CLASS PROPOSALS—WEEK 5, MATHCAMP 2013

Contents

Coming Events!

On **Tuesday**, we will feature **Future of You**. Come to learn about the future! In this colloquium we'll tell you about the differences between liberal arts colleges and state universities, about study abroad and options for taking time off before college, about what life in college is like and what factors to consider when making a choice of school. Everyone is welcome, whether you've already decided on a school or haven't even begun thinking about it.

On **Wednesday**, we will feature **Future of Mathcamp**. This is an event for all of us to come together as a community and discuss our favorite moments at Mathcamp, as well as suggestions for future years. Be part of a creative process that will build a better camp!

On **Thursday** Mathcamp visitor, alum, and veteran Dan Zaharopol will give a colloquium followed by a discussion:

A New Mathematical Pathway: How Underserved Students Can Enter Mathematics. (*Daniel Zaharopol*, Thursday)
Why is it so hard for low-income and minority students to become scientists and mathematicians? Have you every wondered what it must be like to be a smart kid interested in math who just can't seem to penetrate into the ecosystem of opportunities that we all know and are a part of? What can we do, as a community of people who love math, to help these students succeed?
For the past three years, I've been running the Summer Program in Mathematical Problem Solving for underserved New York City middle school students with talent in math. In the process, I've learned a lot about these students—as well as about myself, and how we all learn math (and learn to love math). Come hear about how mathematics is done at a very different summer program, and how we might some day make programs like Mathcamp accessible to everyone.

Finally, on **Friday** will be the **Project Fair**, where you can share your work and see what campers have been working on! From knitted orientable surfaces to computer programs, from combinatorial games to machine learning, from musical mathematical babbling to poset reasearch—Project Fair will be a festival of all the math that you've seen and wished that you had been able to see. This will take place in classrooms on the ground floor of Diamond. Come see campers' demos and posters—and be sure to ask questions!

(Unless otherwise stated, Week 5 classes have no homework.)

9:10 Classes

5 Ways of Looking at the FTA. (🌶🌶🌶🌶, Alfonso, Ari, Matt, Nic, and Zach, Wed)
The Fundamental Theorem of Algebra (FTA) says that every polynomial (with coefficients in $\mathbb{C}$) has a complex root. To see why this result is so fundamental, this course will explore the FTA's connections to many and distant branches of mathematics. In particular, we will offer five different proofs of the FTA, using methods from Complex Analysis, Algebraic Topology, Combinatorics, Point-Set Topology, and Algebra (surprise!). With a total of 5 proofs, 5 instructors, and 50 minutes, this whirlwind tour promises to be Fabulously Tantalizing Awesomeness.

Homework: Recommended

Prerequisites: Some subset of Complex Analysis, Algebraic Topology, Combinatorics, Point-Set Topology, and Algebra

50 Definitions in 50 Minutes. (🌶, Aaron, Adam, Alex, Kevin, Matt, Pesto, Ruthi, Zach, Fri)
Who needs theorems? This will be a class entirely driven by definitions. Several of us will deliver rapid-fire explanations of some of our favorite definitions in mathematics. Our definitions of "favorite" may vary.

Prerequisites: None

Applications of Transfinite Induction. (🌶🌶🌶, Waffle, Thu–Fri)
In this class we'll look at some applications of transfinite induction outside of set theory, some well-known and less well-known. On the first day, we'll see how transfinite induction can be used to prove some basic facts in calculus, such as the Intermediate Value Theorem or the fact that a function whose derivative is 0 everywhere must be constant. On the second day, we'll look at applications in algebra, proving that any vector space has a basis. If we have time, we may prove Zorn's Lemma, and explain how it is just a packaging up of a common pattern of transfinite induction arguments.

Homework: Optional

Prerequisites: Set theory, or a willingness to accept transfinite induction on faith. The first day will use a small amount of calculus/real analysis in the applications, and the second day will use some linear and abstract algebra in the applications.

