CLASS PROPOSALS—WEEK 5, MATHCAMP 2013

Why is this so long?

These are all the possible Week 5 classes. Since there are too many to all happen, we need you to vote on them.

How do I vote?

Go to `https://appsys.mathcamp.org/ac/week5/`. We're using approval voting so you can vote for as many or as few classes as you want. However: Voting for every class is equivalent to voting for none, so pick only the ones you want. And since staff will be sad without students, especially near the end of camp, we're asking you to ask yourself the following question to test whether or not you should vote for a class: "Would I get up and go to this class even at 9am on Friday?" Your votes will be used not only to decide which classes will run, but also to decide which other classes they conflict with. We'll do our best to minimize conflicts. **Voting closes on W4 Wednesday at 11:59pm.**

What about these classes proposed by campers?

They begin with infinitely many votes, so they are guaranteed to run, but you should still vote for them if you want to attend so that they will be less likely to conflict with other classes you want to take.

Contents

Classes Proposed by Campers

Bill Kuszmaul's Class

A Combinatorial Solution to the Subset-Sum Problem. (🌶🌶, Bill Kuszmaul, 1 day)
Consider the following problem. Let $S = \{0, 1, \ldots, n-1\}$. For how many subsets $U \subseteq S$ is the sum of the elements of U congruent to i mod n? When n is prime there are several slick solutions to this problem. But when n isn't, all the easy arguments break down... or do they?

I will show you a combinatorial trick that allows us to take an argument that is only supposed to work for n prime, and use it for arbitrary n. The same idea can be used for other problems of the same flavor. In fact, we'll build a set of tools that just in the last six months or so have been used to solve several previously open problems.

This class will contain some of the prettiest results that I've obtained this year in my research under the MIT PRIMES program.

Homework: Optional

Prerequisites: None

Milica Kolundžija's Class

Complex Numbers in Geometry. (🌶, Milica Kolundžija, 1 day)
Complex numbers are a powerful, general tool in geometry, and for some problems they yield simpler and nicer solutions. An example of such a problem: If $A_0A_1A_2A_3A_4A_5A_6$ is a regular heptagon, prove that $\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_3}$. We will use complex numbers to describe geometric transformations and relations. Then we'll prove a theorem about a large number of intersecting circles that complex numbers make it easier to deal with.

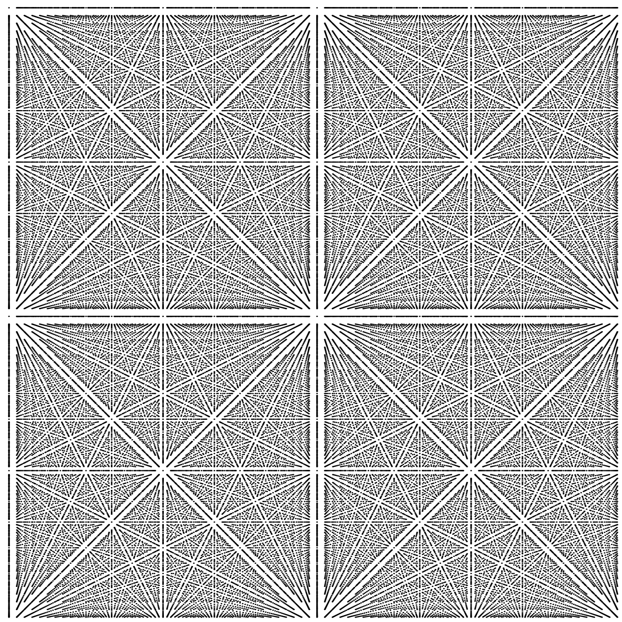
Homework: Recommended

Prerequisites: basic operations with complex numbers and familiarity with the complex plane.

Classes Proposed by One or Two Staves

Aaron's Classes

Farey Lace. (🌶🌶, Aaron, 1 day)
If you take the lattice $\mathbb{Z}^2$, scale it by $\frac{1}{n}$ for each n from 1 to 40, and plot all the results on top of each other, you'll see this.



Surprised? Intrigued? Come to this class!

Homework: None

Prerequisites: None

Holomorphic Quantization. (🌶🌶🌶🌶, Aaron, 3 to 4 days)
Have you ever wished you could make quantum versions of of all your favorite linear systems from classical mechanics—including marbles in bowls, ripples on ponds, and rays of light? Do you wield complex analysis and linear algebra like a pair of gleaming scimitars? Are you willing to learn to pronounce words like "symplectic"? If you answered "yes" to all these questions, this is just the class for you!

Homework: None

Prerequisites: Comfort with complex differentiation and integration, power series, infinite-dimensional inner product spaces, and quotients of vector spaces.

What's An Elementary Particle? (🌶🌶🌶, Aaron, 1 to 2 days)
For centuries, natural philosophy, chemistry, and physics have been driven by the idea that nature might be made of elementary building blocks that can't be broken down into smaller parts. With the advent of particle physics, the importance of elementary particles in science is greater than ever, but our understanding of what "elementary" means has subtly shifted. In this course, I'll show you how modern physicists think of elementary particles, how our current understanding is different from the original one, and what that means for our search for irreducible building blocks in nature.

Homework: None

Prerequisites: We'll be looking at a lot of different examples, and you don't have to understand all of them. You should probably have some background in at least one of the following: (1) Classical mechanics, (2) Markov chains, (3) Representation theory.

Alex's Classes

Galois Theory. (🌶🌶🌶🌶, Alex, 4 days)
Galois Theory studies how we can describe extensions of fields using group theory. Here's an example: suppose we have the real numbers $\mathbb{R}$ which live in the complex numbers $\mathbb{C}$. Then, the only automorphisms of $\mathbb{C}$ which fix $\mathbb{R}$ are the identity map and complex conjugation. Together, the identity map and complex conjugation form a group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We could use this group to understand how the complex numbers extend the real numbers. In this course, we would take this approach to understand field extensions. This framework, in turn, can be used to address why we can't trisect an angle with a ruler and compass, why the quintic is unsolvable, and many other questions.

Homework: Recommended

Prerequisites: Linear Algebra, Group Theory

Matroid Theory. (🌶🌶🌶, Alex, 4 days)
In linear algebra, we are interested in when we have a linear dependence among a set of vectors. In graph theory, we are interested in when a set of edges forms a cycle. What do these two phenomena have in common? Both involve notions of dependence among members of their respective sets, all of which is described by a structure called a matroid. In this course, we will describe matroids, and then figure out what types of dependence can be described by graphs and by vectors over a given field (in particular, over certain finite fields). If we have time, we will discuss how matroids can be applied to combinatorial optimization.

Homework: Recommended

Prerequisites: Linear algebra. Some knowledge of graph theory is also helpful.

Alfonso's Classes

Burnside's Lemma. (🌶🌶, Alfonso, 1-2 days)
How many different necklaces can you build with 6 pebbles, if you have a large number of black and white pebbles? Notice that you won't be able to tell apart two necklaces that are the same up to rotation and reflection.

