CLASS DESCRIPTIONS—WEEK 5, MATHCAMP 2014

Contents

1. Classes Taught by Students

1.1. Conrad's Proposals.

Constructing a Peano Curve. (🌶🌶, Conrad, 1 day)
A Peano curve is a function $f : [0,1] \to A \subset \mathbb{R}^2$ where A is a shape with positive area and f is continuous and surjective. If this does not sound pathological, read it again.

Come to this class to learn the black magic of constructing Peano curves! First, I will show how to build it when $A = [0,1]^2$, then when A is any compact, convex subset of $\mathbb{R}^n$, and for the final twist, I will make f infinity-to-one.

Homework: Optional

Prerequisites: None. You don't need to know what compact or convex mean to appreciate this class.

1.2. Yonah's Proposals.

The Art Gallery Problem. (🌶🌶🌶, Yonah, 1 day)
What do we do when people try to steal from the largest art museum in Northern New England? We place guards! But how many guards do we need in order to ensure sure that they can see everything, so no priceless works can be stolen?

More specifically, given a (not necessarily convex) n-sided polygon, we want to place "guards" at its vertices so that any point inside the polygon is within the unobstructed view of a guard, who can see in all directions but cannot move. The Art Gallery Problem (now a theorem) gives a formula minimizing the number of guards necessary, and is well-known for having a fantastically short proof that makes the theorem seem obvious. We will show that proof, as well as the proof for the related Fortress Problem, which is instead concerned with seeing the *outside* of polygons. If we have time, I will also present other related results and generalizations, some of which are still open.

All the math presented here is relatively recent; the standard Art Gallery Theorem, which started the chain of results, was only proved in 1975. So if you think that all new math uses words like "homological algebra" or "algebraic variety," come to this class to see how much more simple stuff there still is to figure out.

Homework: None

Prerequisites: Knowing what a graph is.

Related to (but not required for): Graph theory classes

2. Classes Proposed by Visitors

2.1. **Dan Zaharopol's Proposals.**

A New Mathematical Pathway: How Underserved Students can Enter Mathematics. (🌶, Dan Zaharopol, 1 day)
Why is it so hard for low-income and minority students to become scientists and mathematicians? Have you ever wondered what it must be like to be a smart kid interested in math who just can't seem to penetrate into the ecosystem of opportunities that we all know and are a part of? What can we, as a community of people who love math, do to help these students succeed?

For the past three years, I've been running the Summer Program in Mathematical Problem Solving (SPMPS) for underserved New York City middle school students with talent in math. In the process, I've learned a lot about these students—as well as about myself, and how we all learn math (and learn to love math). Come hear about how mathematics is done at a very different summer program, and how we might some day make programs like Mathcamp accessible to everyone.

Differential Topology. (🌶🌶🌶🌶, Dan Zaharopol, 3-4 days)

We'll take calculus and vastly generalize it to the world of *manifolds*. Manifolds are topological spaces that have just barely enough structure on them to understand what tangent vectors are, what derivatives ought to be (in terms of tangent vectors!), and what it means for a function to be *smooth*. We'll work with these manifolds, discovering all sorts of strange things about what you can do with them.

Prerequisites: Calculus, preferably (but not necessarily) multivariable; also, ideally, some point-set topology or basic real analysis, though you can get away without that if you're willing to ignore some details.

Interaction between Time and Space Complexity. (🌶🌶🌶, Dan Zaharopol, 2-4 days)
A big question in computer science is how fast various problems can be solved and how fast various algorithms can run. But time is not the only limited resource: the amount of memory that your computer has may be restricted as well. It turns out that time and space restrictions are linked to one another and there are many open questions about how exactly the connection works.

In this class, we'll study everything from polynomial time to exponential time and from logarithmic space to exponential space. The first day we'll review the basics on computational complexity, so if you've seen the beginnings of Pesto's Week 2 MCSP class or Paddy's Week 4 "NP-completeness and Latin Squares", you should be fine showing up on Day 2.

Isometries of $\mathbb{R}^n$. (🌶🌶, Dan Zaharopol, 4 days)

Consider the following amazing theorem: any function $\mathbb{R}^n \to \mathbb{R}^n$ that preserves distance is a composition of at most $n+1$ reflections.

It's pretty awesome. It tells us that every *isometry* (that's a map that preserves distance) is generated by reflections, which is kind-of surprising when you think about it. (This generalizes the fact that, in two dimensions, translations and rotations are compositions of two reflections, and glide reflections are compositions of three reflections.) It works for every dimension: any isometry you can think of for $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$, and so forth are all combinations of reflections.

The goal of this class is for *you* to prove the theorem. A small group of us will gather, and I'll provide hints if you need them, but the insights and progress will all be a result of your work together.

I won't lecture: you'll come up with the big ideas and see them go into action. This is your chance to do big, interesting theorems in a group together.

Once you've proved the theorem, we'll study isometries in more detail, either of Euclidean space or of other, stranger spaces, and we'll see how they relate.

Prerequisites: None, but some group theory might be helpful to understand the later material on a deeper level.

2.2. **Josh Tenenbaum's Proposals.**

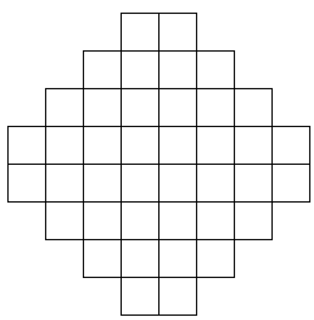
Toward a Human-level AI. (🌶, Josh Tenenbaum, 1 day)

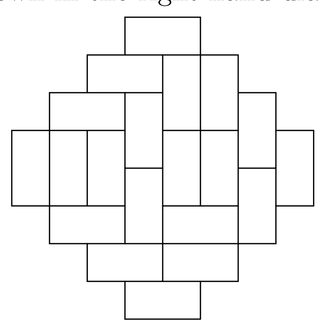
What will it take to to build a human-level AI, why aren't we there yet, and how long will it be until we are? Let's have a speculative but informed discussion of these questions, driven by your ideas and proposals, but grounded in the latest research in human cognitive science and neuroscience as well as computer science and mathematics relevant to AI. Along the way we'll also talk about some of the most important things scientists have come to understand in the last decade about how human intelligence works, as well as the most important scientific questions about the mind that we don't yet understand.

2.3. **Julian Gilbey's Proposals.**

Aztec Diamonds. (🌶🌶, Julian, 2 days)

An Aztec diamond is a diamond shape made of squares, as shown in the left hand diagram for a diamond of order 4. It can be covered in dominoes as shown in the right hand diagram.





How many ways are there of covering such a shape with dominoes? And what would a random such tiling look like on average? These seemingly hard questions have been investigated in the past twenty years and have revealed some beautiful mathematics (as well as astonishing pictures).

In this class, we will explore some of the hidden secrets of the Aztecs; there will be a few proofs and sketch proofs, but the emphasis will be on the beautiful known results.

Homework: None

Prerequisites: None

BIG Numbers! (🌶, Julian, 1 day)

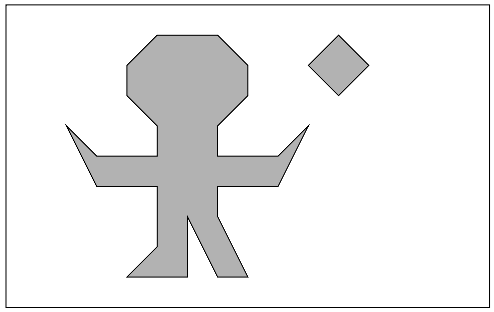
We'll be learning the original proof of van der Waerden's theorem: if we colour the natural numbers with c colours, there are monochromatic arithmetic progressions of arbitrary (finite) length.