BIG Numbers! (🌶, Julian, Thu)
We'll be learning the original proof of van der Waerden's theorem: if we colour the natural numbers with c colours, there are monochromatic arithmetic progressions of arbitrary (finite) length. (A recent strengthening of this theorem was the remarkable result of Ben Green and Terry Tao that the primes contain arbitrarily long arithmetic progressions.)

BIG WARNING: This class will involve big numbers. I mean, really huge numbers. Astonishingly colossal numbers. In fact, numbers so large that this class comes with a health warning: Do not attend this class if you are of a nervous disposition or are scared of large numbers!

Prerequisites: No fear of big numbers

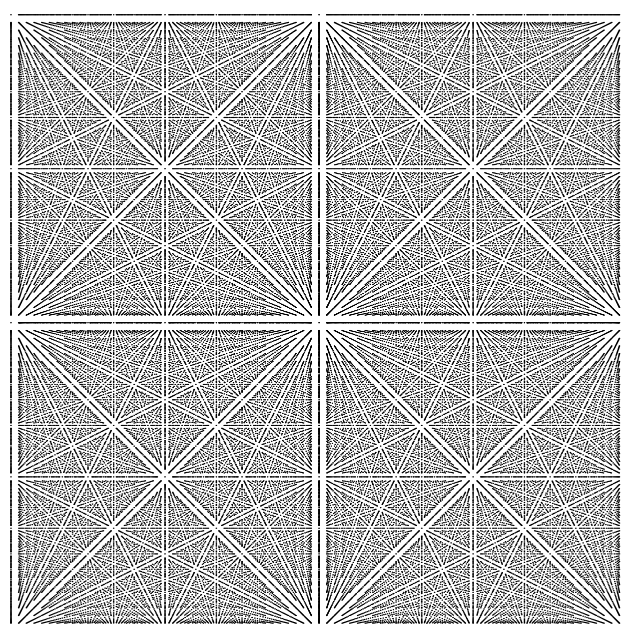
Decomposing Polynomials. (🌶🌶, Kevin, Wed)
Say we have two polynomials $f(x)$ and $g(x)$. Then you certainly know how to find their composition $h(x) = f(g(x))$.

Say we have a polynomial $h(x)$. Can we find a way to decompose $h(x)$ as $f(g(x))$?

Well, we have to define our problem a little better—we could always take $g(x) = x$ and $f(x) = h(x)$. But with the right setup, we'll see that whenever this decomposition exists, it's unique. And I'll teach you how to find the decomposition in your head, with some practice and a little mental calculation.
Prerequisites: None

Euclid. (🌶, Mark, Thu)
Euclid is known for the geometry in the *Elements*, but somehow his name comes up in number theory as well. What does his work actually look like, and to what extent do we know what he did?
Prerequisites: Basically none, although it would be nice if you knew the "Euclidean algorithm."
Related to: Perfect numbers (from Trail Mix)

Farey Lace. (🌶🌶, Aaron, Wed)
If you take the lattice $\mathbb{Z}^2$, scale it by $\frac{1}{n}$ for each n from 1 to 40, and plot all the results on top of each other, you'll see this.



Surprised? Intrigued? Come to this class!
Prerequisites: None

More Posets! Binomial. (🌶🌶🌶, Kevin, Fri)
Think about your favorite kinds of generating functions. Ever wonder why the denominators 1 and $n!$ seem to appear in so many of them? No? Well, start wondering! Why are those denominators useful? What other kinds of creatures could we build generating functions out of? It turns out that binomial posets provide a somewhat unified framework for answering this question.

My PhD thesis in 50 minutes. (🌶🌶🌶🌶, Alfonso, Thu)
This is probably a very bad idea.

In this class, I will explain the background and main result (without proof) of my PhD thesis.

