrix groups.

Matrix groups form a rich class of examples that includes important objects like GL_n, O_n, U_n and many more (these terms will be defined). We will explore:

- algebraic stuff like conjugacy classes and orbits,
- topological stuff like connected components and compactness,
- geometric stuff like the distance between two matrices, and why the rotation group in three dimensions is itself almost a sphere but not quite,
- and analytic stuff like exponentials.

If you like to see magic happen when you combine several different areas of mathematics, put them in a cauldron, and boil and stir, then this class is for you.

Homework: Recommended

Prerequisites: Linear algebra, Group theory, Metric spaces helpful but not required.

Related to: From Linear Algebra and Group Theory to the Hydrogen Atom.

Proofs and Problem Solving. (🌶🌶, Mira, Tue–Sat)
The goal of this class is to help improve your ability to write rigorous, elegant, and easily comprehensible proofs. Although the class has "problem solving" in its title, it is not just for people interested in math contests: after all, much of mathematics consists of solving problems and then proving that your solutions are correct.

We won't be covering much theory in this class. Most of the time you'll be solving problems (similar to the ones on the MC Qualifying Quiz), writing proofs, and reading and discussing other people's proofs. The problems should be fun but not too difficult (it's a 🌶🌶 class), and the emphasis will be not on how to figure out the answer but on how to turn your intuitive solution into a rigorous argument. (Although, you'll often find that thinking about the proof will help you uncover and fix a flaw in your original solution.) Each student in the class will be assigned a "proof advisor" (mentor or faculty) who will give you detailed individual feedback on your written work.

So should you take this class? Ask yourself these questions:

- Once you've more or less figured out how to solve a problem, how confident are you that you can write down your solution as a rigorous proof?
- When you are reading a proof (your own or someone else's), how confident are you that you can judge whether the proof is rigorous or not?
- Do you use the various proof techniques as a way to *help* you solve problems, rather than as a scary hurdle to overcome?

If you've answered "yes" to all, then you do not need to take this course. But bear in mind: every year, there are students who post on AoPS forums saying "I think I solved all the problems on the Mathcamp Qualifying Quiz", when in fact they weren't even close. The ability to evaluate one's own proofs for rigor is one of the skills that I hope you can improve by coming to this class.

Homework: Required

Prerequisites: None

Related to: Everything.

Set Theory. (🌶🌶🌶, Waffle, Tue–Sat)
Do you like counting? Well, I don't blame you if you don't—the way it's usually done is pretty boring, and I'm not much good at it anyways. Counting is traditionally done with the natural numbers:

$$0, 1, 2, 3, \ldots$$

These are fine for counting mundane boring finite things, but I like counting more exciting *infinite* things, and for these, sometimes the natural numbers just aren't good enough. You can use the natural numbers to count the integers, and if you're pretty clever you can even use them to count the rational numbers, but a famous theorem of Cantor says that no matter how hard you try, you can't use them to count how many real numbers there are. One implication of this theorem is that there are different sizes of infinity, and the reals are a strictly larger infinity than the natural numbers.

In this class, we'll learn about a generalized kind of number, called *ordinal numbers*, that can be used to count *anything*, even the reals or sets much much much bigger than the reals. We'll then use the ordinals to understand the different sizes that infinite sets can have, and how to do arithmetic with these different infinities.

Homework: Recommended

Prerequisites: None

Symmetric Polynomials and Schubert Calculus (week 1 of 2). (🌶🌶🌶, Kevin & Nic, Tue–Sat)
Take four lines in three-dimensional space. How many lines intersect all four? The answer can sometimes depend on the way the lines are arranged—if they're all parallel, it's 0, and if they all lie in the same plane, it's infinity. But most ways to arrange the lines, it turns out that the answer is exactly 2. Similarly, for most arrangements of six planes in 4-space, there are 5 lines that intersect all of them.

Schubert calculus is about answering questions of this form, where the dimension of the space might be different and the lines are replaced by points or planes or other linear subspaces. We'll start with the seemingly unrelated theory of symmetric polynomials, which are polynomials in several variables that stay the same when you switch any two of them. We'll show that both Schubert calculus and the algebra of symmetric polynomials are intimately related to tiny pictures made out of little boxes with numbers in them, and this will give us a systematic way to answer Schubert problems by multiplying certain symmetric polynomials.