You probably can answer the above question by counting carefully, but what if we are building necklaces with 20 pebbles and we have pebbles of 8 different colours?

There is a nifty little result in group theory that allows us to solve this type of problem very quickly and without a calculator. Come to learn it!

Homework: Optional

Prerequisites: Basic group theory

The Redfield-Polya Theorem. (🌶🌶🌶, Alfonso, 2 days)
The Redfield-Polya Theorem is like Burnside's Lemma on steroids.

If you have taken a group theory class, or maybe some combinatorics class, you are Probably familiar with Burnside's Lemma (a.k.a. many other things): If G is a group acting on a set X, then the number of orbits of the action is given by

$$\frac{1}{|G|}|\operatorname{Fix}(g)| \tag{1}$$

It is a nice result, but not enough. For instance, how many different necklaces can you build with 20 beads out of very large supplies of red, green, and blue beads? With the help of (1), it will take you less than 5 minutes to calculate that the answer is 87230157 with just pen and paper. But what if I

ask you to tell me how many such necklaces are there with R red beads, B blue beads, and G green beads, for *each* value of R, B, and G? If you think there is no way to avoid using brute force to count in this problem, think again! You can still answer it in less than 5 minutes, but you will need the full force of Redfield-Polya. Come receive it!
Homework: Optional
Prerequisites: You need to understand (1) and its proof, or take my class on the Burnside's Lemma.

My PhD thesis in 50 minutes. (, Alfonso, 1 day)
This is probably a very bad idea.

In this class, I will explain the background and main result (without proof) of my PhD thesis.

What is it about? See for yourself at `http://aif.cedram.org/item?id=AIF_2005__55_7_2257_0`
Homework: None
Prerequisites: Multivariable calculus, linear algebra

Nets, Filters, and Waffles (or Why You Should Never Trust Sequences). (, Alfonso, 4 days)
You know what a topological space is. You also know how to define convergence of a sequence in an arbitrary topology. As it turns out, sequences are very sneaky objects. They pretend they are your friend, they tell you that all your topological problems can be solved with them, but they are lying.

For example, consider the following definition. Given a sequence (x_n) in a topological space X, a point $a \in X$ is an *accumulation point* of the sequence if, for every open neighborhood U of a, there are infinitely many points x_n in the sequence such that $x_n \in U$. At first sight, it looks like being an accumulation point of the sequence is the same as being a limit of a subsequence. Don't be fooled! Surprising as it may seem, being an accumulation point of a sequence is not the same as limit of a subsequence.

Or consider the following notion. A map $f : X \to Y$ between topological spaces is *sequentially continuous* when for every sequence (x_n) in X convergent to a point $a \in X$, the image sequence $f(x_n)$ in Y converges to the point $f(a) \in Y$. Sequences would like you to believe that sequentially continuous is the same as continuous, but this is just another of their lies.

It turns out that sequences are a flawed notion. They should be done away with and replaced with any of the much more reliable notions of nets, filters, or waffles (yes, that is the technical term).
Homework: Required
Prerequisites: Point-set topology. Specifically, you need to be able to follow this abstract.

The Banach-Tarski Paradox. (, Alfonso, 2 days)
You may have heard it before: we can take a sphere, divide it into five pieces, rearrange them, and get two spheres of the same size as the original one. Nifty trick, but how does it work? Come learn it!

PS: What is a good anagram of "Banach-Tarski"? "Banach-Tarski-Banach-Tarski"!
Homework: None
Prerequisites: Basic group theory

Ari's Classes

Fudge Donuts (i.e., Classification of Surfaces via Morse Theory). (🌶, Ari, 2 days)
A "Morse function" is a way of dipping donuts in fudge. We will assume that these exist, because life would be too sad to contemplate if they did not. Then we'll use them to classify all two-dimensional compact manifolds by counting critical points. Delicious!

Homework: Recommended

Prerequisites: None

Related to: other classes on surfaces

Geometric Algebra. (🌶🌶🌶, Ari, 4 days)
You've probably seen that complex numbers can be used to represent two-dimensional vectors and rotations. You may even have seen that quaternions do something similar for three-dimensional space. But why do they work this way? Why should the complex numbers and the quaternions be the objects that describe certain symmetries of space? In this class, we will explore the "real" geometric reasons for the behavior of these algebraic objects. We'll also dispel several myths, including "you can't multiply two vectors together", "you can't add a vector and a scalar", "you can't divide by a vector", and "in the quaternions, $ijk = 1$." (It should be $+1$.) Our principal tool will be the geometric product of vectors:

$$uv = u \cdot v + u \wedge v$$

What on earth does this mean? Come find out!

Homework: Recommended

Prerequisites: Linear algebra. You don't need to know what a quaternion is.

Knot Theory. (🌶, Ari, 2 to 4 days)
"So, what kind of math do you like?"
"Knot theory."
"Yeah, me neither."

Homework: Optional

Prerequisites: None

Asi's Classes

Extensions. (🌶🌶🌶🌶, Asi, 2 days)
Take any two abelian groups, for instance $G = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$. Then the product group $G \times H$ has the property that $G \hookrightarrow G \times H$ is an injective map, and $G \times H \to H$ is a surjective map, and the kernel of the second map is exactly the image of the first map.

Based on this structure, we define an *extension* of H by G to be a new abelian group E such that there is an injection $G \to E$ and a surjection $E \to H$, and the kernel of the second map is precisely the image of the first. Do there exist extensions other than the product extension shown above? How many are there? Are some of them isomorphic? How can we tell?

In this class we will study part of the story of extensions of abelian groups, and some beautiful underlying structure that fits them all together.

Homework: None

Prerequisites: Group theory, ring theory/commutative algebra will be helpful.

Related to: Commutative algebra, Category theory

Expressing Yourself Regularly. (🌶, Asi, 1 day)
Suppose you have written a 200-page PhD thesis. Can you convert it to pig latin with one quick command?

This is a somewhat contrived example, but in real life one often comes across patterns that one wants to search for or replace with other patterns. *Regular expressions* are a powerful way of searching for complicated patterns in blocks of text.

In this class we will see a quick glimpse of regular expressions, and get a sense of what patterns they can and cannot match. However, the goal of class is practical, not theoretical: we will really get our hands dirty by learning to use *grep*, which is a popular utility to carry out regular expression searches. At the end of the class, you will be able to tell me exactly which words in the SOWPODS scrabble dictionary are eight letters long, whose middle letters are equal but not equal to the first or the last letters, which contain exactly two vowels, one on either side of the two middle letters, contain no S or P, and contain at least one each of O, R, and T. And with a little more work you may even be able to convert your PhD thesis to pig latin.

Homework: None

Prerequisites: None

Related to: Simple Models of Computation

Platonic Rotations. (🌶🌶🌶, Asi, 2 days)
Groups of rotations in two and three dimensions are continuous groups that have many infinite subgroups. In this class we will tackle an easier problem: can we classify *all* possible finite subgroups of the two and three dimensional rotation groups?