What is it about? See for yourself at `http://aif.cedram.org/item?id=AIF_2005__55_7_2257_0`
Prerequisites: Multivariable calculus, linear algebra

Primitive Roots. (🌶🌶, Mark, Wed)
Suppose you are working modulo n and you start with some integer a and multiply it by itself repeatedly. For instance, if $n = 17$ and $a = 2$ you get $2, 4, 8, 16, 15, 13, 9, 1$ and then you're back where you started. Note that on the way we haven't seen all the nonzero integers mod 17; however, if we had used $a = 3$ instead we would have gotten $3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$ and cycled through all the nonzero integers mod 17. In general we can ask when (that is, for what values of n) you can find an a such that every integer mod n that's relatively prime to n shows up as a power of a (such an a is called a *primitive root* mod n). We may not get much beyond the case that n is prime, but even in that case the analysis is interesting. In particular, we'll be able to show that a exists in that case without having any idea of how to find a, other than the flat-footed method of trying $2, 3, \ldots$ in turn until you find a primitive root.

Prerequisites: Modular arithmetic; a bit more number theory wouldn't hurt.

The Sphere in Infinite Dimensions. (🌶🌶, Daniel Zaharopol, Fri)
What does it mean to be in infinite dimensions? What does it mean to be a sphere in infinite dimensions? And how could you possibly study such a thing?

We'll use the example of spheres (and the infinite-dimensional sphere in particular) to overview several cool ideas in topology, including homology and colimits. Despite the fancy terminology, this will be a non-rigorous overview outlining some beautiful results and ways of thinking about mathematics.

Prerequisites: None.

10:10 Classes

Galois Theory. (🌶🌶🌶🌶, Alex, Tue–Fri)
Galois Theory studies how we can describe extensions of fields using group theory. Here's an example: suppose we have the real numbers $\mathbb{R}$ which live in the complex numbers $\mathbb{C}$. Then, the only automorphisms of $\mathbb{C}$ which fix $\mathbb{R}$ are the identity map and complex conjugation. Together, the identity map and complex conjugation form a group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We could use this group to understand how the complex numbers extend the real numbers. In this course, we would take this approach to understand field extensions. This framework, in turn, can be used to address why we can't trisect an angle with a ruler and compass, why the quintic is unsolvable, and many other questions.

Homework: Recommended

Prerequisites: Linear Algebra, Group Theory

How To Add. (🌶🌶, Matt & Zach, Tue–Fri)
You've probably seen the Fibonacci numbers, the nth of which is obtained by adding together the previous two. We can write this as a "recurrence relation":

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2}. \end{aligned}$$

You may also have heard that it has a surprising "closed form": we can directly find the nth Fibonacci number, without calculating any of the ones before it.

You may also know that

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2},$$

which looks suspiciously similar to the familiar fact from calculus that

$$\int_0^y x\,dx = \frac{y^2}{2}.$$

Is this a coincidence? (Spoiler alert: no.)

We'll see systematic ways of taking recurrence relations and sums, and getting closed forms from them. These will include generating functions, a hammer we can use to hit all kinds of recurrence relations, as well as "finite calculus," a system that resembles ordinary calculus but works with sums instead of integrals.

Homework: Recommended

Prerequisites: Basic calculus

Knot Theory. (🌶🌶, Ari, Tue–Fri)
"So, what kind of math do you like?"
"Knot theory."
"Yeah, me neither."

Homework: Optional

Prerequisites: None

Problem Solving: Inequalities. (🌶🌶🌶🌶, Pesto, Tue–Fri)
High-school Olympiads usually try to choose problems relying on as little prior knowledge as possible. In inequalities problems, they usually fail completely; training is necessary to solve most and sufficient to solve many of them. We'll go over the common Olympiad-style inequalities, and solve problems like the following:

(1) Prove that if a, b, and c are positive and $ab+bc+cd+da=1$, then $\frac{a^3}{b+c+d}+\frac{b^3}{a+c+d}+\frac{c^3}{a+b+d}+\frac{d^3}{a+b+c} \geq \frac{1}{3}$.
(2) [USAMO 2004] Prove that if a, b, and c are positive, then $(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3$

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to Olympiad-style problems you'll've solved as homework the previous day.

If you've taken an inequalities problem-solving class from me and want to take this one, poke me and I may be able to make this one disjoint.

