WEEK 5 CLASS PROPOSALS, MATHCAMP 2015

Voting takes place at `appsys.mathcamp.org` and closes at 11:59pm on Wednesday.

Contents

Alfonso's Classes	1
Anti's Classes	3
Asilata's Classes	4
Chris's Classes	5
Don's Classes	7
J-Lo's Classes	8
Jalex's Classes	9
Jeff's Classes	10
John Suh's Classes	12
Josh Tenenbaum's Classes	12
Kevin's Classes	12
Marisa's Classes	14
Mark's Classes	15
Milica's Classes	17
Mira's Classes	18
Misha's Classes	19
Nancy's Classes	20
Paweł Piwek's Classes	21
Pesto's Classes	21
Ruthi's Classes	23
Sachi's Classes	26
Sam's Classes	27
Steve's Classes	29
Susan's Classes	30
Yuval's Classes	31
Co-taught Classes	32

Alfonso's Classes

Burnside's Lemma. (🌶🌶–🌶🌶🌶, Alfonso, 1–2 days)
How many different necklaces can you build with 6 pebbles, if you have a large number of black and white pebbles? Notice that you won't be able to tell apart two necklaces that are the same up to rotation or reflection. You probably can answer the above question by counting carefully, but what if we are building necklaces with 20 pebbles and we have pebbles of 8 different colours?

There is a nifty little result in group theory that allows us to solve this type of problem very quickly and without a calculator. Come and learn it!

Homework: Recommended

Prerequisites: Basic group theory

Combinatorial Game Theory. (🌶🌶, Alfonso, 4 days)
We have a plate with blueberries, blackberries, and raspberries. We take turns eating them. In your turn, you may eat as many berries as you want, at least one, but they all have to be of the same type. Then it is my turn. Unless you are Marisa, the goal is not to eat a lot of berries: the winner is the person who eats the last one. Will you beat me?

The above is an example of an impartial combinatorial game. There are tons where it came from, and you have probably encountered some. To solve them all, there are basically only two skills that you need to learn. If you want enlightenment, my young grasshopper, avoid spoilers and come to this class. I will motivate the two skills and I will guide you all to figure them out. The beauty of this topic is as much in the final results as it is in the journey, and I do not want to deprive you of the pleasure of discovering it slowly. We will also attack many examples, from easy ones to actual open problems.

Homework: Recommended

Prerequisites: None.

Fractal dimensions. (🌶–🌶🌶, Alfonso, 1 day)
A line has dimension 1, a plane has dimension 2, and the space we live in has dimension 3. Can you think of something of dimension 1.5? What does it mean to have dimension 1.5? Actually, what does it even mean to have dimension 2?

In this class, I will give you one possible definition of dimension and we will compute the dimension of a few objects, including some with non-integer dimensions.

Homework: Recommended

Prerequisites: You need to be comfortable with logarithms and limits.

The Redfield-Pólya Theorem. (🌶🌶🌶, Alfonso, 2 days)
If you have taken a group theory class, or maybe some combinatorics class, you are probably familiar with Burnside's Lemma (a.k.a. many other things): If G is a group acting on a set A, then the number of orbits of the action is given by

$$\frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix}(g)| \tag{1}$$

It is a nice result, but not enough. For instance, how many different necklaces can you build with 20 beads out of very large supplies of red, green, and blue beads? With the help of (1), it will take you less than 5 minutes to calculate that the answer is 87230157 with just pen and paper. But what if I asked you to tell me how many such necklaces are there with R red beads, B blue beads, and G green beads, for *each* triple of values (R, B, G)? If you think there is no way to avoid using brute force to count in this problem, think again! You can still answer it in less than 5 minutes, but you will need the full force of Redfield-Pólya. Come receive it!

Homework: Recommended

Prerequisites: You need to understand (1) and its proof. One way to do this is to take my class on Burnside's Lemma.