We will solve this problem by looking at the five platonic solids: the regular tetrahedron, cube, octahedron, dodecahedron and icosahedron. We will guess the rotational symmetry groups of these objects, and then figure out why they are what they are. Using simple but elegant arguments, we will be able to find a complete solution to this problem.

Homework: None

Prerequisites: Group theory

The Ring Swap-out Cop-out. (🌶🌶🌶, Asi, 1 day)
You may have heard of the Cayley-Hamilton theorem, which states that any square matrix satisfies its own characteristic polynomial. This theorem is true for matrices with real or complex entries. But in fact, it is also true for matrices with entries over *any* commutative ring with identity. Normally, a statement like this might be proved using some abstract properties of rings. Instead, I will show you a proof that appears to be a cop-out at first glance, but secretly turns out to be awesome. There will be some rings, much linear algebra, and perhaps surprisingly, some epsilons and deltas.

Homework: None

Prerequisites: Linear Algebra, familiarity with continuous functions and the definition of a ring.

Jeff's Class

The Euler Characteristic of a Torus. (🌶🌶🌶→🌶🌶🌶🌶, Jeff, 2 days)
(Day 1, 🌶🌶🌶) The Euler characteristic is a pretty big deal! You've probably heard of it in at least one context—for planar graphs, there is a magical formula

$$V - E + F = 2,$$

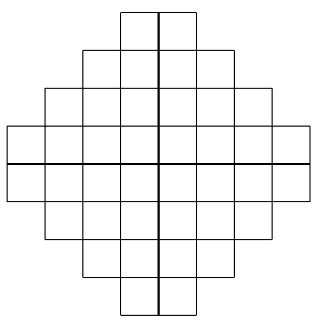
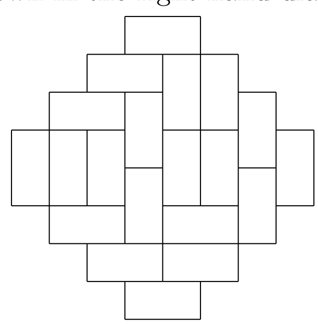
where V stands for the number of vertices, E the number of edges, F the number of faces. The last number, 2 is called the *Euler characteristic* of the plane. However, if we draw (nice) graphs on the surface of a donut, we have a slightly different formula,

$$V - E + F = 0.$$

The number 0 in this formula is the Euler characteristic of the torus. You may already have known this. You might be able to prove it. But how many ways can you show that the alternating sum of these three numbers is zero? In this class, we will show how integrating local properties like curvature of a surface can give us topological invariants, and we will also introduce our first homology theory!
(Day 2, 🌶🌶🌶🌶) We continue our quest to find ways to make $V - E + F = 0$! On our journey, we will cancel out finite dimensional vector spaces of triangles, infinite-dimensional vector spaces of vector fields, look at divergences and gradients, and use a snake in an unexpected fashion to show that combinatorially-constructed and differential-geometrically-constructed topological invariants are related to each other in a deep way.
Homework: None
Prerequisites: Linear Algebra, Multivariable Calculus
Related to: Cohomology of Surfaces, Fudge Donuts, Curvature of Polyhedra

JULIAN'S CLASSES

Aztec Diamonds. (🌶🌶, Julian, 2 days)
An Aztec diamond is a diamond shape made of squares, as shown in the left hand diagram for a diamond of order 4. It can be covered in dominoes as shown in the right hand diagram.

How many ways are there of covering such a shape with dominoes? And what would a random such tiling look like on average? These seemingly hard questions have been investigated in the past twenty years and have revealed some beautiful mathematics (as well as astonishing pictures).

In this class, we will explore some of the hidden secrets of the Aztecs; there will be a few proofs and sketch proofs, but the emphasis will be on the beautiful known results.
Homework: None
Prerequisites: None

BIG Numbers! (🌶, Julian, 1 day)
We'll be learning the original proof of van der Waerden's theorem: if we colour the natural numbers with c colours, there are monochromatic arithmetic progressions of arbitrary (finite) length. (A recent strengthening of this theorem was the remarkable result of Ben Green and Terry Tao that the primes contain arbitrarily long arithmetic progressions.)

BIG WARNING: This class will involve big numbers. I mean, really huge numbers. Astonishingly colossal numbers. In fact, numbers so large that this class comes with a health warning: Do not attend this class if you are of a nervous disposition or are scared of large numbers!
Homework: None
Prerequisites: No fear of big numbers

How a Mathematician Reads a Newspaper. (🌶, Julian, 1 day)
Maths is full of hypotheses, theorems and logical arguments. What happens when we apply our thinking to a piece of text, say from a newspaper?

You will need an open mind, and a willingness to explore an article logically! This session will be heavily based on work by Bandler and Grinder, who developed a model for analysing text based on Chomsky's theory of transformational grammar.

Homework: Optional

Prerequisites: None

Reflection Groups. (🌶🌶🌶, Julian, 4 days)
Take a cube. Look at all of the reflections that preserve the cube—there are quite a few of them! Together they generate a group, called a *reflection group.*

What would happen if we started with a different set of reflections in $\mathbb{R}^n$ and only demanded that the generated group was finite? How many possible groups could we make (up to isomorphism)?

We'll explore with group theory, some linear algebra and maybe some Zome as well!

Homework: None

Prerequisites: Group theory, linear algebra

Squaring the Square. (🌶, Julian, 2 days)
The square is one of those perfect shapes, a thing of beauty. So who would dare cut it up? We will! And not only will we cut it up, but we'll aim to chop it into smaller squares, each of a different size.

If it's even possible, that is.

This is a story of how some undergraduates had fun playing with a problem just because it was there.

Homework: Recommended

Prerequisites: None

The Kakeya Needle Problem. (🌶🌶, Julian, 2 days)
In 1917, Japanese mathematician S. Kakeya proposed a problem: What is the smallest area through which a needle of length one can be rotated 360 degrees? Clearly a circle of diameter length one—of area $\pi/4$—would do the trick: just pivot the needle about its centre. But we can do better than that: if we take an equilateral triangle of altitude 1 (and hence side length $2/\sqrt{3}$), we can slide the line segment up one side of the triangle, rotate it in the vertex, slide it down the next vertex, and so forth, until the needle is fully rotated. This triangle has area $1/\sqrt{3}$, which is less than $\pi/4$. In fact, a shape called a deltoid, which looks like a triangle whose edges are curved inward, does the trick—and its area is just $\pi/8$—less than $1/\sqrt{3}$, and just half the area of the circle. Can we do better? In 1928 the mathematician A.S. Besicovich came up with two different answers to Kakeya's problem ... but you'll have to come to the class to find out what he did!

Homework: None

Prerequisites: Fractals useful but not necessary

Related to: Fractals

Kevin's Classes

Decomposing Polynomials. (🌶🌶, Kevin, 1 day)
Say we have two polynomials $f(x)$ and $g(x)$. Then you certainly know how to find their composition $h(x) = f(g(x))$.

Say we have a polynomial $h(x)$. Can we find a way to decompose $h(x)$ as $f(g(x))$?