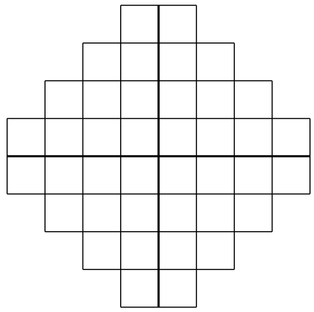
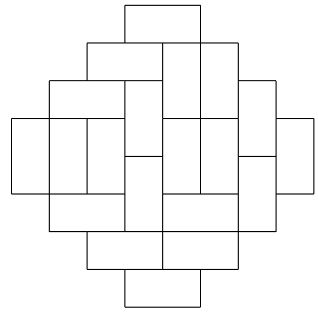
Homework: Required

Prerequisites: None

Related to: Other problem-solving

11:10 Classes

Aztec Diamonds. (🌶🌶, Julian, Thu–Fri)
An Aztec diamond is a diamond shape made of squares, as shown in the left hand diagram for a diamond of order 4. It can be covered in dominoes as shown in the right hand diagram.

How many ways are there of covering such a shape with dominoes? And what would a random such tiling look like on average? These seemingly hard questions have been investigated in the past twenty years and have revealed some beautiful mathematics (as well as astonishing pictures).

In this class, we will explore some of the hidden secrets of the Aztecs; there will be a few proofs and sketch proofs, but the emphasis will be on the beautiful known results.

Prerequisites: None

Burnside's Lemma. (🌶🌶🌶, Alfonso, Wed)

How many different necklaces can you build with 6 pebbles, if you have a large number of black and white pebbles? Notice that you won't be able to tell apart two necklaces that are the same up to rotation and reflection.

You probably can answer the above question by counting carefully, but what if we are building necklaces with 20 pebbles and we have pebbles of 8 different colours?

There is a nifty little result in group theory that allows us to solve this type of problem very quickly and without a calculator. Come to learn it!

Homework: Optional

Prerequisites: Basic group theory

The Classification of Finite Abelian Groups. (🌶🌶🌶, Waffle, Tue–Wed)

One of the basic questions of group theory is to classify all finite groups up to isomorphism. This question is universally considered to be impossibly hard, and one of the great achievements of 20th-century mathematics was the classification of just the finite *simple* groups (a group is simple if it has no nontrivial normal subgroups; this roughly means that it can't be broken down into smaller pieces).

On the other hand, for *abelian* groups, this question is much nicer and easier to answer. The answer is that every finite abelian group is isomorphic to an essentially unique direct product of cyclic groups. This result is extremely useful throughout mathematics, as whenever you have some finite abelian group, you automatically have a good idea of what it has to look like. In this class, we will prove this classification of finite abelian groups and (time permitting) sketch some ways in which it generalizes to other areas of abstract algebra.

Prerequisites: Group theory

The Delta "Function". (🌶🌶🌶, Mark, Tue)

In introductory books on quantum mechanics you can find "definitions" of a "function", called the delta function, which has two apparently contradictory properties: Its value is zero for all nonzero x, and yet its integral over any interval containing 0 is 1. This "function" was introduced by the theoretical physicist Dirac and it turns out to be quite useful in physics, but how can we make any mathematical sense of such a creature?

Prerequisites: Integration by parts

The Four Cube-Face Fun Fact of "The Barn". (🌶🌶🌶🌶, Yuval & Zach, Tue)
If you wish to say that some int n is the sum of many a cube-face, how many of a cube-face must you use? As a test case, look at this year, 2013: we can find that this is the sum

$$2013 = 38^2 + 22^2 + 9^2 + 2^2.$$

In fact, we can do even more good; this year is the sum of just two-plus-one of a cube-face (try it!), but if we try the same with the int 7, we find that we do need four of a cube-face. Is four good all of the time, or do some ints need even more? In this talk, we will see once and *four* all that we will not ever need five or more. "The Barn," a very good math guy of yore, did show this fun fact in the year $42^2 + 2^2 + 1^2 + 1^2$. We will show it too, but he used more of a long word than we will use.
Must Know: How to work in mod p
Has to do With: Many a Fact on Nums, Many a Fact on Nums (but with Alg!)