BIG WARNING: This class will involve big numbers. I mean, really huge numbers. Astonishingly colossal numbers. In fact, numbers so large that this class comes with a health warning: Do not attend this class if you are of a nervous disposition or are scared of large numbers!

Homework: None
Prerequisites: No fear of big numbers

Just one cut? Are you serious? (, Julian, 1 day)

Here's a piece of paper with a figure cut out of it.



Would you believe me if I told you I cut this figure out with one single solitary straight snip of my scissors?
Homework: None
Prerequisites: None

Rational Trigonometry and Universal Geometry. (, Julian, 2 days)

Know the intersecting chords theorem? (See below if you don't.) A nice result. At least on the Euclidean plane, where we measure distances with real numbers. But what would happen if we were to measure "distances" with integers modulo 7, or the complex numbers? Can we still do geometry? What would the statement of the theorem even mean? And could it still, in some sense, be true?

And is there a way of doing some trigonometry (triangle measuring questions) in this bizarre setting?

Come and learn how to do trigonometry without a calculator, and then how it can be extended to wild and wonderful scenarios!

(The intersecting chords theorem: if AB and CD are two chords of a circle which intersect at P, then $AP \cdot PB = CP \cdot PD$.)
Homework: None
Prerequisites: Modular arithmetic; complex numbers are useful but not essential

Squaring the Square. (, Julian, 2 days)

The square is one of those perfect shapes, a thing of beauty. So who would dare cut it up? We will! And not only will we cut it up, but we'll aim to chop it into smaller squares, each of a different size.

If it's even possible, that is.

This is a story of how some undergraduates had fun playing with a problem just because it was there.
Homework: Optional
Prerequisites: None

3. Classes Proposed by Staff

3.1. **Aaron's Proposals.**

Why Maximize Entropy? (, Aaron, 2–3 days)
Suppose you learn something about a random variable. How should you use your newfound knowledge to update the probability distribution representing your beliefs? A statistician will say, "Use conditional probability!" A physicist will say, "Maximize entropy!" Under certain circumstances, the physicist's recommendation is an almost perfect approximation to the statistician's, and you can use it to solve probability problems that might otherwise look almost intractable.
Homework: Recommended
Prerequisites: Know what a random variable is. Know that expectation is linear. The Bell Curve (Week 4) is more than sufficient.

Circles in $\mathbb{C}$. (, Aaron, 1–2 days)
If you don't look too closely, circles in the plane all look pretty much the same. But did you know that circles band together into families? That they have their own dot product? That there's an underworld of circles with imaginary radii? Through a little-known correspondence between circles and 2×2 hermitian matrices (or, if you'd rather avoid coordinates, conjugate-symmetric sesquilinear forms), we'll see all these hidden structures and more.
Homework: None
Prerequisites: Linear algebra, complex numbers.

Not Solving Differential Equations. (, Aaron, 2–3 days)
If you study math, science, or engineering for long enough, you'll probably end up taking a class on solving differential equations. In this class, you'll learn a skill that's less widely taught, but just as important: not solving differential equations. I'll show you several techniques for getting information about the solutions of a differential equation without being able to calculate the solutions explicitly. Topics might include:

- Proving trig identities without knowing how to calculate any trig functions.
- Deriving a formula for the exponential function from the fact that the exponential function exists.
- Predicting the long-term behavior of every one-dimensional dynamical system at the same time.

Homework: None
Prerequisites: Knowing what a derivative is, and how to calculate some simple ones.

3.2. **Alfonso's Proposals.**

Math Teaching Workshop. (, Alfonso, 1 day)
Teaching is not just for professional teachers. Have you ever wanted to present a solution to a class? Show a friend something cool you've learned? Or maybe explain to a non-mathy person what's so interesting about what you do? In this class, we will explore the essential skill of communicating mathematics face-to-face with speech and writing. Great teachers make teaching seem entirely natural; so natural, in fact, that their students often overlook the skills that go into a successful presentation. In this class, we'll get into the details of organizing material, board technique, verbal communication, classroom management, pacing, and more. Sound like a lot to handle? Relax – even the best teachers rarely get every detail right. Our goal is to give you some awareness of these issues so that you can start improving and take your first steps on the road of teaching. The format will be part lecture and part workshop, in which you will prepare and deliver miniature lectures for the class.
Homework: Required
Prerequisites: None

The Banach-Tarski Theorem. (🌶🌶🌶🌶, Alfonso, 2 days)
You heard it before during Yvonne's opening colloquium: we can take a sphere, divide it into five pieces, rearrange them, and get two spheres of the same size as the original one. Nifty trick, but how does it work? Do you want to learn and understand the full details of the proof? Then come to this two-hour class! If you took Mark Sapir's course at the beginning of camp, skip this one.

Homework: None

Prerequisites: Basic group theory

The Coin Game. (🌶, Alfonso, 1 days)
Take a bunch of coins and arrange them in piles.

(1) Order the piles from larger to smaller.
(2) Take the top coin from each pile to make a new pile.
(3) Go back to Step 1.

What happens? This very simple game produces various interesting combinatorial patterns. In this class, you will learn these patterns by discovery. This means that you will spend most of the time exploring and trying to answer questions about the game while I move around and help you.

Homework: None

Prerequisites: None

The Redfield-Pólya Theorem. (🌶🌶🌶, Alfonso, 1 day)
If you have taken a group theory class, or maybe some combinatorics class, or maybe my one-day class on Burnside's Lemma, you are probably familiar with Burnside's Lemma (a.k.a. many other things): If G is a group acting on a set A, then the number of orbits of the action is given by

$$\frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix}(g)| \tag{1}$$

It is a nice result, but not enough. For instance, how many different necklaces can you build with 20 beads out of very large supplies of red, green, and blue beads? With the help of (1), it will take you less than 5 minutes to calculate that the answer is 87230157 with just pen and paper. But what if I asked you to tell me how many such necklaces are there with R red beads, B blue beads, and G green beads, for *each* triple of values (R, B, G)? If you think there is no way to avoid using brute force to count in this problem, think again! You can still answer it in less than 5 minutes, but you will need the full force of Redfield-Pólya. Come receive it!

Homework: Optional

Prerequisites: You need to understand (1) and its proof. One way to do this is to take my class on Burnside's Lemma.

Burnside's Lemma. (🌶🌶🌶, Alfonso, 1 day)
How many different necklaces can you build with 6 pebbles, if you have a large number of black and white pebbles? Notice that you won't be able to tell apart two necklaces that are the same up to rotation or reflection. You probably can answer the above question by counting carefully, but what if we are building necklaces with 20 pebbles and we have pebbles of 8 different colours? There is a nifty little result in group theory that allows us to solve this type of problem very quickly and without a calculator. Come and learn it!

Homework: Optional

Prerequisites: Basic group theory

The Stable Marriage Problem. (🌶, Alfonso, 1 days)
N single men and N single women want to pair up and get married. These are their names and preferences:

- Don: Marisa, Ruthi, Lisa, Susan, Angela.
- Jeff: Angela, Marisa, Susan, Ruthi, Lisa.
- Kevin: Angela, Susan, Ruthi, Marisa, Lisa.
- Pesto: Lisa, Ruthi, Susan, Angela, Marisa.
- Tim!: Lisa, Angela, Marisa, Susan, Ruthi.