Well, we have to define our problem a little better—we could always take $g(x) = x$ and $f(x) = h(x)$. But with the right setup, we'll see that whenever this decomposition exists, it's unique. And I'll teach you how to find the decomposition in your head, with some practice and a little mental calculation.

Homework: None

Prerequisites: None

Generating Functions and Complex Analysis. (, Kevin, 1 to 2 days)

If you've seen generatingfunctionology before, then you should believe it's magic.

If you've seen complex analysis before, then you should believe it's magic.

We're going to mix magic with magic.

We'll be able to get closed forms for tricky generating functions by taking contour integrals. And for those where we can't find a closed form, we'll use complex analysis to get some pretty good asymptotic estimates.

Homework: None

Prerequisites: Complex analysis and some familiarity with generating functions.

More Posets! (, Kevin, variable days)

This class will talk about more fun things relating to partially ordered sets! Depending on interest, I will talk about some subset of the following topics:

(1) Differential Posets (1 day): The derivative from calculus has the nice property that

$$\frac{d}{dx}(xf(x)) - x\frac{d}{dx}f(x) = f(x),$$

or, in other words, that $\frac{d}{dx}x - x\frac{d}{dx}$ is the identity operator on functions. In a locally finite poset, we can define operators called up and down; then differential posets are those that satisfy the similar looking property

$$DU - UD = \mathrm{Id}.$$

We'll see how just this one property can work a lot of magic.

(2) Distributive Lattices (1-2 days): Lattices are nice posets that have operations that "look like" union and intersection. Distributive lattices are lattices where these operations look even more like union and intersection! In fact, every finite distributive lattice arises from a finite topology, where the operations are honestly union and intersection. We could, if we were so inclined, use fancy words like "equivalence of categories".

But wait, there's more! It turns out that finite distributive lattices are naturally in bijection with… arbitrary finite posets. This is my favorite example of a natural bijection between a (necessarily infinite) set and a proper subset!

(3) Binomial Posets (1 day): Think about your favorite kinds of generating functions. Ever wonder why the denominators 1 and $n!$ seem to appear in so many of them? No? Well, start wondering! Why are those denominators useful? What other kinds of creatures could we build generating functions out of? It turns out that binomial posets provide a somewhat unified framework for answering this question.

(4) Hyperplane Arrangements (2-3 days): Remember the relay problem asking how many pieces of cheese you could make with 5 cuts? OK, you probably don't. But this is a question answered by studying hyperplane arrangements!

A hyperplane arrangement is just a collection of hyperplanes in some Euclidean space. For example, in two dimensions, a hyperplane arrangement is just a bunch of lines; in three dimensions, it's a bunch of planes. By studying a magical poset called the intersection poset, we can easily count the number of regions the hyperplane arrangements divide space into. In doing so, we'll see that with n cuts, we can cut that block of cheese into at most

$$\sum_{k=0}^{3} \binom{n}{k}$$

pieces.

Homework: None

Prerequisites: The Posets class from Week 1 is helpful but not required.

Mark's Classes

Elliptic Functions. (🌶🌶🌶🌶, Mark, 4 days)
If you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (people were interested in the arc length of an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first noticed as inverse functions of those integrals), elliptic functions don't have much to do with ellipses.

Instead, they are closely related to cubic curves. If time permits, we may be able to use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k),$$

where $\sigma_i(k)$ is the sum of the i-th powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1],$$

which you can check some time if you're bored.)

Homework: Recommended

Prerequisites: Functions of a complex variable

Euclid. (🌶, Mark, 1 day)
Euclid is known for the geometry in the *Elements*, but somehow his name comes up in number theory as well. What does his work actually look like, and to what extent do we know what he did?

Homework: None

Prerequisites: Basically none, although it would be nice if you knew the "Euclidean algorithm."

Related to: Perfect numbers (from Trail Mix)

Fermat. (🌶, Mark, 1 day)
So you've heard of Fermat's Last Theorem (a.k.a. the Fermat-Wiles-Taylor Theorem), and maybe you know how to prove his Little Theorem, but do you know more about the context of his life and work in- and outside number theory? Although he was of an earlier generation than Newton and Leibniz, he had already developed some "calculus" methods. If you want to know more, you know what to do.

Homework: None

Prerequisites: Rudimentary calculus

Fun((Counting Involving) Parentheses). (🌶, Mark, 1 day)
If you shorten the words in the title of this class to their first letters, to get $f((ci)p)$, this would be one of the five ways to interpret the ambiguous product $fcip$ if you didn't know multiplication was associative. There are also five ways to arrange three pairs of parentheses if nesting is allowed but you may never put a closing parenthesis ") " before the corresponding opening " (" ; here are those five ways:

$$((())), (()()), (())(), ()(()), ()()().$$

In this class we'll see what happens in general and how to make a direct connection between the two problems. (Part of this is an observation I was able to publish less than fifteen years ago, but that I hope you will rediscover independently by exploring in small groups for a little while. Please don't work on it in advance, though.)
Homework: None
Prerequisites: None

Multiplicative Functions. (🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such *multiplicative* functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.
Homework: Optional
Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)
Related to: Perfect numbers (from Trail Mix)

Primitive Roots. (🌶🌶, Mark, 1 day)
Suppose you are working modulo n and you start with some integer a and multiply it by itself repeatedly. For instance, if $n = 17$ and $a = 2$ you get $2, 4, 8, 16, 15, 13, 9, 1$ and then you're back where you started. Note that on the way we haven't seen all the nonzero integers mod 17; however, if we had used $a = 3$ instead we would have gotten $3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$ and cycled through all the nonzero integers mod 17. In general we can ask when (that is, for what values of n) you can find an a such that every integer mod n that's relatively prime to n shows up as a power of a (such an a is called a *primitive root* mod n). We may not get much beyond the case that n is prime, but even in that case the analysis is interesting. In particular, we'll be able to show that a exists in that case without having any idea of how to find a, other than the flat-footed method of trying $2, 3, \dots$ in turn until you find a primitive root.
Homework: None
Prerequisites: Modular arithmetic; a bit more number theory wouldn't hurt.

Rescuing Divergent Series. (🌶🌶, Mark, 1 day)
Consider the infinite series

$$1 - 1 + 1 - 1 + \dots .$$

What is its sum? Maybe

$$(1 - 1) + (1 - 1) + \cdots = 0,$$

maybe

$$1 + (-1 + 1) + (-1 + 1) + \cdots = 1?$$

At one time mathematicians were quite perplexed by this, and one even thought the issue had theological significance. Now presumably it's nonsense to think that the "real" answer is $\frac{1}{2}$ just because the answers 0 and 1 seem equally good, right? After all, how could the sum of a series of integers be anything other than an integer?

Homework: None

Prerequisites: A bit of experience with the idea of convergence.

The Delta "Function". (, Mark, 1 day)
In introductory books on quantum mechanics you can find "definitions" of a "function", called the delta function, which has two apparently contradictory properties: Its value is zero for all nonzero x, and yet its integral over any interval containing 0 is 1. This "function" was introduced by the theoretical physicist Dirac and it turns out to be quite useful in physics, but how can we make any mathematical sense of such a creature?