Expressing Yourself Regularly. (🌶, Asi, Tue)
Suppose you have written a 200-page PhD thesis. Can you convert it to pig latin with one quick command?

This is a somewhat contrived example, but in real life one often comes across patterns that one wants to search for or replace with other patterns. *Regular expressions* are a powerful way of searching for complicated patterns in blocks of text.

In this class we will see a quick glimpse of regular expressions, and get a sense of what patterns they can and cannot match. However, the goal of class is practical, not theoretical: we will really get our hands dirty by learning to use *grep*, which is a popular utility to carry out regular expression searches. At the end of the class, you will be able to tell me exactly which words in the SOWPODS scrabble dictionary are eight letters long, whose middle letters are equal but not equal to the first or the last letters, which contain exactly two vowels, one on either side of the two middle letters, contain no S or P, and contain at least one each of O, R, and T. And with a little more work you may even be able to convert your PhD thesis to pig latin.
Prerequisites: None
Related to: Simple Models of Computation

How a Mathematician Reads a Newspaper. (🌶, Julian, Wed)
Maths is full of hypotheses, theorems and logical arguments. What happens when we apply our thinking to a piece of text, say from a newspaper?

You will need an open mind, and a willingness to explore an article logically! This session will be heavily based on work by Bandler and Grinder, who developed a model for analysing text based on Chomsky's theory of transformational grammar.
Homework: Optional
Prerequisites: None

The Redfield-Polya Theorem. (🌶🌶🌶, Alfonso, Thu–Fri)
The Redfield-Polya Theorem is like Burnside's Lemma on steroids.

If you have taken a group theory class, or maybe some combinatorics class, you are Probably familiar with Burnside's Lemma (a.k.a. many other things): If G is a group acting on a set X, then the number of orbits of the action is given by

$$\frac{1}{|G|}|\operatorname{Fix}(g)| \tag{1}$$

It is a nice result, but not enough. For instance, how many different necklaces can you build with 20 beads out of very large supplies of red, green, and blue beads? With the help of (1), it will take you less than 5 minutes to calculate that the answer is 87230157 with just pen and paper. But what if I ask you to tell me how many such necklaces are there with R red beads, B blue beads, and G green beads, for *each* value of R, B, and G? If you think there is no way to avoid using brute force to count in this problem, think again! You can still answer it in less than 5 minutes, but you will need the full force of Redfield-Polya. Come receive it!

Homework: Optional

Prerequisites: You need to understand (1) and its proof, or take my class on the Burnside's Lemma.

Rescuing Divergent Series. (, Mark, Wed)
Consider the infinite series

$$1 - 1 + 1 - 1 + \dots .$$

What is its sum? Maybe

$$(1-1) + (1-1) + \cdots = 0,$$

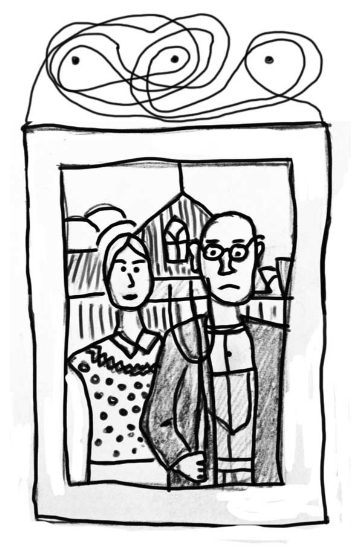
maybe

$$1 + (-1+1) + (-1+1) + \cdots = 1?$$

At one time mathematicians were quite perplexed by this, and one even thought the issue had theological significance. Now presumably it's nonsense to think that the "real" answer is $\frac{1}{2}$ just because the answers 0 and 1 seem equally good, right? After all, how could the sum of a series of integers be anything other than an integer?