- Angela: Don, Pesto, Tim!, Kevin, Jeff.
- Lisa: Kevin, Jeff, Don, Tim!, Pesto.
- Marisa: Kevin, Pesto, Tim!, Jeff, Don.
- Ruthi: Kevin, Tim!, Don, Jeff, Pesto.
- Susan: Kevin, Pesto, Jeff, Tim!, Don.

Is it possible to make everybody happy? Obviously not since almost everybody wants to marry Kevin. But is it possible to at least create a *stable* situation? For instance, it is a bad idea for Kevin to marry Ruthi and for Susan to marry Pesto, because then Kevin and Susan would prefer each other rather than staying with their partners, so they will run away together. How can we at least avoid having a run-away couple? Is there more than one way to do it? What is the *best* way to do it? And what if Ruthi and Susan realize that marrying each other is better than marrying Kevin?

Homework: None

Prerequisites: None

3.3. **Don's Proposals.**

Ax-Groethendieck. (🌶🌶🌶, Don, 1 day)
You are walking through the woods, when you see a polynomial injection from $\mathbb{C}^n$ to itself. You'd sure like to prove that this injection is a surjection, but it's not an easy result - if you extend from polynomials to holomorphic maps, for example, this isn't true. Luckily, you have your handy Ax-Groethendieck with you, and you use it to chop $\mathbb{C}$ into bite-sized chunks of finite fields. Proving injections are also surjections is easy for these finite bits, and we'll show why that's good enough for the whole thing!

Homework: Recommended

Prerequisites: Ring Theory

Hyperfinite Approximation. (🌶🌶🌶, Don, 2–3 days)
Hyperfinite sets are like finite sets, but more so.
Also, they're really good at approximating things.

Homework: None

Prerequisites: None

Infinitesimals. (🌶🌶, Don, 2–3 days)
When calculus first came into the world, those who created it did so using a fairly controversial tool: infinitesimal values. Infinitesimals were useful, because there's no need to define limits when you can work directly with a notion of "incredibly close." When infinitesimals are allowed, derivatives can be computed directly, and dx is an actual value instead of a vague concept. Ultimately, the debate over infinitesimals was won by their opponents, who claimed that these things could not be made rigorous, and thus had no place in mathematics. The debate lay dormant for centuries, until a few

mathematicians realized that, even though $\mathbb{R}$ contains no infinitesimals, there are reasonable systems of mathematics that do. In this class, we'll be constructing an alternative to standard calculus, in which infinitesimals create no contradictions. In order to get there, we'll have to throw out a few axioms - you know, minor things like $x \vee \neg x$ being a tautology - but once we've done so, we'll be in a beautiful place, where every function is smooth, and the Banach-Tarski paradox is no more.

Homework: None

Prerequisites: Calculus is suggested, but not required

Topoi. (🌶🌶🌶🌶, Don, 1–2 days)
A Topos is a category which has all limits and what is known as a "power object," which does the same work that the powerset does in Sets. Just as set theory provides a foundation for much of modern mathematics, any Topos you choose has the ability to provide a different foundation. The mathematical worlds provided by these topoi often differ wildly; some topoi provide the axiom of choice, while others have its negation. Some topoi even reject the law of the excluded middle, which means that the only acceptable proofs are those which are directly constructive.

Homework: None

Prerequisites: Category Theory

3.4. Jeff's Proposals.

The Tutte Polynomial. (🌶🌶, Jeff, 2 days)
Suppose I give you a graph G. Here are a few common questions from different branches of graph theory that you may be interested in answering:

- **(Coloring)** How many different ways are there of coloring the graph with k colors?
- **(Network Reliability)** Suppose I remove edges with probability p. Will the graph remain connected?
- **(Knot Theory)** If this graph represents a knot, what invariant is Jeff thinking of which is a conjectured to distinguish it from the unknot? (Ok, not really a knot problem, but still really cool!)

Each of these problems can be solved by decomposing graphs into smaller components systematically, and solving the problem on the smaller parts. In class, we will go over a general technique for keeping track of these decompositions of graphs. On the way, we will find surprising connections between these seemingly disparate problems.

Homework: Recommended

Prerequisites: Graph Theory

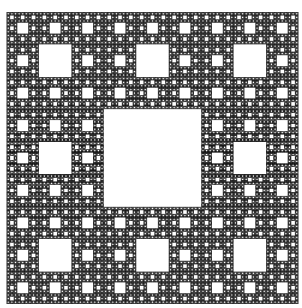
Euler Characteristic Sampler. (🌶 → 🌶🌶🌶🌶, Jeff, 3–4 days)
Curtains open on a small french restaurant... a young couple sit at a candle lit table, clearly famished for some topology. Accordion plays softly in the background.
Waiter: Monsieur, mademoiselle, welcome. Could I interested you in some hors d'oeuvres before the main course, or perhaps a glass of wine? Today the chef's special is *Euler Characteristic a la Graph Theory*, which pairs fantastically with the simple, yet complex *Simplicial-Complexes 1982*, if you are interested.
Young Man: Euler Characteristic a la Graph Theory? That was our main course last week, we were hoping for something... (*gazing into young lady's eyes*) a little more special.
Waiter: I understand. May I suggest the *Euler Characteristic with Discrete Curvature and Basil*?
Young Woman: Do you have something a bit smoother?
Waiter: Ah, I know just the thing. Perhaps I can get you started with *Euler Characteristic served*

with 4 chili integrated curvature sauce? It is a little spicier. We have our differential geometry flown in from the hills of Nice daily.
Young Man: I was hoping for something a little less spicy. Do you have any fish?
Waiter: Ah. The fish of the day is *Sous-vide Euler Characteristic Transversality Style*, which is an elegant blend of topology and algebra garnished with 3 different peppers. It goes will with our *Bell Pepper Discrete Curvature Salad*.
Young Woman: I think we will have that one.
Waiter: Allow me to get you started with a glass of the *Morse Theory 1975*, and place your order...

∼

This 3-4 day class will start at a 🌶 level, and each day will be a spicier variant of the Euler Characteristic.
Homework: Optional
Prerequisites: You can show up to days independently of each other if you know at least 1 definition of the Euler characteristic.

Measure Theory. (🌶🌶🌶, Jeff, 4 days)
Every year, more and more students are afflicted with the Riemannianintegrationitis. You may have this if you exhibit any of the following symptoms:

- Have you ever wanted to integrate
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
and found that you just didn't have the right tools to do it?
- Do you ever feel like

needs to have an area assigned to it, but you just aren't in a position to fix this problem?
- Does the fact that the set of integrable functions fail to form a complete metric space keep you awake late at night?

If you answered yes to any of these questions, you may be eligible for *Measure Theory*. Our 4-day program is proven to help relieve and cure some of the these symptoms. 4 out of 5 professional mathematicians suggest using *Measure Theory* to solve life's little difficulties.[1] Talk to your academic advisor today about how *Measure Theory* can improve your life.[2]
Homework: Optional
Prerequisites: You must be able to write an ε-δ proof.

[1]According to poll of main office, last week.
[2]Common side effect may include doubting the axiom of choice, seeing double spheres, being unable to distinguish functions that differ on only countably many points, overuse of the word generic, mild headaches, nonconstructable sets and infinite dimensional vector spaces.

Flow To The Max! (🌶🌶, Jeff, 2 days)
In this class, we will apply the Max-flow Min-cut theorem to the following real-world problems:

- An evil mentor is building a Maximum Capacity System of Pipes to distribute Kool-Aid to every camper's room. What is the best way for the JCs to disable the network?
- Inspired by the large wedding last week, you are trying to maximize the number of marriages at camp. Everybody submits a "Marriage Compatibility Sheet of Preferences" which lists who they would be willing to marry. How many newlyweds will there be this year?
- You start ranking the week 5 classes by constructing a Math Camp Schedule Poset, but realize that none of Jeff's classes are comparable. What is the largest number of mutually noncomparable classes?