Homework: None

Prerequisites: Integration by parts

The Jacobian Determinant. (, Mark, 1 day)
How do you change variables in a multiple integral? In the "crash course" in week 1 we saw that a somewhat mysterious factor r comes in when you change to polar coordinates. This is a special case of an important general fact involving a determinant of a matrix of partial derivatives. We'll see how and roughly why this works; if time permits, we'll use the technique to evaluate the famous sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(You may well know the answer, but do you know a proof? If so, do you know a proof that doesn't involve complex analysis or Fourier series?)

Homework: None

Prerequisites: Multivariable calculus; some experience with determinants

The Quadratic Reciprocity Law. (, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

Question 1: "Is q a square modulo p?"

Question 2: "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. If all goes well you'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square mod 9973, you'll be able to answer a lot more quickly, whether or not you use technology.

Homework: Optional

Prerequisites: Some basic number theory, specifically Fermat's little theorem.

The Sylow Theorems. (, Mark, 2 days)
The converse of Lagrange's theorem is false: if a finite group G has order n and d is a divisor of n, in general there may not be a subgroup of G whose order is d. However, there are important special cases in which the converse is true, in particular the case when d is the highest power of some prime p that divides n. In this class, you should see proofs of this and related statements, and then see them used to show that groups with some plausible properties actually can't exist.

Homework: None

Prerequisites: Group Theory

Matt's Classes

How Fast Can We Multiply? (, Matt, 4 days)
Way back in elementary school, you (probably) learned how to multiply two numbers. A little thought should convince you that the amount of time it takes to multiply two n-bit numbers that way is roughly proportional to n^2. Can we do better?

Surprisingly, we can! We'll see a bunch of ways we can improve on the algorithm we learned, including the best-known algorithm for multiplying large numbers: Fast Fourier transform multiplication.

We'll also see a way that a lot of really tiny computers (whose role will be played by you, the campers) can together multiply even faster. The more people who show up to the class, the bigger the numbers we'll be able to multiply!

Homework: None

Prerequisites: None

How Not To Prove The P=NP Conjecture. (, Matt, 4 days)
Susan told us how not to prove the continuum hypothesis. But there are plenty of other things in math we know how not to prove, too! And surprisingly, when it comes to the P=NP conjecture, we know *a lot* about how not to prove it.

The P=NP conjecture is probably the most important open question in computer science; it asks, essentially, whether being able to efficiently check an answer to a problem means that we can *solve* it efficiently. Pretty much everyone thinks it's false: after all, it seems like factoring a number is *much* harder than multiplying together the factors to check them. But nobody yet has managed to prove it.

Over the years, all kinds of different proof techniques have been tried. In the 70s, people tried to use "simulation" arguments, which had been successful at attacking lots of similar-looking problems. But in 1975, three people published a paper showing that no such argument could possibly show that P≠NP.

After the bad news, people attempted to prove it in ways that had more of a combinatorial flavour. But once again, this entire line of reasoning was killed off: in 1996, it was shown that pretty much all of the proof techniques that had been tried so far couldn't possibly settle the P=NP conjecture.

We'll look at how exactly we can prove that a method of proof can't work, and then see if there are any ways that might still have some hope for settling the conjecture.

Homework: Recommended

Prerequisites: Some knowledge of computability. (Simple Models of Computation or Reverse Mathematics are sufficient.)

Related to: Reverse Mathematics, Simple Models of Computation.

How To Add. (, Matt & Zach, 4 days)
You've probably seen the Fibonacci numbers, the nth of which is obtained by adding together the

previous two. We can write this as a "recurrence relation":

$$\begin{aligned}F_0 =& 0\\ F_1 =& 1\\ F_n =& F_{n-1} + F_{n-2}.\end{aligned}$$

You may also have heard that it has a surprising "closed form": we can directly find the nth Fibonacci number, without calculating any of the ones before it.

You may also know that

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2},$$

which looks suspiciously similar to the familiar fact from calculus that

$$\int_0^y x\,dx = \frac{y^2}{2}.$$

Is this a coincidence? (Spoiler alert: no.)

We'll see systematic ways of taking recurrence relations and sums, and getting closed forms from them. These will include generating functions, a hammer we can use to hit all kinds of recurrence relations, as well as "finite calculus," a system that resembles ordinary calculus but works with sums instead of integrals.

Homework: Recommended

Prerequisites: Basic calculus

Turning Paradoxes into Theorems. (🌶🌶🌶, Matt, 4 days)
You may have heard about Berry's paradox: "the smallest natural number that can't be defined in fewer than thirty words" is problematic, because… well, we just defined it in fewer than thirty words. Oops.

You may also have heard about the "Surprise Examination Paradox": a teacher announces that there will be a pop quiz at the beginning of class some day next week, but that the students won't expect it. The students reason that it can't be on Friday: after all, if it's class time on Friday and it hasn't happened yet, the students will be expecting it. But it also can't happen on Thursday, or Wednesday, or Tuesday, or Monday by similar logic. Overconfident in their reasoning that the quiz can't happen at all, the students are then *very* surprised when the quiz is given on Wedesday! What went wrong?

You may also have heard of Gödel's incompleteness theorems, which among other things show that there are statements in mathematics that can't be proved true or false and that (in some sense) we can never prove mathematically that mathematics works.

What you probably *haven't* heard is that both of these paradoxes can be turned into proofs of Gödel's incompleteness theorems! This class will talk about what the incompleteness theorems are, what they imply, and how to prove them using paradoxes. As an added bonus, we'll see how we can turn things around and use the second incompleteness theorem to resolve the surprise examination paradox!

Homework: Optional

Prerequisites: None

Related to: Reverse Mathematics, Models of Computation

Mira's Classes

Does ESP Exist? or, What's Wrong With Statistics? (🌶🌶, Mira, 3 to 4 days)
In 2011, Daryl Bem, a professor of psychology at Cornell, published an unusual paper in the *Journal of Personality and Social Psychology*, titled "Feeling the Future: Experimental Evidence for Anomalous Retroactive Influences on Cognition and Affect". In this paper, Bem presents results from 9 experiments testing for "psi", aka ESP. For instance, in the first experiment, participants are instructed as follows:

> On each trial of the experiment, pictures of two curtains will appear on the screen side by side. One of them has a picture behind it; the other has a blank wall behind it. Your task is to click on the curtain that you feel has the picture behind it. The curtain will then open, permitting you to see if you selected the correct curtain.

Bem's analysis of the data shows that participants were able to predict the location of the picture at a rate significantly higher than chance, but only if the picture involved sex or violence. For other pictures, they did no better than chance. The follow-up experiments were designed to determine whether the psi effect was due to clairvoyance, psychokinesis, or retroactive influence in which "the direction of the causal arrow has been reversed".