Prerequisites: A bit of experience with the idea of convergence.

String Theory. (, Sachi & Tim!, Thu–Fri)
Let's say you want to hang a picture in your room, and you are worried that the 2,000 fans you rented to beat the heat will blow it off the wall, so you want to hang it very securely. You start by wrapping string around some nails, like this:



You figure it will be super-duper secure, because you wrapped it around three nails a lot of times. Yuval, angry about the electricity bills from running all your fans, comes over and pulls one of your nails out of the wall. . . and your picture comes crashing to the ground! Actually, no matter which nail he removes, it falls! What happened?

In this class, you'll be playing with string to solve puzzles like this one, and we'll explore fundamental groups, homology, and monotone boolean functions.
Prerequisites: None

What's An Elementary Particle? (🌶🌶🌶, Aaron, Thu–Fri)
For centuries, natural philosophy, chemistry, and physics have been driven by the idea that nature might be made of elementary building blocks that can't be broken down into smaller parts. With the advent of particle physics, the importance of elementary particles in science is greater than ever, but our understanding of what "elementary" means has subtly shifted. In this course, I'll show you how modern physicists think of elementary particles, how our current understanding is different from the original one, and what that means for our search for irreducible building blocks in nature.
Prerequisites: We'll be looking at a lot of different examples, and you don't have to understand all of them. You should probably have some background in at least one of the following: (1) Classical mechanics, (2) Markov chains, (3) Representation theory.

1:10 Classes

Does ESP Exist? or, What's Wrong With Statistics? (🌶🌶, Mira, Tue–Fri)
In 2011, Daryl Bem, a professor of psychology at Cornell, published an unusual paper in the *Journal of Personality and Social Psychology*, titled "Feeling the Future: Experimental Evidence for Anomalous Retroactive Influences on Cognition and Affect". In this paper, Bem presents results from 9 experiments testing for "psi", aka ESP. For instance, in the first experiment, participants are instructed as follows:

> On each trial of the experiment, pictures of two curtains will appear on the screen side by side. One of them has a picture behind it; the other has a blank wall behind it. Your task is to click on the curtain that you feel has the picture behind it. The curtain will then open, permitting you to see if you selected the correct curtain.

Bem's analysis of the data shows that participants were able to predict the location of the picture at a rate significantly higher than chance, but only if the picture involved sex or violence. For other pictures, they did no better than chance. The follow-up experiments were designed to determine whether the psi effect was due to clairvoyance, psychokinesis, or retroactive influence in which "the direction of the causal arrow has been reversed".

Note: this is not just an individual loony. This is a highly respected professor of psychology, from one of the top departments in the country, writing for one of the top journals in his field, and being, if anything, more scrupulous in laying out his methodology than is standard practice. The journal editor wrote a preface to Bem's paper, stating that although he himself does not believe in the existence of psi, neither he nor the other reviewers could find a flaw in Bem's statistical analysis. Therefore, says the editor, scientific honesty compelled him to publish the paper, so that it could be discussed in a wider forum.

If you find all of this shocking, you are not alone. The entire psychology profession cringed in mortification and blamed the editor for not exercising better judgment.

Yet the real culprit here is not the author (who is free to pursue whatever research he wants, and whom no one suspects of violating scientific ethics) nor the editor (who really was just trying to adhere to the standard practice of his field), but the statistical methodology currently employed by most scientists. A famous 2005 paper published in a medical (!) journal states the problem very starkly in its title: "Why most published research findings are false".

In this course, we'll discuss the (many) problems with how statistics is currently done (not just in psychology). We'll talk about the historical reasons for these problems and present an alternative: Bayesian statistics—a methodology that is actually based on math and dates back to Gauss and

Laplace. In particular, we'll apply a Bayesian analysis to Bem's results. (Spoiler alert: they give no evidence for the existence of ESP. Sorry to disappoint you.)

Homework: Recommended

Prerequisites: Calculus and basic probability. No background in statistics required, though if you do have such a background, you may appreciate the de-brainwashing.