Homework: Recommended

Prerequisites: None

3.5. **Kevin's Proposals.**

A Difficult Definite Integral. (🌶🌶🌶🌶, Kevin, 1 day)
In this class, we will show that

$$\int_0^1 \frac{\log(1+x^{2+\sqrt{3}})}{1+x}\,dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3}).$$

We'll need a healthy dose of clever tricks. Oh, and some heavy-duty algebraic number theory. (This class is based on a StackExchange post by David Speyer.)

Homework: Optional

Prerequisites: A strong background in algebraic number theory, or a willingness to believe in the facts that I'll use. Familiarity with rings and ideals (specifically rings of integers) will help.

Related to (but not required for): When Factoring Goes Wrong

Combinatorial Trigonometry. (🌶🌶🌶, Kevin, 1–2 days)
A *zigzag permutation* is a permutation $a_1, a_2, a_3, \ldots, a_n$ of the numbers $1, 2, 3, \ldots, n$ such that $a_1 < a_2 > a_3 < \ldots$. The number of zigzag permutations for even n are called the zig numbers, and the number of zigzag permutations for odd n are called the zag numbers. Let's define $S(x)$ and $T(x)$ to be exponential generating functions for the zig and zag numbers respectively. Then fun things happen! These functions satisfy identities like $T(x)^2+1 = S(x)^2$, and their derivatives are related to each other in suspicious ways, like $T'(x) = S(x)^2$. Are you starting to guess what's going on? (Spoilers: Read the title!)

Homework: Optional

Prerequisites: Familiarity with generating functions and differential calculus.

Euclidean Algorithms. (🌶🌶🌶, Kevin, 1–4 days)
The integers we know and love have the following division algorithm. Given positive numbers n and k, we can always find nonnegative integers q and r so that $n = qk + r$ with $r < k$. By iterating this process (using k as our new n and r as our new k), we will eventually be left with a remainder of zero and a quotient equal to the greatest common divisor of our original n and k. In general, a Euclidean algorithm on a ring of integers A is a function $h : A \to \mathbb{N}$ such that, given n and k in A, we can find q and r with $h(r) < h(k)$ such that $n = qk + r$. For example, there are "faster" Euclidean algorithms on the integers! Let's take $n = 124$ and $k = 50$. The standard Euclidean algorithm finds their GCD

in 4 steps:

$$\begin{aligned} 34 &= 1 \times 20 + 14 \\ 20 &= 1 \times 14 + 6 \\ 14 &= 2 \times 6 + 2 \\ 6 &= 3 \times 2 + 0. \end{aligned}$$

But we could have done it in three:

$$\begin{aligned} 34 &= 2 \times 20 - 6 \\ 20 &= -3 \times -6 + 2 \\ -6 &= -3 \times 2 + 0. \end{aligned}$$

We'll see in this class that if a Euclidean algorithm exists, then there's always a fastest Euclidean algorithm. We'll also learn techniques to prove that no Euclidean algorithms exist, and we'll maybe see how the Generalized Riemann Hypothesis implies that there are more Euclidean domains than we might expect!
Homework: Optional
Prerequisites: Ring theory or familiarity with rings of integers like $\mathbb{Z}[i]$.
Related to (but not required for): When Factoring Goes Wrong

Fun with the Möbius Function. (🌶🌶, Kevin, 1 day)
The standard Möbius function of number theory is defined as

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree with } k \text{ prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

After briefly reviewing Möbius inversion, we'll do fun things with μ. Fun things may or may not include:

- Showing that
$$\sum_{\gcd(m,n)=1} \frac{1}{m^2 n^2}$$
is, surprisingly, an integer;
- Showing how the Möbius function is related to the Riemann zeta function;
- Showing how the Möbius function is related to Pólya's Conjecture, whose first counterexample is 906150257 (found after 60 years of being open!).

Come see what this amazing function can do!
Homework: Optional
Prerequisites: None

Hyperplane Arrangements. (🌶🌶🌶, Kevin, 2–4 days)
Suppose I want to cut a watermelon into pieces. With 0 cuts, I can make 1 piece: the entire watermelon. With 1 straight cut, I can make 2 pieces. With 2 cuts, I can make 4, and with 3 cuts, I can make 8. I'm sure you see the pattern by now: with k cuts, we can make $\frac{1}{6}(n^3+5n+6)$ pieces. These cuts are an example of a *hyperplane arrangement*, which is simply a collection of k hyperplanes in n-dimensional space. We'll learn some powerful techniques for counting the number of regions that a hyperplane arrangement determines, and we'll maybe even see some connections to graph theory. Throughout, posets will abound!
Homework: Optional
Prerequisites: None

3.6. **Mark's Proposals.**

Integration by Parts and the Wallis Product. (🌶🌶🌶, Mark, 2 days)
Integration by parts is one of only two truly general techniques known for finding antiderivatives (the other is integration by substitution). In this class you'll see (or review) this method, and two of its applications: How to extend the factorial function, so that there is actually something like (1/2)! (although the commonly used notation and terminology is a bit different), and how to derive the famous product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots ,$$

which was first stated by John Wallis in 1655.

Homework: None

Prerequisites: Basic single-variable calculus

Multiplicative Functions. (🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such multiplicative functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.

Homework: Optional

Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)

Perfect Numbers. (🌶, Mark, 1 day)
Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called *perfect*, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, the so-called Mersenne primes - a search that has largely been carried out, with considerable success, by a far-flung network of individual "volunteer" computers.

Homework: None

Prerequisites: None

Primitive Roots. (🌶🌶, Mark, 1 day)
Remember the "number wheels" you get by multiplying by a modulo n, where a is relatively prime to n? In this class we'll explore when (that is, for what values of n) you can find an a such that every integer modulo n that's relatively prime to n shows up on a single wheel (such an a is called a *primitive root* modulo n). We may not get much beyond the case that n is prime, but even in that case the analysis is interesting. In particular, we'll be able to show that a exists in that case without having any idea of how to find a, other than the flat-footed method of trying $2, 3, \ldots, p-1$ until we find a number that works.

Homework: None

Prerequisites: Quick intro. to number theory

Quadratic Reciprocity. (🌶🌶, Mark, 2–3 day)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions? Question 1: "Is q a square modulo p?" Question 2: "Is p a square modulo q?" In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square mod 9973, you'll be able to answer a lot more quickly, whether or not you use technology. If time permits, we'll also explore a different proof that is somewhat shorter and that uses something John Conway pointed out to me: In one sense, the theorem isn't about primes, after all!

Homework: Optional

Prerequisites: A bit of number theory, specifically Fermat's little theorem.

The Cayley-Hamilton Theorem. (🌶🌶🌶, Mark, 1 day)
Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$. (The roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4-by-4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 8X - 17$, compute $A^4 - 6A^3 - A^2 + 8A - 17I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Homework: None

Prerequisites: Linear algebra, including a solid grasp of determinants (the "magic of determinants" class would certainly take care of that).

Related to (but not required for): The magic of determinants

The Magic of Determinants. (🌶🌶, Mark, 3 days)
Determinants, which are numbers associated to square matrices, have many useful properties; for example, they are needed to give a general formula for the inverse of a matrix. Unfortunately, determinants are often defined in a very *ad hoc* way (using Laplace expansion) which may obscure what is really happening. This class will give a "better" theoretical framework as well as some geometric intuition, and I'll try to at least give an outline of the proofs of all the main computational properties of determinants, such as the Laplace expansion.