The Stable Marriage Problem. (🌶, Alfonso, 1 day)

N single men and N single women want to pair up and get married. These are their names and preferences:

- Chris: Marisa > Ruthi > Nancy > Susan > Angela
- Don: Angela > Marisa > Susan > Ruthi > Nancy

- Jeff: Angela > Susan > Ruthi > Marisa > Nancy
- Kevin: Nancy > Ruthi > Susan > Angela > Marisa
- Pesto: Nancy > Angela > Marisa > Susan > Ruthi

- Angela: Chris > Kevin > Pesto > Jeff > Don
- Nancy: Jeff > Don > Chris > Pesto > Kevin
- Marisa: Jeff > Kevin > Pesto > Don > Chris
- Ruthi: Jeff > Pesto > Chris > Don > Kevin
- Susan: Jeff > Kevin > Don > Pesto > Chris

Is it possible to make everybody happy? Obviously not since almost everybody wants to marry Jeff. But is it possible to at least create a stable situation? For instance, it is a bad idea for Jeff to marry Ruthi and for Susan to marry Kevin, because then Jeff and Susan would prefer each other rather than staying with their partners, so they will run away together. How can we at least avoid having a run-away couple? Is there more than one way to do it? What is the best way to do it? And what if, following the recent Supreme Court Ruling, Marisa and Ruthi decide that marrying each other is better than marrying Jeff?

Homework: None

Prerequisites: None.

When sequences are useless: Topology done right. (🌶🌶🌶🌶, Alfonso, 4 days)
On any topological space we can talk about convergence of sequences. In the standard topology in $\mathbb{R}^n$ sequences are enough: everything topological you want to do, we can do it with sequences. However, for other topological spaces, sequences are useless. Examples:

- In the standard topology, a set A is *sequentially closed* (i.e., if a sequence in A converges, the limit has to be in A) iff it is closed. This is false in general.
- If we stick to the standard topology, a map is continuous (i.e., the preimage of an open set is open) iff it is sequentially continuous (i.e., the image of a convergent sequence is convergent). This is false in general.

What to do? We need to substitute a better notion of convergence for sequences. The three standard ones are nets, filters, and waffles. In this class we will master waffles.

As a corollary, we may prove the Axiom of Choice on the last day.

Homework: Required

Prerequisites: Some point-set topology; specifically, you need to know the notions of continuous map, closed set, compact set, and convergence of a sequence in an arbitrary topological space. Nancy's class suffices.

Anti's Classes

Differentials and higher differentials. (🌶🌶–🌶🌶🌶, Anti, 1 day)
A differential is a thing like $\mathrm{d}x$ or $\mathrm{d}y$. You've probably seen them appearing in pairs in Leibniz' notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the derivative, but appearing singly they are rarely given much attention in calculus classes. However, a highfalutin' generalization of them called "exterior differential forms" plays an important role in differential geometry.

A *higher* differential is a thing like d^2y or $\mathrm{d}x^2$. You may have seen them appearing pairwise in Leibniz' notation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for the second derivative, but for the most part they have been banished from appearing singly in modern mathematics. However, this prejudice is unjustified! I'll explain about how to make sense of them, and why they make life much easier when computing higher derivatives and integrals. In particular, we'll see that Leibniz' notation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is actually *wrong*!

Homework: None

Prerequisites: Calculus. Multivariable calculus recommended, but not required.

Asilata's Classes

A Dynkin diagram miscellany. (🌶🌶–🌶🌶🌶, Asilata, 1–2 days)
Dynkin diagrams are a few innocuous graphs that show up in a whole bunch of mathematical classification problems.

In this class, we'll take a stab at a couple of problems that are not at all about graphs, but for which (spoiler alert!) Dynkin diagrams ultimately come along and save the day.

There is no actual overlap between this class and Lie Algebras or Reflection Groups, so you can show up whether or not you took those classes!

Homework: Recommended

Prerequisites: Linear algebra. Group theory recommended but not required.

Related to (but not required for): Ring Theory (W1); Reflection Groups (W2); Lie Algebras (W3); Representation Theory of Finite Groups (Week 1 of 2) (W3); Advanced Linear Algebra (W4); Representation Theory (Week 2 of 2) (W4); Algebraic Groups (W5)

All aboard the complex projective plane! (🌶🌶–🌶🌶🌶🌶, Asilata, 1–2 days)
If you've read Sachi's class blurb about the projective plane and are confused about why you're still reading mine, don't be turned away by the similarity!

The complex projective plane (written $\mathbb{C}\mathbf{P}^1$) is a space that is constructed by adding a single point at infinity to the real plane. The name and construction are similar to the real projective plane, but they give you something strikingly different.