Related to: Machine Learning

Greek Impossibilities. (🌶🌶🌶, Nic, Tue–Fri)
The ancient Greeks (and probably also your high school geometry teacher) were concerned with constructing geometric objects using a compass and straight-edge. They accomplished a lot with these tools, but there were three things that they just couldn't manage to do:

(1) Given a square that's the face of some cube, construct a face of a cube of twice the volume.
(2) Given an angle of measure θ, construct an angle of measure $\frac{1}{3}\theta$.
(3) Given a circle, construct a square of the same area.

It turns out there's a good reason they failed to do these things: they're all impossible. In this class, you'll learn why.

Homework: Recommended

Prerequisites: Linear Algebra, specifically the definitions of linear independence, span, and dimension.

Topology Your Friend. (🌶🌶🌶, Ruthi, Tue–Fri)
The theorems about continuous functions from calculus rely on a notion of what it means for things to be close. If you've studied more advanced calculus, you've been introduced to how this is formalized with the "epsilon-delta" approach. This can sometimes get, how shall we put it, messy? In this class we will discuss how notions of closeness can be generalized to defining a topology on a set, focusing on relating it back to calculus. In particular, we will aim to give alternative (very slick!) approaches to the proofs of the intermediate and extreme value theorems.

Homework: Recommended

Prerequisites: Calculus. Some proofs will involve epsilons, some understanding of this will help.

Related to: Surfaces and Symmetries, Epsilon the Enemy, Topology of Surfaces, Metric Spaces

Turning Paradoxes into Theorems. (🌶🌶🌶, Matt, Tue–Fri)
You may have heard about Berry's paradox: "the smallest natural number that can't be defined in fewer than thirty words" is problematic, because… well, we just defined it in fewer than thirty words. Oops.

You may also have heard about the "Surprise Examination Paradox": a teacher announces that there will be a pop quiz at the beginning of class some day next week, but that the students won't expect it. The students reason that it can't be on Friday: after all, if it's class time on Friday and it hasn't happened yet, the students will be expecting it. But it also can't happen on Thursday, or Wednesday, or Tuesday, or Monday by similar logic. Overconfident in their reasoning that the quiz can't happen at all, the students are then *very* surprised when the quiz is given on Wednesday! What went wrong?

You may also have heard of Gödel's incompleteness theorems, which among other things show that there are statements in mathematics that can't be proved true or false and that (in some sense) we can never prove mathematically that mathematics works.

What you probably *haven't* heard is that both of these paradoxes can be turned into proofs of Gödel's incompleteness theorems! This class will talk about what the incompleteness theorems are, what they imply, and how to prove them using paradoxes. As an added bonus, we'll see how we can turn things around and use the second incompleteness theorem to resolve the surprise examination paradox!

Homework: Optional

Prerequisites: None

Related to: Reverse Mathematics, Models of Computation

2:10 CLASSES

The Banach-Tarski Paradox. (🌶🌶🌶🌶, Alfonso, Tue–Wed)
You may have heard it before: we can take a sphere, divide it into five pieces, rearrange them, and get two spheres of the same size as the original one. Nifty trick, but how does it work? Come learn it!

PS: What is a good anagram of "Banach-Tarski"? "Banach-Tarski-Banach-Tarski"!

Prerequisites: Basic group theory

A Combinatorial Solution to the Subset-Sum Problem. (🌶🌶, Bill Kuszmaul, Thu)
Consider the following problem. Let $S = \{0, 1, \ldots, n-1\}$. For how many subsets $U \subseteq S$ is the sum of the elements of U congruent to i mod n? When n is prime there are several slick solutions to this problem. But when n isn't, all the easy arguments break down... or do they?

I will show you a combinatorial trick that allows us to take an argument that is only supposed to work for n prime, and use it for arbitrary n. The same idea can be used for other problems of the same flavor. In fact, we'll build a set of tools that just in the last six months or so have been used to solve several previously open problems.

This class will contain some of the prettiest results that I've obtained this year in my research under the MIT PRIMES program.