Homework: Recommended

Prerequisites: Linear Algebra

11:10 AM Classes

Chip Firing, Sandpiles, and all that Jazz! (🌶🌶½, *Dave Jensen & Sam Payne*, Tue–Sat)
The chip-firing game is played with a collection of poker chips arranged on the vertices of a graph. While simple to learn, it has deep applications to combinatorics, algebra, and geometry. In this class, we will introduce this game, study its fundamental properties, and discuss some of these applications. The class will be 2.5 chiles for the first four days. Then, after Friday's colloquium, it will get Thai-spicy-hot for the final hour, as we discuss relations to linear series on algebraic curves and the Brill-Noether Theorem.

Homework: None

Prerequisites: Basic graph theory and linear algebra up to the determinant

Many Campers Sort Piles. (🌶, The JCs, Sat)
Oh no! It appears this blurb needs sorting. Come and help us sort it.

quickly. In explore a of sorting comparing the for someone do a you in sort How algorithms, dumps do You it? your nicely Imagine of cards well runtimes it a pile this lap. variety each. ordered . . . want to class, deck big into

Homework: None

Prerequisites: None

Not Just Scratching the Surface: Cohomology and Topology of Surfaces. (🌶🌶🌶, *Scott Taylor*, Tue–Sat)
Cutting & gluing, twisting, crushing are all ways to deform surfaces. Can these deformations turn a Frisbee into a washer or a bicycle tire into a ball? Can you deform a disc so that no point ends up exactly where it began? "Cohomology" is a powerful set of algebraic tools for answering these (and many more!) questions. We'll take a combinatorial approach to the subject of cohomology: to each triangulated surface we'll associate certain vector spaces constructed from objects called "combinatorial vector fields" and "combinatorial scalar fields". Inspired by physics and multivariable calculus we'll look at combinatorial versions of derivatives and integrals and use them to construct yet more vector spaces (the "cohomology groups"). The relationships between those vector spaces are intimately linked to the topology of the surface. So grab a (chalk)board and let's surf(ace)!

In addition to the regular class time, this class will meet for the first hour of TAU in the same classroom.

Homework: Required

Prerequisites: Linear Algebra. Some calculus might be helpful, but is not strictly necessary.

Permutations and Bruhat Orders. (🌶🌶🌶, *Ben Elias*, Thu–Fri)
A permutation of the numbers $\{1, 2, 3, \ldots, n\}$ is an ordering on those numbers; there are $n!$ such orders. My favorite way to encode a permutation is with a "crossing diagram," which looks like a picture of braided hair. By keeping track of which strands cross each other, one can put a partial order on the set of permutations, called the Bruhat order. This is a very important, well-known topic in algebra.

However, there is more than one crossing diagram one can draw for a given permutation. By keeping track of "crossings of crossings," one can put a partial order on the set of minimal crossing diagrams. But this partial order looks just like the Bruhat order? How very meta. Now we look at crossings of crossings of crossings... and we discover the higher Bruhat orders. These lovely things are not as well-known as they should be!
Homework: Recommended
Prerequisites: Read about partially ordered sets and Hasse diagrams on Wikipedia.
Related to: Partially Ordered Sets

Reverse Mathematics. (🌶🌶, Matt & Steve, Tue–Sat)
Most mathematicians take a bunch of axioms and ask, "What theorems can we prove from these?" But we could always go the other way: start with some theorems, and try to figure out what axioms we need to prove them.

In this class, we'll introduce you to this "Reverse Mathematics." We'll look at various nice theorems, and talk about how difficult they are to prove in a precise way; sometimes, two different theorems will turn out to be secretly the same! At the end of this class, we'll address the crazy question: how one can show that one theorem is *strictly* more difficult than another? And what does that even mean?

(This class will use the word "computable" sometimes, but no background in logic or computer science will be needed.)
Homework: Optional
Prerequisites: Some group theory or ring theory.

Using Linear Algebra. (🌶🌶🌶, *Po-Shen Loh*, Tue–Wed)
This will be an "applied" math class. We will apply the theory of Linear Algebra to some interesting problems which, at first glance, do not involve matrices at all. Disclaimer: our main areas of application will be in pure mathematics, primarily combinatorics.
Homework: Optional
Prerequisites: None

1:10 PM Classes

Cut That Out! (🌶🌶, Zach, Tue–Sat)
I hand you a square of paper and a pair of scissors, and your goal is to cut the square into some number of "puzzle pieces" that can be rearranged precisely into an equilateral triangle. Can you do it? Perhaps surprisingly this *is* possible, and even more surprisingly, you can do it with just 4 pieces. Try it! More generally, is it always possible to dissect any polygon into any other (assuming they have the same area)? What about a square and a circle? What about a cube and a tetrahedron? What if we allow infinitely many cuts, or fractal cuts, or. . . ? In addition to discussing all of these questions and more, this class will feature a plethora of puzzles and a panoply of pretty pictures.
Homework: Optional
Prerequisites: None.