Homework: Optional

Prerequisites: Matrix multiplication, and the idea of a linear transformation. (No previous knowledge of determinants is required.)

The Nine-Point Circle. (🌶🌶, Mark, 1 day)
There is some beautiful geometry hidden in and around every triangle. In particular, there are several points that can qualify as "centers" of the triangle, but that are different unless the triangle is equilateral. One of those points is the center of a circle that goes through nine related points, so it's not too surprising that it's called the nine-point circle. If you haven't seen this but you like plane Euclidean geometry, you're in for a treat!

Homework: None

Prerequisites: None

The Pruefer Correspondence. (🌶🌶, Mark, 1 day)
Suppose you have n points around a circle, with every pair of points connected by a line segment. (If

you like, you have the complete graph K_n). Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments, but in only one way. (This tree will be a *spanning tree* for K_n.) How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll go through a combinatorial proof that is especially cool.

Homework: None

Prerequisites: None

The Riemann Zeta Function. (🌶🌶🌶, Mark, 2–3 days)
Many highly qualified people believe that the most important open question in pure mathematics is the Riemann hypothesis, a conjecture about the zeros of the Riemann zeta function; the conjecture had its 150th birthday properly celebrated in 2009. So what's the zeta function, and what's the conjecture? By the end of this class you should have a pretty good idea. You'll also have seen a variety of related cool things, such as the probability that a "random" positive integer is not divisible by a perfect square (beyond 1), and an evaluation of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Homework: None

Prerequisites: Single-variable calculus, including infinite series. Complex analysis would help, but is not required.

Related to (but not required for): Bernoulli numbers; Integration by parts and the Wallis product

3.7. **Mira's Proposals.**

A Math Wrangle. (🌶🌶 or 🌶🌶🌶, Mira, 1–4 days)
A Math Wrangle is a type of math competition popular in Russia, which is just beginning to take root in the US. In addition to problem-solving, it has elements of debate, strategy, and spectacle. Mathcamp students have been invited to participate in a Math Wrangle at MathFest (the big math conference happening in Portland) on the morning of Saturday, August 9. While the timing is not ideal (it's the morning after the Talent Show and the day before the 47-week field trip), I figured that some people may want to give it a shot. If so, we should clearly give you a chance to practice beforehand. And even if no one wants to go to MathFest, it seems like a fun thing to try for ourselves.

In a Math Wrangle, two teams of three players have an hour to work on a set of 8 proof-based problems ahead of time. During the Wrangle itself, the two teams take turns challenging each other to present their solutions in front of a live audience and a panel of judges. The team that has been challenged may choose to *accept* the challenge or to *return* it. If the challenge is returned, the point-value of the problem goes up, and the team that issued the original challenge must present their own solution. In any case, after a solution is presented by one team, the other team has a chance to present a rebuttal, where they show how to correct or improve on their opponents' solution. The strategy involved in a Math Wrangle is quite complex. Suppose Team A has challenged Team B to solve a problem to which Team B does not have a full solution. If they accept the challenge, the best they can hope for is partial credit. On the other hand, if they return the challenge and Team A has a proof, then they could fall behind by up to 10 points. But Team A might be bluffing, or might not have spotted a crucial difficulty in the problem. In that case, Team B will still have a chance to present their partial solution as a rebuttal and/or to explain what is wrong with Team A's approach. Thus, accepting or returning a challenge requires insight into the difficulty of a problem and the strength of the opposing team.

The entire process happens before an audience, who may be consulted by the judges or even called on to volunteer answers that the two teams have not been able to find.

Depending on interest, we can run 0-4 Math Wrangles in Week 5. If you vote for this "class", please indicate whether you are interested in being a participant or an audience member (or both). Also, please indicate if you are more interested in a 🌶🌶 or 🌶🌶🌶 version.

Homework: Recommended

Prerequisites: None

Related to (but not required for): Participating in the Math Wrangle at MathFest on Saturday, August 9th. (But if you play in the Mathcamp Math Wrangle, you do *not* have to go to MathFest.)

A Game You Can't Play, but would Win if You Could. (🌶, Mira, 1 day)
Once upon a time, in the Kingdom of Aleph, lived a cruel King who decided to put his 100 wise men to a test. He had 100 identical rooms constructed in his palace. In each room the king placed an infinite sequence of boxes; in each box he put a real number. The sequence of numbers was exactly the same in each room, but otherwise completely arbitrary.

The king told the wise men that when they were ready, each of them would be locked in one of the 100 rooms. Each of them would be allowed to open all but one of the boxes in the room. (This, of course, would take an infinite amount of time, but in the Kingdom of Aleph, they're pretty cavalier about infinities.) Finally, each wise man would be required to name the number in the box that he did *not* open. If more than one wise man names the wrong number, they would lose their jobs and their lives.

There is no reason for anyone to hurry in the Kingdom of Aleph, and the king gives the wise men an infinite amount of time to work out a strategy. Do they have any hope of making it through his cruel test alive? What should they do?

SVD and its Amazing Applications. (🌶🌶🌶, Mira, 3–4 days)
In this class, we'll focus on a powerful linear algebra technique called *singular value decomposition* (a.k.a. SVD). You can think of SVD as a way of finding eigenvectors for matrices that don't have any. (Actually, you find the eigenvectors for a different matrix, which is closely related to the original one.)

Applications of SVD are everywhere. After explaining the theory, we'll look at examples from (some subset of) image processing, genetics, web search, psychology, and the Mathcamp Week 5 schedule. In the homework assignments, you'll get to try some of these applications for yourself: expect to spend at least some of TAU working in the computer lab.

Homework: Recommended

Prerequisites: Enough linear algebra to understand the statement of the Spectral Theorem: "Let A be an $n \times n$ matrix. A is symmetric iff there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A". The Mathcamp linear algebra class from Week 1 should be sufficient, assuming you followed it all the way to the end and remember it well.

Information Theory and the Redundancy of English. (🌶🌶, Mira, 4 days)

NWSFLSH: NGLSH S RDNDNT!! (DN'T TLL YR NGLSH TCHR SD THT)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting or to have a conversation in a crowded room. The redundancy is a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

How redundant is English? There are two ways to interpret this question:

- How much information is conveyed by a single letter of English text, relative to how much could theoreticaly be conveyed? (But what *is* information? How do you measure it?)

- How much can we compress English text? If we encode it using a really clever encoding scheme, can we reduce the length of the message by a factor of 2? 10? 100? (But how will we ever know if our encoding is the cleverest possible one?)

Fortunately, the two interpretations are related. In this class, we will first derive a mathematical definition of information, based on our intuitive notions of what this word *should* mean. Then we'll prove the Noiseless Coding Theorem: the degree to which a piece of text (or any other data stream) can be compressed is governed by the actual amount of information that it contains. We'll also talk about Huffman codes: the optimal way of compressing data if you know enough about its source. (That's a big "if", but it's still a very cool method.) Finally, we'll answer our original question – how redundant is English? – in the way that Claude Shannon originally answered it: by playing a game I call Shannon's Hangman and using it as a way of communicating with our imaginary identical clones!

The class is 4 days long, but you can skip some of the days and still come to the others. Here's how it works:

Day 1: *Introduction and definition of information.* Required for the rest of the class, unless you've seen some information theory before.

Days 2, 3: *Noiseless coding and Huffman codes.* The mathematical heart of the class, where we'll prove the Noiseless Coding Theorem.