Note: this is not just an individual loony. This is a highly respected professor of psychology, from one of the top departments in the country, writing for one of the top journals in his field, and being, if anything, more scrupulous in laying out his methodology than is standard practice. The journal editor wrote a preface to Bem's paper, stating that although he himself does not believe in the existence of psi, neither he nor the other reviewers could find a flaw in Bem's statistical analysis. Therefore, says the editor, scientific honesty compelled him to publish the paper, so that it could be discussed in a wider forum.

If you find all of this shocking, you are not alone. The entire psychology profession cringed in mortification and blamed the editor for not exercising better judgment.

Yet the real culprit here is not the author (who is free to pursue whatever research he wants, and whom no one suspects of violating scientific ethics) nor the editor (who really was just trying to adhere to the standard practice of his field), but the statistical methodology currently employed by most scientists. A famous 2005 paper published in a medical (!) journal states the problem very starkly in its title: "Why most published research findings are false".

In this course, we'll discuss the (many) problems with how statistics is currently done (not just in psychology). We'll talk about the historical reasons for these problems and present an alternative: Bayesian statistics—a methodology that is actually based on math and dates back to Gauss and Laplace. In particular, we'll apply a Bayesian analysis to Bem's results. (Spoiler alert: they give no evidence for the existence of ESP. Sorry to disappoint you.)

Homework: Recommended

Prerequisites: Calculus and basic probability. No background in statistics required, though if you do have such a background, you may appreciate the de-brainwashing.

Related to: Machine Learning

The Bell Curve. (🌶🌶🌶, Mira, 4 days)
Even non-mathy people have heard of the "bell curve"—the function that describes the distribution of heights, IQs, and exam scores, among many many other things. (When your teacher says that the class is "graded on a curve", this is the curve they are referring to, though they may not know it themselves.)

The real name of the bell curve is the *normal* or *Gaussian* distribution. Gauss discovered it when he was 23, under very interesting circumstances that immediately made him an international scientific rock-star. (Come to the class to find out more.) Gauss used the normal distribution to motivate his *method of least squares* (which we'll talk about in the class), but he himself did not fully realize the implication of what he had found.

So why is the normal distribution so important and so ubiquitous? A large part of the answer is the *Central Limit Theorem*, proved by Pierre-Simon Laplace in 1810. If you've taken statistics, you've probably heard of this theorem (and you've certainly used it implicitly). But you probably haven't seen the proof, which is usually omitted even in standard undergraduate probability courses. In this class, we'll develop the necessary tools to prove the Central Limit Theorem and will discuss some of its implications. We'll also look at other contexts in which the Gaussian distribution comes up and other ways in which it is special.

Homework: Optional

Prerequisites: Single-variable calculus (derivatives, integrals, power series)

Related to: Machine Learning

Nic's Classes

Greek Impossibilities. (🌶🌶🌶, Nic, 4 days)
The ancient Greeks (and probably also your high school geometry teacher) were concerned with constructing geometric objects using a compass and straight-edge. They accomplished a lot with these tools, but there were three things that they just couldn't manage to do:

(1) Given a square that's the face of some cube, construct a face of a cube of twice the volume.
(2) Given an angle of measure θ, construct an angle of measure $\frac{1}{3}\theta$.
(3) Given a circle, construct a square of the same area.

It turns out there's a good reason they failed to do these things: they're all impossible. In this class, you'll learn why.

Homework: Recommended

Prerequisites: Linear Algebra, specifically the definitions of linear independence, span, and dimension.

Modules Over a Principal Ideal Domain. (🌶🌶🌶, Nic, 4 days)
Groups are extraordinarily complicated objects; even finite groups were only "classified" relatively recently, and it's probably fair to say that there isn't a single person on Earth who understands the entire proof. Finite *abelian* groups, on the other hand, are pretty simple: the only ones are cyclic groups and their products.

In this class, we'll not only be able to prove this fact, but also a whole lot more. It turns out to fit into a much larger framework that also, with almost no effort, produces a proof of the fact that, for every linear map on a vector space, there's a basis in which the matrix for the map is in *Jordan canonical form*, an almost-diagonal matrix that carries all the information about the map's eigenvalues even when it isn't diagonalizable.

These two facts and more follow from the classification of finitely-generated modules over a principal ideal domain, a result that's also interesting in its own right. First we'll prove the result, and then we'll see all the exciting things it can do. Feel free to come even if you don't know what a module or a principal ideal domain is; we'll define everything in class.

Homework: Optional

Prerequisites: Ring theory, including an understanding of quotient rings.

Special Relativity. (🌶🌶, Nic, 3 days)
Around the beginning of the twentieth century, physics was undergoing some drastic changes. The

brand-new theory of electromagnetism made very accurate predictions, but it forced physicists to come to grips with a strange new truth: there is no such thing as absolute space, and there is no such thing as absolute time. Depending on their relative velocities, different observers can disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics has to be rebuilt to accommodate these observations. We will see how, as Minkowski said, "space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." Along the way, we will also have to revise the classical notions of momentum and energy, allowing us to derive the famous relation $E = mc^2$. If there's time at the end, we might discuss some other topics in modern physics.

Homework: Recommended

Prerequisites: Probably none, maybe calculus at the very end.

The Line Without a Length. (, Nic, 4 days)
Take your favorite subset of $\mathbb{R}$. How long is it? For most of the sets you might think to try, like intervals, or unions of intervals, this question has a clear answer. But not every set looks like a union of intervals. What's the length of the rationals, or the set of numbers whose decimal expansion doesn't contain any 7's?

In this class we'll build up a way to answer this question using a piece of machinery called *Lebesgue measure*. We'll find out how to find the lengths of many subsets of $\mathbb{R}$ and, perhaps surprisingly, we'll find some sets that can't be assigned lengths at all.

Homework: Recommended

Prerequisites: None

Pesto's Classes

Premodern Cryptography. (, Pesto, 2 days)
UI UOHH JFHQI SGXEKFNGMT ELZZHIJ HOWI KRI FDIJ KRMK XFL TONRK AODP OD M DIUJEMEIG, AOGJK CX KGOMH MDP IGGFG, KRID CX KISRDOBLIJ KRMK YLJK IVKIG-TODMKI KRIT. Then we'll look at some of the premodern code-writers' attempts at strengthened versions, and see why they're still breakable.

Homework: Optional

Prerequisites: None

Problem Solving: Inequalities. (, Pesto, 2 to 4 days)
High-school Olympiads usually try to choose problems relying on as little prior knowledge as possible. In inequalities problems, they usually fail completely; training is necessary to solve most and sufficient to solve many of them. We'll go over the common Olympiad-style inequalities, and solve problems like the following:

(1) Prove that if a, b, and c are positive and $ab+bc+cd+da=1$, then $\frac{a^3}{b+c+d}+\frac{b^3}{a+c+d}+\frac{c^3}{a+b+d}+\frac{d^3}{a+b+c}\geq\frac{1}{3}$.

(2) [USAMO 2004] Prove that if a, b, and c are positive, then $(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3)\geq(a+b+c)^3$

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to Olympiad-style problems you'll've solved as homework the previous day.