$\mathbb{C}\mathbf{P}^1$ is a space absolutely fundamental to modern mathematics, but which was also studied more than a thousand years ago. Depending on your mood, interests, and backgrounds, we could put on our Euclidean geometry hats, differential geometry hats, our algebraic geometry hats, our symplectic geometry hats, or our topology hats to study $\mathbb{C}\mathbf{P}^1$. Perhaps we can do all of these things! Come get your hands dirty with this space that mathematicians just can't live without.

Homework: Recommended

Prerequisites: Ring theory helpful but not necessary. Calculus helpful but not necessary.

Dunkin' Donuts, aka Morse Theory. (🌶🌶🌶–🌶🌶🌶🌶, Asilata, 1–4 days)
Topologists often want to study complicated spaces by breaking them up into pieces. But there's a problem with this idea: it's not at all clear *how* to break up the space into nice, easy pieces.

Turns out the answer is dunkin' donuts. I don't mean the coffee shop! I mean actually dunking your donuts into your coffee.

The real answer is that Morse theory in some sense captures what happens when you slowly dunk a donut into a cup of coffee, and uses that information to break up your space into convenient chunks. We can even prove that the chunks are convenient, whatever that means. Time permitting, I will also tell you how I use Morse theory in my own research.

Homework: Recommended

Prerequisites: Point-set topology, Multivariable Crash Course. It is helpful (but not necessary) to know what a manifold is.

Extensions. (🌶🌶🌶🌶, Asilata, 2 days)
Take any two abelian groups, for instance $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Then the product group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has the

property that $\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is an injective homomorphism, and $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is a surjective homomorphism, and the kernel of the second map is exactly the image of the first map.

Based on this example, we define an *extension* of H by G to be any new abelian group E for which there is an injective homomorphism $G \hookrightarrow E$ and a surjective homomorphism $E \twoheadrightarrow H$, and the kernel of the second homomorphism is precisely the image of the first.

Are there any other extensions than the "product extension" shown above? What does "other" mean? How many are there, and what do they all look like?

We'll think about all of this. But first, hang on tight, it's about to get meta: not only are there (sometimes) lots of different extensions, but the *set of extensions* itself forms an abelian group! Doesn't that sound pretty? Let's learn about it!

Homework: Required

Prerequisites: None.

Learn to code! (🌶, Asilata, 1–4 days)

This class will be a gentle introduction to programming and basic principles of coding, in one of my favourite programming languages: Haskell. If you know how to code in another language, this class is probably not for you. (You can feel free to show up to learn the basics of Haskell, but remember that it will be gentle.)

If you haven't coded much or never coded at all before, this class is definitely for you! We'll start with some basic general principles of coding and do hands-on examples before moving on to Haskell-specific features. Haskell is a language that is particularly satisfying if you are mathematically inclined, and we'll see why. We'll also discover that Haskell is lazy, and how we can exploit that to our advantage.

Here is some bonus pretty code taken straight from the Haskell homepage. The following lines generate an *infinite list* (yes you read that right!) called "primes". This is exactly a list of all the prime numbers, obtained by implementing Eratosthenes' sieve!

```
primes = filterPrime [2..]
  where filterPrime (p:xs) = p : filterPrime [x | x <- xs, x `mod` p /= 0]
```

Homework: Recommended

Prerequisites: None

The Robinson-Schensted Correspondence. (🌶–🌶🌶, Asilata, 1 day)
The Schensted algorithm is a simple and beautiful recipe that turns permutations into pictures, also known as standard Young tableaux.

The algorithm is tricky, but simple to describe. Amazingly, it can be turned into a reversible procedure that sends any permutation to a *pair* of Young tableaux and vice-versa. This is known as the Robinson–Schensted correspondence, which has applications into some unexpectedly deep mathematics.

Expect a fun hour with delightful combinatorial surprises and lots of examples. It'll be a treat.

Homework: Recommended

Prerequisites: None. Group theory helpful but not required.

Chris's Classes

Abstract Nonsense. (🌶🌶🌶🌶, Chris, 2–4 days)
So you've learned some category theory but you still feel like that's too concrete for you? Then this

class is the answer! Abstract Nonsense is the semi-official title of categorical proving-techniques that on first sight feel completely non-sensical, but apply to a wide range of situations due to their abstract nature.