Day 4: *Shannon's Hangman and the redundancy of English.* You can come to this class even if you don't come on Days 2 and 3 – you just need the material from Day 1.

Homework: Recommended
Prerequisites: None

Dominoes on a Chess Board. (, Mira, 3–4 days)
Seemingly innocent question: How many ways are there to cover an $M \times N$ chessboard with non-overlapping dominoes?

Seemingly insane answer:

$$\prod_{m=1}^{M}\prod_{n=1}^{N}(4\cos^2\frac{m\pi}{M+1}+4\cos^2\frac{m\pi}{N+1})^{1/4}$$

"Wait", you say, "that doesn't even look like an integer!" You're right, it doesn't – but it is! Come find out where all those cosines come from.

More Bell Curve! (, Mira, 1–3 days)
You want more Bell Curve? I got more Bell Curve

- Find out why the method of least squares (also due to Gauss) makes sense.
- Meet the Cauchy Distribution, which dares to defy the Central Limit Theorem.
- Learn about the role that the normal distribution plays in Bayesian statistics and information theory.

Homework: Optional
Prerequisites: The Bell Curve (Week 4)

3.8. **Misha's Proposals.**

Exotic Random Graphs. (, Misha, 4 days)
There are lots of unusual models of random graphs designed to produce graphs with different properties. We will begin by playing around with a few of these to see what happens. The second half of the class will be a more in-depth study of random regular graphs, whose structure we will analyze by abusing conditional probability almost past its breaking point.

Homework: Recommended

Prerequisites: Some probabilistic techniques we developed in Random Graphs. If you want to join without having taken Random Graphs, talk to me at TAU.

Memorylessness: The Exponential Distribution. (🌶🌶, Misha, 4 days)
Every second that Paddy spends in the main lounge, there is a small but constant chance that he will be drawn on by campers. How long will Paddy have to wait for this to happen? The exponential distribution answers this question. In this class, we will use it to find shortest paths in graphs, analyze the load on a server, and find the mean lifetime of a supply of radioactive uranium.

Homework: Recommended

Prerequisites: None

Entropy Compression. (🌶🌶🌶🌶, Misha, 1 day)
The entropy compression method is a tool for proving that certain randomized algorithms work by showing that if they didn't, we would have a universal compression algorithm. We will use this to derive the existence of some cool combinatorial objects and also prove the Lovasz Local Lemma, a powerful tool of the probabilistic method.

Homework: Optional

Prerequisites: None

The Harmonic Series. (🌶🌶, Misha, 1 day)
The series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges. How quickly? We will introduce and use some techniques of asymptotic analysis to estimate the sum of the first n terms of this series, with applications to unusual shuffling methods and the stamp collector problem.

Homework: None

Prerequisites: None

The Transfer Matrix Method. (🌶🌶, Misha, 1 day)
The campers and staff are passing around a coconut, but each person is only willing to hand it off to someone they're good friends with. Knowing the complete friendship network of Mathcamp, can we count the number of ways to pass the coconut 10 times? Using the transfer matrix method (not to be confused with the method of the same name in physics), we will find a generating function to solve problems of this type.

Homework: None

Prerequisites: Knowledge of generating functions and of matrix multiplication

3.9. **Paddy's Proposals.**

Algebra without (Much) Algebra, or; Graph Theory is Best Theory. (, Paddy, 1 day)

```
A GRAPH THEORIST and an ALGEBRAIST are walking to TAU from
opposite directions, looking at their class notes.

            ALGEBRAIST
    Mmm, group theory!

            GRAPH THEORIST
    Mmm, graph theory!

They COLLIDE.

            ALGEBRAIST
    You got graphs in my groups!

            GRAPH THEORIST
    You got groups in my graphs!

            NARRATOR
    Groups and graphs: two great tastes that go great together!

            ALGEBRAIST, GRAPH THEORIST
    AAAAAAAA TERRIFYING VOICE FROM THE SKY FLEE FLEE
```

Somewhat more seriously: in this class, we're going to define Cayley and Schreier graphs, deduce a number of results about these objects, and use our observations to prove that the subgroup of any free group is free! This is a surprisingly non-obvious theorem in algebra, that is incredibly awful to prove without using techniques from other fields – we're going to do it using graphs, using techniques that are both remarkably useful in designing computer-aided algebra algorithms and also satisfying to compute by hand.

Homework: Recommended

Prerequisites: Graph theory, group theory

Related to (but not required for): Graph theory classes, group theory classes

How to (Not) Prove the Four-Color Theorem, or; Graph Theory is Best Theory. (, Paddy, 2–3 days)

The **four-color theorem** is a famous result in mathematics that (roughly speaking) claims that in any map, we can assign colors to countries in such a way that no two adjacent countries get the same color, while using at most four colors through our entire map. Mathematicians worked on this problem for almost a full century, creating huge portions of modern graph theory along the way, to finally prove that this result is true via a combination of cleverness and exhaustive computer search in 1976. In this class, we will **not** prove this result!

Homework: Recommended

Prerequisites: Graph theory

Related to (but not required for): Graph theory classes

Many Campers Sort Piles. (🌶, Paddy, 1 day)
Some days you wake up and everything's just perfectly aligned: your problem sets are stacked by order of difficulty, your Alpha-Bits cereal is alphabetically sorted in your spoon, and the sun streaming through your conveniently prismatic window is sorted into a beautiful rainbow of its spectra. Don't you wish every day could be like that? Don't you wish you had some way to remove the disorder of day-to-day life, to purge this world of its chaotic unpredictability, and replace it with a new dawn of order and structure? This class... will not help you do this. But it will help you sort lists of numbers!

Homework: Recommended

Prerequisites: None!

Related to (but not required for): Algorithms classes, computer science classes

Sequential Dynamical Systems, or; Graph Theory is Best Theory. (🌶🌶🌶, Paddy, 1 day)
The field of sequential dynamical systems is a fairly new one; it was first developed in a series of papers spanning 99-03, and while it has a number of fairly exciting/useful applications (neural networks! traffic simulation!) it is pretty much a wide-open field of study. This is... somewhat surprising, when you consider how simple the concepts behind sequential dynamical systems are! Basically, they consist of the following:

- A space you want to study, modeled by some graph G.
- A set of possible vertex states K.
- A collection of local update functions f_v, one for every vertex in our graph.
- A way to "combine" these local update functions in some sequence to create a global update function.

We're going to study these systems in this class, prove a few of the most beautiful known results about such systems, and discuss related open problems.

Homework: Recommended

Prerequisites: Graph theory, being OK with permutation groups

Related to (but not required for): Graph and group theory classes, dynamical system classes

The Jordan Curve Theorem, or; Graph Theory is Best Theory. (🌶🌶🌶, Paddy, 1 day)
The Jordan curve theorem is perhaps one of the best known "obvious but awful" theorems in mathematics. On one hand, its claim — any non-self intersecting loop drawn in $\mathbb{R}^2$ divides the plane into two regions, an "interior" and an "exterior" — seems so simple as to not even need a proof. How could a closed loop **not** divide the plane into two pieces? On the other hand: many proofs of this are **awful**, especially when you look at the earliest successful papers on the theorem, and required heavy machinery from either algebraic topology or analysis. Fun fact that makes me incredibly happy: you can prove this with graph theory! We're going to do this here. It's going to be great!

Homework: Recommended

Prerequisites: Graph theory

Related to (but not required for): Graph theory, topology, anytime you want to assert that the indoors and outdoors are different things.