Finding the Perfect Match. (🌶🌶, Alex, Tue–Sat)
(1) Suppose we have n campers and n different Mathcamp t-shirts, and each camper only likes some of the shirts. Under what conditions can we assign shirts so that each camper likes his or her shirt?
(2) There's a class project where students must work in pairs. How can the students form teams so that as many people as possible can work with their friends?

(3) Maybe this class has a billion students. How can a computer solve the above question quickly?
(4) How can a bunch of people get married so that no one has an incentive to run away with someone else's spouse?

We will answer these questions and more in our class on matchings, an area of graph theory with applications in computer science, economics and many other fields.

Homework: Optional

Prerequisites: Some previous exposure to graph theory would be helpful. If you haven't seen the definition of a graph and would like to take this class, come and speak to me!

Related to: Flows

Functions of a Complex Variable. (🌶🌶🌶, Mark, Tue–Sat)
Spectacular things happen in calculus when you allow the variable (now to be called $z = x + iy$ instead of x) to take on complex values. For example, functions that are "differentiable" in a region of the complex plane now automatically have power series expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both in- and outside math. For example, complex analysis was used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if a and b are positive integers with $gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, \ldots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet's theorem is certainly beyond the scope of this class and heat conduction probably is too, but we should see a proof of the so-called "Fundamental Theorem of Algebra", which states that any nonconstant polynomial (with real or even complex coefficients) has a root in the complex numbers. We should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and venturing boldly forth into the complex plane! This class runs for two weeks, but it should definitely be worth it. (If you can take only the first week, you'll still get to see one or two of the things mentioned above; check with me for details.)

Homework: Recommended

Prerequisites: Multivariable Calculus (the week 1 crash course will do).

The Probabilistic Method. (🌶🌶, Tim!, Tue–Sat)
Prove that there exists a graph with 1,000,000 vertices such that every set of 40 vertices has a pair of adjacent vertices and a pair of nonadjacent vertices.

Here's one strategy: Consider all graphs with 1,000,000 vertices. Calculate the probability that a graph chosen at random will have this property. If there were no graphs with this property, then that probability would be zero. So, conversely, if you show that a randomly chosen graph has this property with some nonzero probability, then in particular, a graph with this property must exist! One might be worried that such a probabilistic proof has a chance of not working; but even though the proof uses probability, the final result is true with absolute certainty.

In this class, we'll explore this strategy, as well as other probability-based approaches to solving problems (whose statements often don't reference probability at all!). Part of the class time will consist of campers working on problems in groups and presenting solutions.

Homework: Recommended

Prerequisites: None

Related to: Extreme Probability, Graph Theory

Superclasses

Partizan Game Theory. (🌶🌶, Alfonso, Tue–Sat)
So you know how to win at nim and you know how to transform any impartial combinatorial game into nim, my young grasshopper, but you want more. What do you do when your game is not impartial? How do you analyze Hackenbush, chess, go, domineering, or amazons?

There are techniques to analyze these games, but they are a bit more complicated and less clean than those for impartial games. You are not afraid of getting dirty, are you? That is what you will learn this week.

Homework: Recommended

Prerequisites: You need to have taken my Week 1-class "Combinatorial Game Theory" this Summer, or the equivalent class in a previous Summer, or be familiar with Nim, nim-values and the MEX rule.