One aspect of this are so-called simplicial sets — the most abstract and weird way to describe topological spaces. In this class we will be studying these simplicial sets and maybe in the end use them to define the classifying space of a category. Yes, that's right – we will turn a category into a space. Why? Because we can! And also because we then can define K-Theory (maybe).

Homework: Recommended

Prerequisites: Category Theory (Categories, Functors)

(Co)Homology — How it all began. (, Chris, 4 days)

There are many reasons why you would be interested in this class: Maybe you were in Ruthi's Galois Cohomology or Jeff's Fundamental Theorem on Manifolds class and are confused why everyone gets so excited about cohomology. Or you were in Sachi's Fundamental Group or my Homotopy Theory class and are wondering if there's an easier way to tell two spaces apart since homotopy groups are really hard to calculate. Or you really just want to learn some algebraic topology and are wondering how one would strictly show why two spheres of different dimension cannot be deformed into each other.

(Co)Homology was historically first defined in the context of topology: Given a topological space X and a group G, we can assign to them many groups $H^n(X, G)$ such that homotopy equivalent spaces (that is spaces that can be transformed into each other) have the same (co)homology groups. Although these groups are somewhat hard to define, they are fairly easy to calculate and extremely powerful in telling non-equivalent spaces apart. They have proven so useful that mathematicians are making great efforts to define cohomology theories in many different areas of mathematics such as motivic cohomology in algebraic geometry, Heegaard-Floer cohomology in knot theory or Galois cohomology in number theory.

In this class, we will learn the basic definition of the (co)homology groups of a topological space and we will see how powerful they are and why everyone wants to define something similar in their field.

Homework: Recommended

Prerequisites: Basic group theory, some notion of topological space and continuity

∞-Categories. (, Chris, 2–4 days)

Doing math you will very often run across the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{e} & D. \end{array}$$

In a perfect world this will always commute, meaning $h \circ f = e \circ g$. But unfortunately the world is not always perfect. So you hope for the next best thing: You hope that $h \circ f$ can be transformed (i.e. by a homotopy) into $e \circ g$. Luckily this does happen fairly often which makes you happy. But your usual category theory does not give you good tools to deal with this almost-commuting-situation which makes you unhappy. And that's where ∞-categories come in. They give you exactly the right context for doing category-theory "up to transformations" and provide a theory so flexible that an ∞-category can describe anything between a topological space and a classical category.

Homework: Recommended

Prerequisites: Category Theory (Categories, Functors)

Don's Classes

1D Manifolds. (🌶🌶🌶, Don, 1 day)
Manifolds are at the heart of modern topology; structured enough that we know what they look like locally, yet loose enough to cover a wide variety of applications. So, like many things in modern math, they're really hard to define.

That is, unless you restrict your attention to the one dimensional case.Then, the definition is pretty easy to work with, and we'll even have time to classify them all — a straightforward result used every day by low dimensional topologists.

Homework: None

Prerequisites: Calculus, Topology

Abelian Groups. (🌶🌶🌶, Don, 1–2 days)
I love group theory, but I don't love when elements of groups don't commute with each other; that's not good for the environment!

Even with all of our groups being abelian, though, algebraists are still far from a complete understanding. What we can do is look at a variety of different extra restrictions, and classify the abelian groups meeting those restrictions; say, the finitely generated abelian groups, or the divisible abelian groups.

Homework: None

Prerequisites: Group Theory

Related to (but not required for): Introduction to Groups (W1); Galois Cohomology (W2); Profinite groups (W5); What is the deal with Homology? (W5)

A Card Trick, and a Set. (🌶🌶, Don, 1 day)
The Gilbreath principle says that when you shuffle a deck a certain way, much of its structure is preserved; you can use the similarity of the shuffled deck to the original to perform a number of magic tricks.

The Mandelbrot set is a marvelously complicated set in $\mathbb{C}$ that is infinitely self-similar yet not regular.

These two notions of self similarity are, in a certain sense, the same! We'll see why in this class.