The Unit Distance Graph Problem And The Axiom Of Choice, or; Graph Theory is Best Theory. (🌶🌶🌶, Paddy, 3–4 days)
The **unit distance graph** is the following graph:

- Vertices: the elements of $\mathbb{R}^2$.
- Edges: connect any two vertices whenever their distance in $\mathbb{R}^2$ is exactly 1. For example, $(1, 0)$ and $(\sqrt{2}/2, \sqrt{2}/2)$ are both connected to $(0, 0)$.

Here's an open research problem: find the chromatic number of this graph! In this class, we're not going to answer this problem (though we will find the strongest known bounds currently known to mathematics.) Instead, we're going to focus on the following incredible conjecture: the chromatic number of this graph? It probably depends on the axiom of choice. Vote for this class to see why!

Homework: Recommended

Prerequisites: Graph theory.

Related to (but not required for): Graph theory classes, being paranoid about the axiom of choice in all contexts

3.10. **Pesto's Proposals.**

Character Table Sudoku. (🌶, Pesto, 1 day)
A character table is a grid of numbers with some constraints; for instance, this is a character table:

1	1	1	1	1	1
1	-1	-1	-1	1	1
2	0	0	0	-1	-1

If you carefully choose a subset of the grid and delete the rest, filling it back in is a fun logic puzzle. For instance, can you infer enough constraints from the previous character table to fill in this one?

1	1	1	1	1	1	1	
1	1	1	1				
1	1			1	1		
1	1						

Character tables are important in *representation theory*, a combination of group theory and linear algebra. We won't do any representation theory, but familiarity with character tables makes learning representation theory later a bit easier.

Homework: None

Prerequisites: None. If you know enough about character tables to object that the examples are presented wrong, you probably won't enjoy this class.

Related to (but not required for): Group Theory, Linear Algebra, Sylow

Nonzero-Sum Games. (🌶, Pesto, 2 days)
Sarah wants to take the Chess class and Stephanie wants to take the Go class, but they conflict. Even more than either of them wants to take their preferred class, though, they want to be in the same class, to confuse their teachers. Games like chess and go, with only one winner, have optimal strategies, although they may be hard to find. In Sarah and Stephanie's class selection game, it's not

clear what an optimal strategy means. We'll talk about and play games where everyone could win, or everyone could lose, or anything in between. We'll also discuss the relevance of the golden rule and Kant's categorical imperative to game theory.

Homework: Recommended

Prerequisites: None

Quantum Factoring. (🌶🌶🌶🌶, Pesto, 1 day)
With the best currently known algorithms, all the world's supercomputers working together for a year couldn't factor a 4000-digit integer into prime factors. Even a small[3] computer that takes advantage of quantum mechanical phenomena that aren't well approximated by what classical computers do could factor such numbers using "Shor's algorithm". We'll race through a description of how we model quantum computation (without assuming any knowledge of physics), then see Shor's algorithm.

Homework: Recommended

Prerequisites: Number Theory (enough to be comfortable with the statement "the multiplicative group mod n has order $n-1$ iff n is prime"), linear algebra (definition of a linear transformation)

Related to (but not required for): Quantum Tricks

Quantum Tricks. (🌶🌶🌶, Pesto, 1 day)
You have 100 boxes. Some of them hold bombs, but will explode if you open them. You have no way to tell whether a box has a bomb other than by opening it. Can you, with high probability, figure out which boxes have bombs, while having only a low chance of blowing yourself up? We'll talk about this and other tricks (like provably secure cryptography) that quantum computation makes possible.

Homework: Recommended

Prerequisites: Linear algebra (definition of a linear transformation)

Related to (but not required for): Quantum Computation

Time Travel Algorithms. (🌶🌶🌶, Pesto, 2 days)
Achron is a real-time strategy computer game in which you can make changes to the past. Lost a battle? Go back and try a different strategy, and hope your opponents don't do the same. How do you go from a real-time strategy computer game in which you can't make changes to the past to one in which you can? If you're making a bunch of complicated changes to, say, a spreadsheet, and at some point you realize that you made a mistake early on that affected your later work, you'll probably have to reload and start over. Could you design the spreadsheet program in such a way that you can say "Oh, I meant something else then" and have all your subsequent work update automatically? We'll talk about how to make data structures retroactively changeable, without worrying too much about what the data structures actually are.

Homework: Recommended

Prerequisites: None

3.11. Ruthi's Proposals.

Arithmetic Dynamics. (🌶🌶, Ruthi, 4 days)
Let f be a polynomial with coefficients in $\mathbb{Q}$. What kind of things can we say about the sequences

$$q, f(q), f\circ f(q), f\circ f\circ f(q), \ldots$$

[3] "Small" according to some arbitrary attempt to compare quantum computer sizes to classical computer sizes, but still millions of times bigger than any currently-constructible quantum computer.

for any rational number q? If everything were complex, this would fall in the realm of fractals and chaos, but it turns out that arithmetic structure gives us a totally different way of viewing these things. We'll discuss this in class, but only get the opportunity to glance at the rich topic that is arithmetic dynamics.

Homework: None

Prerequisites: None

Schemes. (🌶🌶🌶🌶, Ruthi, 4 days)
Define a commutative ring for me. Go ahead, I dare you. Did you write something down about binary operators that obey certain axioms and so on? I'm here to tell you that this is the *wrong* way to think about commutative rings. The right definition is

A commutative ring is a collection of functions on a space.

Maybe by now you're used to think about polynomials as elements of a ring. But that's not how you first saw them, is it? When you first saw a polynomial, you were probably told it was a function! But what about the integers? They're a ring too, aren't they? What space are they functions on? The answer is: the integers are functions on a *scheme* where its points are the primes. In this class, we'll dissect what this means and why this is the right thing to be thinking about.

Homework: None

Prerequisites: Ring theory.

The Theorem Formerly Known as the Mordell Conjecture. (🌶🌶🌶🌶, Ruthi, 1–3 days)
The premise of arithmetic geometry is this: we can turn questions from algebraic number theory into problems about geometry, and use our geometric knowledge to solve the problem there. The beautiful and surprising way these interplay changed the way people think about number theory and has formed modern research in the field. One of the key theorems is Faltings' Theorem (née the Mordell Conjecture), which states that no curve of genus 2 or larger has infinitely many rational points. In this class we will discuss what this means, some results we can get from it, and some of the key elements of a proof first presented by Bombieri (perhaps touching upon the tour de force which landed Faltings the Fields Medal in 1986).

Homework: None

Prerequisites: Know what an abelian group is. The more algebra you know, the smoother this ride will be.

3.12. Steve's Proposals.

Weird Logics. (🌶🌶, Steve, 2 days)
Consider the sentence "If this statement is true, then Santa Claus exists." Let's call that sentence "P." If P is true, then if P is true, Santa Claus exists; so, since P is true, Santa Claus exists. On the other hand, what if P is false? Well, the only way for an "If A then B"-statement to be false is if A is true and B is false; in this case, since A is "P is true," we have that if P is false then P is true. So clearly P is true and Santa Claus exists. This is a bit of a problem, and has led some people to conclude that natural language is a lot of bunk. But that's no fun. Instead, let's conclude that it is *logic* that is a lot of bunk! And that we should replace it with new, non-bunky logic. In this class, we will de-bunkify logic. Warning: logic is not guaranteed to not be bunk at the end.