If you've taken an inequalities problem-solving class from me and want to take this one, poke me and I may be able to make this one disjoint.

Homework: Required

Prerequisites: None

Related to: Other problem-solving

The Hoffman Singleton Theorem. (🌶🌶🌶, Pesto, 1 day)
There are three known graphs in which

(1) every vertex has the same degree,
(2) the shortest cycle has length 5, and
(3) between every two vertices there's a path of length at most 2.

In those three graphs, every vertex has degree 2, 3, or 7, respectively. It's known that if any other such graphs exist, then every vertex has degree 57. We'll prove it.

Homework: None

Prerequisites: Linear algebra: eigenvectors.

Related to: Intro to Graph Theory

Ruthi's Classes

Congruent Numbers and Elliptic Curves. (🌶🌶🌶🌶, Ruthi, 4 days)
Let n be a positive integer. We call n a congruent number provided that there is a right triangle that has all rational sides and area n. Can we find an easy way of determining whether n is a congruent number? Tunnell's theorem gives us a way, but it uses advanced number theory, in particular the study of elliptic curves. It is also dependent on a weak form of an unsolved conjecture called the Birch and Swinnerton-Dyer Conjecture. (Cool fact: a proof—or counterexample—of this conjecture is worth a million dollars! This is definitely not the easiest way to get rich though.)

In this class we will do the following: discuss congruent numbers and Tunnell's theorem, define elliptic curves and their basic properties, explain why answering questions about elliptic curves will tell us about congruent numbers, and maybe get some idea about how this leads us to Tunnell's result.

Homework: Recommended

Prerequisites: Elementary number theory, know what an abelian group is

Related to: Ring Theory, Algebraic Number Theory, Flag Varieties, Commutative Algebra

Counterexamples in Analysis. (🌶🌶🌶, Ruthi, 4 days)
Sometimes in math, things go horribly terribly wrong. Theorems we wish were true aren't true, functions don't behave the way we want them to, and limits don't always commute. Among other things, we will discuss: a function that is discontinuous on the irrationals and continuous on the rationals, a function that is continuous, has derivative 0 almost everywhere but is 0 at 0 and 1 at 1, and a function that is continuous everywhere and differentiable nowhere.

Homework: Optional

Prerequisites: Calculus. To understand the proofs, you should understand arguments with ε and δ.

Related to: Epsilon the Enemy, Harmonic Analysis

Quadratic Forms. (🌶🌶🌶, Ruthi, 4 days)
Consider the following question: suppose you have an equation of the form $ax^2 + bxy + cy^2$ for some rational numbers a, b, c. Which rational numbers can you get by plugging in rational x, y and z?

In some situations, you may already know how to do this: you could use Pell's equation or ask which number is the sum of two squares. In this class, we will discuss the general case of such equations (called quadratic forms), and some of the interesting theories required to study this, including learning about the p-adics, which surprisingly give us lots of information about the rationals!

Homework: Optional

Prerequisites: Elementary number theory

Related to: Continued Fractions

Topology Your Friend. (🌶🌶🌶, Ruthi, 4 days)
The theorems about continuous functions from calculus rely on a notion of what it means for things to be close. If you've studied more advanced calculus, you've been introduced to how this is formalized with the "epsilon-delta" approach. This can sometimes get, how shall we put it, messy? In this class we will discuss how notions of closeness can be generalized to defining a topology on a set, focusing on relating it back to calculus. In particular, we will aim to give alternative (very slick!) approaches to the proofs of the intermediate and extreme value theorems.

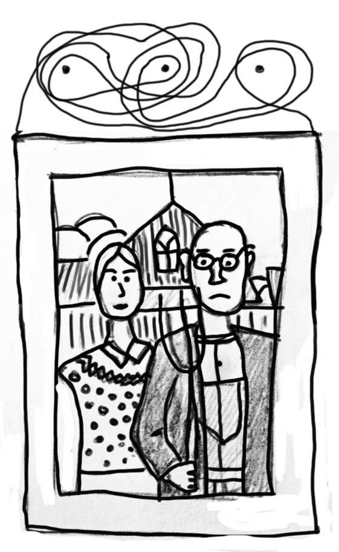
Homework: Recommended

Prerequisites: Calculus. Some proofs will involve epsilons, some understanding of this will help.

Related to: Surfaces and Symmetries, Epsilon the Enemy, Topology of Surfaces, Metric Spaces

Sachi's Classes

String Theory. (🌶🌶, Sachi & Tim!, 2 days)
Let's say you want to hang a picture in your room, and you are worried that the 2,000 fans you rented to beat the heat will blow it off the wall, so you want to hang it very securely. You start by wrapping string around some nails, like this:



You figure it will be super-duper secure, because you wrapped it around three nails a lot of times. Yuval, angry about the electricity bills from running all your fans, comes over and pulls one of your nails out of the wall. . . and your picture comes crashing to the ground! Actually, no matter which nail he removes, it falls! What happened?

In this class, you'll be playing with string to solve puzzles like this one, and we'll explore fundamental groups, homology, and monotone boolean functions.
Homework: None
Prerequisites: None

Tim!'s Classes

Calculus Without Calculus. (🌶🌶, Tim!, 2 days)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. But lots of those problems can actually be solved without any calculus at all. In this class, you'll learn to do common calculus problems without taking any derivatives or integrals at all!

Some example problems that we'll solve without calculus in this class:

- You have 40 meters of fence, and you want to build a rectangular pen, where three sides of the pen will be made of fence, and the last side will be a preexisting brick wall. What is the largest area that your pen can enclose?
- Sachi is $5'2''$ tall and Asi is 157.5 cm tall, and they are standing 5 cubits apart. You want to run a string from the top of Sachi's head to the top of Asi's head that touches the ground in the middle. What is the shortest length of string you can use?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 50 feet way along the shoreline and throws a stick 20 feet out into the water. The dog can run along the shoreline at 5 miles per hour, and can swim at 4 miles per hour. What is the fastest route that the dog can take to get to the stick?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!
Homework: Optional
Prerequisites: None
Related to: Calculus in school.

Fractals Are Like Snowflakes. (🌶🌶🌶, Tim!, 1 day)
A Sierpinski triangle consists of 3 half-size copies of itself, arranged in an equilateral triangle. Is there any other shape that is also made of three half-size copies of itself in an equilateral triangle? If you require that your shape be nonempty and compact, then then answer is no; the Sierpinski triangle is unique in having this property.

Suppose we have a rule making shapes out of smaller copies of itself — the rule can specify number of copies, and the size/position/rotation/stretching of each copy. Is there any (nonempty compact) shape that fits that rule? The answer is that there is always exactly one (and it's often a fractal). Just like no two snowflakes are the same, no two fractals satisfy the same rule. And this is true not just in the plane, but in any complete metric space.