Programming in Haskell. (🌶🌶🌶, Nic, Tue–Sat)
Functional programming languages are ones that encourage the programmer to focus not on which instructions a program is supposed to follow but on what the output is supposed to be. When writing in a functional style, variables can't be modified, data structures can't change their shape, and functions can't do anything with their input but use it to compute and return an output. This might sound like coding with both hands tied behind your back, but once you get used to it, the functional paradigm can be incredibly powerful. For example, the factorial function can be defined in a functional language just by saying:

```
factorial 0 = 1
factorial n = n * factorial (n - 1)
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Notice that there's no loop—you just tell Haskell what the value of the function is and it's in charge of figuring out which computations to perform in what order.

This class meets three hours a day, and most class time will be spent writing code to solve problems that will be provided. You'll be encouraged to solve problems at your own pace; there should be more than enough to keep you occupied for the whole week.

Homework: Recommended

Prerequisites: You should definitely have some experience programming, but it doesn't have to be in a functional language.

Related to: Category Theory

Colloquia

A Tale of Three Knot Invariants. (*Scott Taylor*, Tue)
Bridge number $b(K)$ and *Seifert genus* $g(K)$ are two classical knot invariants (numbers that do not depend on the position of a knot K in space, only on the type of knot). In general there is no relationship between them, however when we take into account a third invariant, *Bridge distance*, we find a startling relationship: if the bridge distance of a knot is at least 3, then $b(K) \leq 4g(K) + 2$. In this talk, I'll define all three invariants and sketch the proof of the theorem. As time allows I will also discuss the relationship of this theorem to some major open problems in knot theory. This is joint work with Ryan Blair, Marion Campisi, Jesse Johnson, and Maggy Tomova.

Counting. (*Po-Shen Loh*, Wed)
How hard can it be to count? The answer to that question may depend on how much stamina you have. It might, for example, take an awfully long time to count how many 125-element subsets there

are of $\{1, 2, \ldots, 250\}$ by simply listing all of them. There are faster ways to calculate this number, but even then, at first glance it may appear to require a substantial amount of computation in order to determine even the last digit of $\binom{250}{125}$. In this talk, we will find this last digit, and explore some related topics. (Hint: the last digit is not zero—that would be too easy!)

TBA. (*David Jordan*, Thu)
TBA.

Chip-firing and Riemann-Roch. (*Dave Jensen and Sam Payne*, Fri)
We will discuss a simple combinatorial game played with poker chips on graphs, with one simple move for each vertex, called chip-firing. Although the game is easy, it has deep hidden structures related to the geometry of algebraic curves. In this colloquium, we will introduce you to the basics of the chip-firing game and sketch some of the big ideas relating this game to the Riemann-Roch Theorem for algebraic curves.

Visitor Bios

Scott Taylor (Colby College) studies topological knot invariants (like bridge number and tunnel number) using modern versions of classical techniques (like thin position and sutured manifold theory). He's always up for combinatorial arguments that give topological insight. He also loves exploring relationships between mathematics, literature, and art.

Po-Shen Loh (Carnegie Mellon) studies questions that lie at the intersection of two branches of mathematics: combinatorics (the study of of discrete systems) and probability theory. Randomness can manifest itself in the construction of a combinatorial system, as in the case of a so-called "random graph", but may also be artificially introduced as a proof technique to solve problems about purely deterministic systems, as was pioneered by Paul Erdös in what is now known as the Probabilistic Method.

Sam Payne (Yale) studies algebraic varieties with techniques from nonarchimedean geometry. He has nothing against archimedean geometry (not really, anyway), and holds a deep and enduring admiration for Archimedes, who single-handedly defended the city of Syracuse against Marcellus and the forces of the Roman Empire. For a while. That ended badly, but the point is that Archimedes discovered the laws of hydrostatics and the principle of the lever, invented devices like The Claw and The Screw and The Spiral and The Method. The last of which is more a mathematical technique than a device—it's basically the Fundamental Theorem of Calculus, but 2000 years before Newton and Leibniz. And he used it to compute the volume of the sphere(!). And then he wrapped it in bacon and shot it into space. What this is all leading up to is that nonarchimedean geometry, although not discovered by Archimedes, really owes its (co)identity to this great man, and if he were here at Mathcamp, he would get along with Sam just fine, and they would take a little while to talk about what's been happening in math for the last 2300 years, and then they would prove something awesome about algebraic curves and surfaces using p-adic analysis.

David Jordan (UT Austin) is interested in representation theory, which is the study of how to make abstract symmetries (of space, time, shapes, varieties) into concrete systems of matrices. He is especially interested in interactions with physics.

His one-week course will be something a primer to thinking about the interactions between pure mathematics and theoretical physics.

When he's not doing math, he likes to build stained glass windows, shoot a bow and arrow, play the mandolin, and go rock climbing.

Dave Jensen (SUNY Stonybrook) is interested in using algebraic and combinatorial techniques to study geometric problems. He enjoys ultimate frisbee and has been known to dress as a math textbook on Halloween.

Ben Elias (MIT) works on representation theory and categorification. Categorification is the art of finding the deeper, richer structure which lies behind well-studied phenomena. This structure has traditionally been studied using geometry, but Ben's style is to find new, easier, and more explicit tools to work with categorifications. In the meantime, Ben is happy to school you in any manner of board game or card game, or to outdance you in DDR.