Homework: Optional

Prerequisites: None

Automatic Structures. (🌶🌶, Steve, 2 days)
A finite-state automaton is a particularly simple model of computation — much simpler and much weaker than any real programming language. A mathematical structure — say, a group, or ring, or field, or something else entirely — is *automatically presentable* if there is some way of describing it completely using finite-state automata. (We will make this notion precise in class.) This is a very strong condition: structures which are automatically presentable tend to be extremely simple. However, there is a surprising amount of diversity among automatically presentable structures. In this class, we will study automatically presentable structures, and try to understand what it might mean to say that one structure is "more automatically presentable" than another

Homework: Recommended

Prerequisites: None

Related to (but not required for): Models of Computation Simpler than Programming

Open Problems in Logic. (🌶🌶🌶🌶, Steve, 4 days)
What exactly *do* logicians do these days? Well, four things they *don't* do is try to solve the following four problems, because they're too hard:

(1) Consider the set of math problems, partially ordered by relative difficulty. Does the resulting poset have any nontrivial automorphisms? (Computability theory.)
(2) Exactly how hard *is* it to prove that set theory is consistent? (Proof theory.)
(3) What is the largest kind of infinity that can exist before set theory breaks? (Set theory.)
(4) How many different ways can a set of axioms be satisfied. . . ? (Model theory.) And, are we sure. . . ?

This class is really four 1-day classes as opposed to one 4-day class: each day, we'll discuss one problem (in the order listed above); we'll state it precisely, explain why it is interesting and what partial progress has been made. Each day is totally independent - you can come to day 1 only, or days 2 and 4, or whatever! It'll be awesome and terrible! Wheeee!

Homework: None

Prerequisites: Some background in logic or set theory; e.g., what the ZFC axioms are.

Related to (but not required for): Continuum hypothesis

3.13. Susan's Proposals.

Diamond and Forcing and Trees—Oh My! (🌶🌶🌶🌶, Susan, 1 day)
If you've been in my classes this summer, you may find yourself with some leftover questions. Like how do we show that the diamond axiom is consistent with ZFC? Or can we find a set theoretic universe in which Suslin trees don't exist? Let's take an hour, tie up some loose ends, and play with the connections between these two classes.

Homework: None

Prerequisites: Infinite Trees, The Continuum Hypothesis

Stupid Games on Uncountable Sets. (🌶🌶, Susan, 2 days)
Let's play a game. You name a countable ordinal number. And then I name a bigger countable ordinal. We'll keep doing this forever. When we're all done, we'll see who wins. In this class we'll be discussing strategies for winning an infinite game played on ω_1. In particular, we'll talk about how to set up the game so that at any point, neither player has a winning strategy.

Homework: None

Prerequisites: Might help to have seen the ordinals. Talk to Susan if you want a quick intro.

Wedderburn's Cute Little Theorem. (🌶🌶🌶, Susan's Nose Function, 1 day)
Wedderburn's Little Theorem states that any finite domain is a field. So you want a finite domain with zero divisors? Too bad! Finite noncommutative division rings? Not gonna happen! The proof of this adorable theorem will be presented by an appropriately adorable entity: Susan's nose fuction.

Homework: Recommended

Prerequisites: Definition of group, ring, field, and domain.

3.14. Tim!'s Proposals.

Calculus without Calculus. (🌶🌶, Tim!, 2–4 days)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- You have 40 meters of fence, and you want to build a rectangular pen, where three sides of the pen will be made of fence, and the last side will be a preexisting brick wall. What is the largest area that your pen can enclose?
- Sachi is 5'2" tall and Lisa is 159 cm tall, and they are standing 5 cubits apart. You want to run a string from the top of Sachi's head to the top of Asi's head that touches the ground in the middle. What is the shortest length of string you can use?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters way along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest angle of your vision?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Optional

Prerequisites: Calculus (derivatives) recommended

More Evasiveness. (🌶🌶🌶🌶, Tim!, 2 days)
On the last day of Evasiveness, I stated a result of my own. In this class, I'll show you the proof. You've already seen several important tools we'll need; it will just need one last big new idea...

Homework: None

Prerequisites: Evasiveness

4. Classes Co-Proposed by Multiple Staff

Facing the Music. (🌶, Aaron + J-Lo, 3–4 days)
Math and music, friends forever! They go together like...

- Chords and trigonometric identities.
- Scales and rational approximations.
- Sanskrit poetry and Fibonacci numbers.
- Folk rhythms and the Euclidean algorithm.
- Modulation and group theory.

- Timbre and harmonic functions.

This class will be a survey of connections between math and music. Each day we'll talk about a musical concept and then introduce a mathematical idea that some insight into it (we won't necessarily cover exactly the topics mentioned here). Each day will be more or less self-contained.

Homework: None

Prerequisites: None

The Many Ways to the FTA. (🌶🌶🌶🌶, Alfonso + friends, 4 days)
The Fundamental Theorem of Algebra (FTA) says that every polynomial (with coefficients in $\mathbb{C}$) has a complex root. There are many ways to prove it, but they all require to go deep into some field of mathematics. And I do mean *some* field, since it appears that advanced knowledge of any subfield of math produces a proof of the FTA. In every meeting of this class I will give you a crash course on a topic, starting from scratch (sort of) that culminates with a proof of the FTA.

- *A crash course on complex analysis* - Let's redo in one hour some of what Kevin did in two weeks and use analysis to prove the FTA!
- *A crash course in algebraic topology* – Let's redo in one hour some of what Jeff did in one week and use the fundamental group to prove the FTA
- *A crash course in Galois theory* – Let's redo in one hour some of what Mark did in two weeks and use it to prove the FTA.
- *A crash course in symmetric polynomials* – Let's redo in one hour what Newton probably did in one hour and use combinatorics to prove the FTA.
- *A crash course in differential geometry* – Let's do in one hour a chunk of a graduate course in Riemannian geometry and use it to prove the FTA.

Join us in this whirlwind tour which promises to be Fabulously Tantalizing Awesomeness.

Homework: None

Prerequisites: No prerequisites: yet at the same time I do not promise full understanding, so if you have some knowledge of these topics, it will help.

Related to (but not required for): Math

Group Cohomology. (🌶🌶🌶🌶, Don + Ruthi, 2 days)
Cohomology theories are terrific at cutting through messy and cumbersome calculation to get at the heart of where structure looks different. Get far enough and you find them simplifying arguments in every part of mathematics. Say you want to study Galois groups? Then you'd better understand group cohomology! It works like this: you have a group G that acts on an abelian group A. Then cohomology spits out an infinite sequence of groups. . . with tons of information about G and A. This gives us simple and elegant ways to study field extensions and the abelian groups they act on (like elliptic curves) — as long as we understand the theory behind the cohomology.

Homework: None

Prerequisites: Week 1 of Galois Theory

String Theory. (🌶🌶, Sachi + Tim!, 2 days)
Let's say you want to hang a picture in your room, and you are worried that the 2,000 fans you rented to beat the heat will blow it off the wall, so you want to hang it very securely. You start by wrapping string around some nails, like this:

You figure it will be super-duper secure, because you wrapped it around three nails a lot of times. Don, angry about the electricity bills from running all your fans, comes over and pulls one of your nails out of the wall… and your picture comes crashing to the ground! Actually, no matter which nail he removes, it falls! What happened?

In this class, you'll be playing with string to solve puzzles like this one, and we'll explore fundamental groups, homology, and monotone boolean functions.
Homework: Optional
Prerequisites: None

The Brauer Group. (🌶🌶🌶🌶, Steve + Don, 2 days)
"The Brauer group of a field F is the abelian group whose elements are equivalence classes of finite-rank central simple algebras over F, and whose operation is tensor product." We're not really sure what any of that meant, but if you want to find out with us, come to this class. Also, geometry! We might say something about geometry.
Homework: None
Prerequisites: Ring Theory