Homework: None

Prerequisites: None

Related to (but not required for): Mathematical Magic (W3)

Algebraic Groups. (🌶🌶🌶🌶, Don, 1–2 days)
Question: What do group theory, linear algebra, lie algebras, lie groups, reflection groups, algebraic geometry, and category theory all have in common?

Answer: Algebraic Groups.

In particular, algebraic groups formalize the idea that there is an analogy between $GL_n(\mathbb{R})$ the group of invertible $n \times n$ matrices over $\mathbb{R}$, $GL_n(\mathbb{C})$ the group of invertible $n \times n$ matrices over $\mathbb{C}$, and $GL_n(\mathbb{Z})$ the group of invertible $n \times n$ matrices over $\mathbb{Z}$. The objects you get out are like groups, parameterized over rings, and you can do wonderful things with them (including taking Galois cohomology, and giving an analogous classification to Lie Algebras).

Homework: None

Prerequisites: Group Theory, Ring Theory, Linear Algebra

Related to (but not required for): Galois Cohomology (W2); Reflection Groups (W2); Lie Algebras (W3); Classifying Spaces (W3); Category Theory in Sets (W4); A Dynkin diagram miscellany (W4)

A Mathematician Goes to Vegas. (🌶🌶–🌶🌶🌶, Don, 1 day)
Many would say that someone who knows math would never gamble.

Some would say that someone who knows math would only gamble if there's a positive expected value.

I would say that simply by existing, we have no choice but to constantly gamble, and so you might as well learn how to do it right.

Learn how to do it right. Come to this class.

Homework: None

Prerequisites: Calculus

Stone Spaces and Boolean Algebra. (🌶🌶🌶🌶, Don, 1 day)
Boolean algebras are like normal algebras, but variables can only take on the values 1 and 0.

A Stone space is a totally disconnected compact Hausdorff topological space.

There's no difference.

Don't believe me?

Too bad.

There's nothing you can do about it (unless you want to disbelieve the axiom of choice).

Homework: None

Prerequisites: Point Set Topology (enough to understand the definition above)

J-Lo's Classes

Cosine Waves on Musical Staves. (🌶–🌶🌶, J-Lo, 1–2 days)
Facts:

- You can hear a trigonometric identity (I used this one to tune the piano in the main lounge).
- The continued fraction expansion of $\log_2 3$ contains a sumamry of the development of musical scales.
- The four-chord harmonic progression you hear in almost every pop song can be interpreted via the symmetries of a regular dodecagon.
- Beethoven's Ninth Symphony almost traces out one of the generators of the fundamental group of a torus.

You will see some subset of these facts explained in greater depth.

Homework: Recommended

Prerequisites: Group Theory (for the latter two facts)

Related to (but not required for): Introduction to Groups (W1); Time-Frequency Analysis (W1); Continued Fractions (Week 1 of 2) (W2); Continued Fractions (Week 2 of 2) (W3); Can You Hear the Shape of a Drum? (W3)

Prime non-numbers. (🌶🌶🌶–🌶🌶🌶🌶, J-Lo, 1–4 days)
You thought integer prime numbers behaved weird? Well you don't have to travel too far from the integers to find numbers which, on the one hand, can't be divided into smaller factors, but on the

other hand, can divide a product ab without dividing either a or b. Yep, it's basically like math is just out to troll us.

In this course, a speedy intro to Algebraic Number Theory, we'll see why numbers stop being nice to us, and why we have to make two new friends to help us explore primality and divisibility: ideals and absolute values.

(P.S. To those of you who attended my Absolute Values class…sorry guys, there actually ARE more places. We only found the rational ones.)

Homework: Recommended

Prerequisites: Intro Ring Theory

Related to (but not required for): Ring Theory (W1); Absolute Values: All the Other Ones (W3)

Jalex's Classes

More Complexity: Randomness. (🌶🌶–🌶🌶🌶, Jalex, 1–2 days)
Can a computer that uses polynomial time and polynomially many coin flips solve more problems than a computer that uses polynomial time but doesn't flip coins? Can it solve as many problems as a computer that uses exponential time?

We don't know the answer to either of these questions, but we do know some things. We'll see why randomness is really useful for problems that can be phrased in terms of polynomial rings over finite fields. By using a Wigderson extractor, we can give an explicit upper bound on how strong these computers are.