Homework: Optional

Prerequisites: None

Calculus Without Calculus. (🌶🌶, Tim!, Tue–Wed)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. But lots of those problems can actually be solved without any calculus at all. In this class, you'll learn to do common calculus problems without taking any derivatives or integrals at all!

Some example problems that we'll solve without calculus in this class:

- You have 40 meters of fence, and you want to build a rectangular pen, where three sides of the pen will be made of fence, and the last side will be a preexisting brick wall. What is the largest area that your pen can enclose?
- Sachi is $5'2''$ tall and Asi is 157.5 cm tall, and they are standing 5 cubits apart. You want to run a string from the top of Sachi's head to the top of Asi's head that touches the ground in the middle. What is the shortest length of string you can use?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 50 feet way along the shoreline and throws a stick 20 feet out into the water. The dog can run along the shoreline at 5 miles per hour, and can swim at 4 miles per hour. What is the fastest route that the dog can take to get to the stick?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Optional

Prerequisites: None

Related to: Calculus in school.

Complex Numbers in Geometry. (🌶, Milica Kolundžija, Fri)
Complex numbers are a powerful, general tool in geometry, and for some problems they yield simpler and nicer solutions. An example of such a problem: If $A_0A_1A_2A_3A_4A_5A_6$ is a regular heptagon, prove that $\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_3}$. We will use complex numbers to describe geometric transformations and relations. Then we'll prove a theorem about a large number of intersecting circles that complex numbers make it easier to deal with.
Homework: Recommended
Prerequisites: basic operations with complex numbers and familiarity with the complex plane.

Extensions. (🌶🌶🌶🌶, Asi, Thu–Fri)
Take any two abelian groups, for instance $G = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$. Then the product group $G \times H$ has the property that $G \hookrightarrow G \times H$ is an injective map, and $G \times H \to H$ is a surjective map, and the kernel of the second map is exactly the image of the first map.

Based on this structure, we define an *extension* of H by G to be a new abelian group E such that there is an injection $G \to E$ and a surjection $E \to H$, and the kernel of the second map is precisely the image of the first. Do there exist extensions other than the product extension shown above? How many are there? Are some of them isomorphic? How can we tell?

In this class we will study part of the story of extensions of abelian groups, and some beautiful underlying structure that fits them all together.
Prerequisites: Group theory, ring theory/commutative algebra will be helpful.
Related to: Commutative algebra, Category theory

Generating Functions and Complex Analysis. (🌶🌶🌶🌶, Kevin, Thu–Fri)
If you've seen generatingfunctionology before, then you should believe it's magic.

If you've seen complex analysis before, then you should believe it's magic.

We're going to mix magic with magic.

We'll be able to get closed forms for tricky generating functions by taking contour integrals. And for those where we can't find a closed form, we'll use complex analysis to get some pretty good asymptotic estimates.
Prerequisites: Complex analysis and some familiarity with generating functions.

Multiplicative Functions. (🌶🌶, Mark, Thu–Fri)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such *multiplicative* functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.
Homework: Optional
Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)
Related to: Perfect numbers (from Trail Mix)

Premodern Cryptography. (🌶, Pesto, Tue–Wed)
UI UOHH JFHQI SGXEKFNGMT ELZZHIJ HOWI KRI FDIJ KRMK XFL TONRK AODP OD M DIUJEMEIG, AOGJK CX KGOMH MDP IGGFG, KRID CX KISRDOBLIJ KRMK YLJK IVKIG-TODMKI KRIT. Then we'll look at some of the premodern code-writers' attempts at strengthened versions, and see why they're still breakable.

Homework: Optional
Prerequisites: None

The Quadratic Reciprocity Law. (🌶🌶, Mark, Tue–Wed)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

Question 1: "Is q a square modulo p?"

Question 2: "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. If all goes well you'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square mod 9973, you'll be able to answer a lot more quickly, whether or not you use technology.
Homework: Optional
Prerequisites: Some basic number theory, specifically Fermat's little theorem.