We'll talk about the contraction mapping principle, then we'll put a metric on the set of nonempty compact subsets of of a complete metric space, and finally we'll prove that every rule has a unique shape that satisfies it.
Homework: None
Prerequisites: Metric Spaces. You should be happy about Cauchy sequences and compact sets.
Related to: Fractal Geometry

String Theory. (🌶🌶, Sachi & Tim!, 2 days)
See Sachi's Classes.

Waffle's Classes

Adjoint Functors. (, Waffle, 2 days)
Adjoint functors are a phenomenon that is pervasive in mathematics and are one of the most fundamental tools in category theory. Loosely speaking, a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are adjoint if for any objects $A \in \mathcal{C}$ and $B \in \mathcal{D}$, maps $A \to GB$ are "the same thing" as maps $FA \to B$.

In this class, we'll make this definition more precise, and then see oodles of examples. Just a few of the many constructions in mathematics that can be seen as examples of adjoint functors are free groups, bases for vector spaces, the abelianization of a group, limits (and colimits), and exponential objects. We'll prove that in an adjunction, the functor F always preserves colimits and G always preserves limits, which is the ultimate explanation for both the distributive law in categories with exponential objects and the fact that limits in most "concrete" categories look the same as limits of sets. If we have time, we'll finish with a sketch of the proof of the adjoint functor theorem, which gives a sort of converse: under some relatively mild assumptions, any functor that preserves (co)limits automatically has an adjoint.
Homework: Optional
Prerequisites: Category Theory

Applications of Transfinite Induction. (, Waffle, 2 days)
In this class we'll look at some applications of transfinite induction outside of set theory, some well-known and less well-known. On the first day, we'll see how transfinite induction can be used to prove some basic facts in calculus, such as the Intermediate Value Theorem or the fact that a function whose derivative is 0 everywhere must be constant. On the second day, we'll look at applications in algebra, proving that any vector space has a basis. If we have time, we may prove Zorn's Lemma, and explain how it is just a packaging up of a common pattern of transfinite induction arguments.
Homework: Optional
Prerequisites: Set theory, or a willingness to accept transfinite induction on faith. The first day will use a small amount of calculus/real analysis in the applications, and the second day will use some linear and abstract algebra in the applications.

The Classification of Finite Abelian Groups. (, Waffle, 2 days)
One of the basic questions of group theory is to classify all finite groups up to isomorphism. This question is universally considered to be impossibly hard, and one of the great achievements of 20th-century mathematics was the classification of just the finite *simple* groups (a group is simple if it has no nontrivial normal subgroups; this roughly means that it can't be broken down into smaller pieces).

On the other hand, for *abelian* groups, this question is much nicer and easier to answer. The answer is that every finite abelian group is isomorphic to an essentially unique direct product of cyclic groups. This result is extremely useful throughout mathematics, as whenever you have some finite abelian group, you automatically have a good idea of what it has to look like. In this class, we will prove this classification of finite abelian groups and (time permitting) sketch some ways in which it generalizes to other areas of abstract algebra.
Homework: None
Prerequisites: Group theory

Yuval's Classes

The Four Cube-Face Fun Fact of "The Barn". (, Yuval & Zach, 1 day)
If you wish to say that some int n is the sum of many a cube-face, how many of a cube-face must you

use? As a test case, look at this year, 2013: we can find that this is the sum

$$2013 = 38^2 + 22^2 + 9^2 + 2^2.$$

In fact, we can do even more good; this year is the sum of just two-plus-one of a cube-face (try it!), but if we try the same with the int 7, we find that we do need four of a cube-face. Is four good all of the time, or do some ints need even more? In this talk, we will see once and *four* all that we will not ever need five or more. "The Barn," a very good math guy of yore, did show this fun fact in the year $42^2 + 2^2 + 1^2 + 1^2$. We will show it too, but he used more of a long word than we will use.

Home work: None

Must Know: How to work in mod p

Has to do With: Many a Fact on Nums, Many a Fact on Nums (but with Alg!)

Zach's Classes

The Four Cube-Face Fun Fact of "The Barn". (🌶🌶🌶🌶, Yuval & Zach, 1 day)
See Yuval's Classes.

How To Add. (🌶🌶, Matt & Zach, 4 days)
See Matt's Classes.

How to Solve Every Twisty Puzzle. (🌶🌶🌶, Zach, 2 days)
Algorithmically speaking, the Rubik's Cube and many of its more complicated cousins are easy to solve: they can be expressed in terms of permutation groups acting on the pieces, and therefore they fall to an elegant and efficient algorithm due to Schreier and Sims. This algorithm is well-suited for many types of queries regarding permutation groups, such as testing membership (can you turn a single corner in a $3 \times 3 \times 3$ cube? Swap two edges in a $4 \times 4 \times 4$ cube?), instantly computing the size of the group (there are precisely $43,252,003,274,489,856,000$ possible $3 \times 3 \times 3$ cube positions, and you can bet it didn't enumerate them all individually!), and many others.

This is *not* a course on how to solve a Rubik's Cube. We will deal more generally with computations in *any* permutation group (or group action), and the resulting algorithm can describe the group generated by *any* desired set of basic permutations. We will see how this can be applied to many types of puzzles, and we will also discuss some limitations of the approach.

Homework: None

Prerequisites: Group Theory

Classes Proposed by Three or More Staves

50 Definitions in 50 Minutes. (🌶, Aaron, Adam, Alex, Kevin, Matt, Pesto, Ruthi, Zach, 1 day)
Who needs theorems? This will be a class entirely driven by definitions. Several of us will deliver rapid-fire explanations of some of our favorite definitions in mathematics. Our definitions of "favorite" may vary.

Homework: None

Prerequisites: None

5 Ways of Looking at the FTA. (🌶🌶🌶🌶, Alfonso, Ari, Matt, Nic, and Zach, 1 day)
The Fundamental Theorem of Algebra (FTA) says that every polynomial (with coefficients in $\mathbb{C}$) has a

complex root. To see why this result is so fundamental, this course will explore the FTA's connections to many and distant branches of mathematics. In particular, we will offer five different proofs of the FTA, using methods from Complex Analysis, Algebraic Topology, Combinatorics, Point-Set Topology, and Algebra (surprise!). With a total of 5 proofs, 5 instructors, and 50 minutes, this whirlwind tour promises to be Fabulously Tantalizing Awesomeness.

Homework: Recommended

Prerequisites: Some subset of Complex Analysis, Algebraic Topology, Combinatorics, Point-Set Topology, and Algebra

What Is *Your* Axiom of Choice? (🌶🌶🌶, The Staff, 1 day)

The Axiom of Choice. What does it state? How do we prove it? Why do we need it? Why it is controversial?

The Mathcamp Staff proudly present controversy, cuteness, and choice.

Act 1: Russell's paradox is discovered. The Zermelo-Fraenkel Axioms boldly enter—cardinals are compared—and the Axiom of Choice claims to be natural.

Act 2: The Well-Ordering Principle, the Axiom of Choice, and Zorn's Lemma have a cute-off.

Act 3: The Banach-Tarski paradox; the Axioms defend their worth. Zermelo and Fraenkel assert their independence!

Homework: None

Prerequisites: None