Homework: Recommended

Prerequisites: Some familiarity with the notion of a complexity class

More Complexity: Why Counting is Really Hard. (🌶🌶🌶–🌶🌶🌶🌶, Jalex, 1–2 days)
Fact 1: There is a polynomial time algorithm to determine whether a given graph can be partitioned into vertex-disjoint cycles (such a partition is known as a cycle cover). There probably isn't a polynomial-time algorithm to determine whether a given Boolean formula is satisfiable. However, counting the number of cycle covers of a graph is just as hard as counting the number of satisfying assignments to a Boolean formula.

Fact 2: Counting cycle covers is as hard as any language describable in second-order logic.

In this class, we'll make the first fact precise and provide a proof, and then give a brief overview of a sketch of an outline of the statement and proof of the second fact, which is known as Toda's theorem.

Homework: Recommended

Prerequisites: Know what NP-complete means.

Qualifying Quiz Problem 6. (🌶🌶, Jalex, 1 day)
Problem 6 on the Qualifying Quiz this year asked you to find an optimal strategy for an n-player game when n was a power of 2. It turns out that the induction part of the intended solution was nonessential — many campers found that more-or-less the same strategy works for any even n. More surprisingly, somebody submitted a nonconstructive proof that there is an optimal strategy for all n. The proof is clean — it's just a few key lemmas about directed multigraphs.

Homework: Recommended

Prerequisites: Know what a directed graph is.

Surreal Numbers. (🌶, Jalex, 1 day)
If you've ever constructed the reals, then you know it's kind of a drag:

- Start with 0.
- Put in 1.
- Use addition to get $\mathbb{N}$
- Use subtraction to get $\mathbb{Z}$.
- Put in fractions to get $\mathbb{Q}$.
- Complete the Cauchy sequences to get $\mathbb{R}$.

That's a big number of steps! In this class, we'll give a single definition that yields not only the reals, but a much richer extension that has a proper subfield of every cardinality. (In other words, it's *huuuge*.)

Homework: Recommended

Prerequisites: None.

Required for: Ordinal Arithmetic (W3)

Tarski's Theorem on Paradoxical Sets. (🌶🌶🌶🌶, Jalex, 1–2 days)
The Banach–Tarski theorem says this: the unit ball in $\mathbb{R}_3$ is SO_3-paradoxical, where SO_3 is the group of isometries of $\mathbb{R}_3$. If you've seen a proof, you know that the basic strategy is to break the ball into pieces which aren't measurable, move them around in ways that confuse the Lebesgue measure, and then stick them back together again. Question: is this the only kind of argument that can show a set is paradoxical? Answer: yes! More precisely, we have the following theorem:

Let G be a group acting on a set X. A subset $A \subseteq X$ is not G-paradoxical iff there exists a G-invariant finitely additive measure $\mu : \mathcal{P}(X) \to [0, \infty]$ such that $\mu(A) = 1$.

In order to get there, we'll need to learn things about ordered abelian semigroups and perfect matchings of bipartite graphs. prereqs

Homework: Recommended

Prerequisites: If you've seen a proof of Banach–Tarski before, this class will be more meaningful to you.

Related to (but not required for): The Banach–Tarski Paradox (W2); Lebesgue Measure (W3)

Jeff's Classes

Combinatorial Topology. (🌶🌶🌶, Jeff, 3–4 days)
So, you want to be a topologist. But, you've never taken point-set topology[1]. How much can you prove about topology?
Turns out, quite a bit. In this class, we'll be developing simplicial complexes, which give combinatorial representations of topological spaces. Then, we'll look at discrete Morse theory, a combinatorial representation of a construction from differential topology. Along the way, we'll draw lots of pictures and diagrams, and get a feel for what topology *should* do, without messing around with all of those icky open sets.

Homework: Recommended

Prerequisites: None.

Polynomials and Graphs. (🌶, Jeff, 2 days)
The Tutte polynomial, $T_G(s, t)$, is a magical polynomial associated to a graph G. By plugging numbers

[1] Disclaimer: I've never taken point-set topology

into s and t, we amazingly start counting things:

$T_G(1, 1/2) =$Related to the probability that G is connected if I remove every edge with probability 1/2.
$T_G(2, 1) =$Number of Trees contained within G.
$T_G(3, 3) =$If G is a grid, the number of ways of covering it with tetrominoes
$|T_G(5, 0)| =$Number of 5 colorings of G.

Homework: Recommended

Prerequisites: Graph Theory

The Quadrature. (🌶🌶, Jeff, 1–2 days)
Here is a fun fact:

> "Every nontrivial knot has a straight line that intersects it at 4 or more points."

What — How weird is that!?

Homework: Recommended

Prerequisites: Fundamental Groups, or willingness to believe a small result.

Related to (but not required for): Fundamental Group (W2); Intro Knot Theory (W2)

What is the deal with Homology? (🌶–🌶🌶🌶🌶, Jeff, 4 days)
Usually in mathematics, when a theorem is false, it is a bad thing. Here are a couple of theorems (3 of which are false.)

(1) (🌶) Let X be a set. If $X = A \cup B$, then the size of X is given by the size of A plus the size of B minus the size of $A \cap B$.
(2) (🌶🌶) Let M be a space. If $M = A \cup B$, then the number of connected components of M is the components of A plus the components of B minus the components of $A \cap B$.
(3) (🌶🌶🌶) Let C be an abelian group. Suppose C is generated by subgroups A and B. Then homomorphisms from C to another group D are given by homomorphisms from A to D, plus the homomorphisms from B to D, minus the homomorphisms from $A \cap B$ to D.
(4) (🌶🌶🌶🌶) Let G be a graph. Suppose $G = G_1 \cup G_2$. Then the number of edges that disconnect G is equal to the number of edges that disconnect G_1 plus the number of edges that disconnect G_2 minus the number of edges that disconnect $G_1 \cap G_2$.

If you are sad because theorems are false, don't be. In fact, the falseness of these theorems should make you happy! By measuring the falseness of each of these theorems, we learn new and wonderful facts about sets, groups, graphs and topological spaces. The tool we need to make this process work is homology, an algebraic framework developed for topology which now shows up in nearly every field of mathematics.

Homework: Recommended

Prerequisites: You'll need to know what a vector space, dimension, kernel and image are. For day 2, no knowledge of topology is required, but may be helpful. For day 3, some basic knowledge of groups. For day 4, intuition on what a tensor product of vector spaces is, and definition of a graph coloring.

Related to (but not required for): Galois Cohomology (W2); Coloring Maps (W2); Fundamental Theorem of Calculus in Dimension n (W3); Putting the Algebra in Algebraic Topology (W4); Abelian Groups (W5)

John Suh's Classes

The monsters of the Axiom of Choice. (🌶🌶, *John Suh*, 1 day)
You have probably heard of the Axiom of Choice and of its controversy. You might even have heard of its statement: given any family of disjoint, non-empty sets, there exists a set containing one (and only one) element from each of the sets. In another words, one can choose one element from each set. Why would such an intuitive-sounding axiom be so controversial?

What paradoxes occurs under the assumption of AC, and what would we lose without it?

In this one day class, we will explore such questions. We will learn about the monsters of the Axiom of Choice, the results that violate everyday intuition.

Homework: Recommended

Prerequisites: If you have heard of the Axiom of Choice, then its good. We need basic linear algebra (the notion of basis) for one of the examples.

Josh Tenenbaum's Classes

Building a machine that learns like a human. (🌶🌶, *Josh Tenenbaum*, 1 day)
People can learn a new concept almost perfectly from just a single example, yet machine learning algorithms typically require tens or hundreds of examples to perform similarly. People can also use what they learn in richer ways than most machines do — to guide their action, imagination, and explanation. I will talk about how we can build machine learning algorithms that better capture these human learning abilities. Our approach represents concepts as simple programs that can generate observed examples, and learning can be described as a search for the program that best explains what you see. I will show results from a series of "visual Turing test" experiments probing this model's ability to classify as well as to create new instances of concepts, showing that in many cases it is indistinguishable from the performance of humans. Time permitting, I will also talk about other exciting recent work in AI that aims to build machines with human-level learning abilities, such as the Atari Video Game player from Google DeepMind.

Homework: None

Prerequisites: Previous classes on probability, statistics, machine learning, Bayesian inference, or Monte Carlo would be helpful but are not necessary.

Kevin's Classes

Analytic Number Theory. (🌶🌶🌶🌶, Kevin, 2–4 days)
Mathematicians have studied the integers and the prime numbers for thousands of years, and Euclid's proof that there are infinitely many prime numbers is still the standard one that everyone knows.

But that's not the number theory we're doing in class. With the development of modern analysis and algebra, we can prove tremendously stronger results. For example, Dirichlet's Theorem states that, given two relatively prime positive integers a and n, there are infinitely many primes congruent to a mod n, a vast generalization of the simple fact that there are infinitely many primes.

To prove things like this, we'll use complex-analytic techniques and ideas to study functions like the Riemann zeta function, and we'll see how the analytic properties of these functions tell us about integers and the primes.

Depending on the length of the class and interest, we will prove one or both of the following:

(1) Dirichlet's Theorem, described above.
(2) The Prime Number Theorem, and how the Riemann Hypothesis gives us the best possible error term.

Homework: Optional

Prerequisites: complex analysis, group theory (if we do Dirichlet's Theorem)

A Very Difficult Definite Integral. (, Kevin, 1 day)
In this class, we will show

$$\int_0^1 = \frac{\log(1+x^{2+\sqrt{3}})}{1+x}\,dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3}).$$

We'll start by turning this Very Difficult Definite Integral into a Very Difficult Series to sum. Then we'll sum it!

We'll need a healthy dose of clever tricks. Sadly, these tricks will not include differentiation under the integral sign, or any of the useful things I taught in Summing Series. Instead, we'll use some heavy-duty algebraic number theory. It will be Very Difficult.

(This class is based on a StackExchange post by David Speyer.)

Homework: None

Prerequisites: A strong background in algebraic number theory, or a willingness to believe in the facts that I'll use. Familiarity with rings and ideals (specifically rings of integers) will help.

Cluster Algebras of Invariants. (, Kevin, 3–4 days)
Here's a question to think about:

Let S and T be k-element subsets of $\{1, 2, \ldots, n\}$. Call S and T *weakly separated* if there do not exist $a, c \in S - T$ and $b, d \in T - S$ such that $a < b < c < d$ or $d < a < b < c$. For example, $\{1, 3, 5\}$ and $\{2, 3, 4\}$ are weakly separated, but $\{1, 3, 4\}$ and $\{2, 3, 5\}$ are not, as $1 < 2 < 4 < 5$. In other words, to be weakly separated, there are no pairs of elements just in S and just in T that interlock.

Let's start with a k-element subset of $\{1, 2, \ldots, n\}$, and let's keep adding k-element subsets which are weakly separated from each of the previous selections. Eventually, we won't be able to do this anymore. How big is the resulting collection of sets? Is it always the same size? How many such collections are there?

While you're thinking about that question, here's some history. One of the major results early on in cluster algebras is Jeanne Scott's proof in her PhD thesis that polynomial vector invariants have the structure of a cluster algebra. In this class, I'll introduce the basics of invariant theory as studied by Hermann Weyl almost 100 years ago, then I'll dive into the beautiful combinatorial structure that emerges when studying these invariants as a cluster algebra.

Time permitting, I'll also discuss the main results of my own PhD thesis which extend this whole story to the wilder world of mixed invariants, where everything is harder but, in the end, fits together all the more beautifully.

Oh, and we'll see why all of this cluster algebra nonsense solves the purely combinatorial questions I posed at the start of the blurb.

Homework: Optional

Prerequisites: Laurent Phenomenon

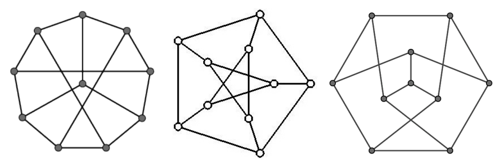
Posets. (, Kevin, 4 days)
What do the principle of inclusion-exclusion, divisors, graph colorings, and polytopes have in common? We can study and prove results about these things using partially ordered sets! And the list goes on and on: subgroup structures, partitions, permutations... so many things come naturally with a partial ordering!

Unlike a total ordering, we can't always compare two elements in a poset. For example, it's natural to order sets based on containment, but then the sets $\{1, 2\}$ and $\{1, 3\}$ are incomparable—neither

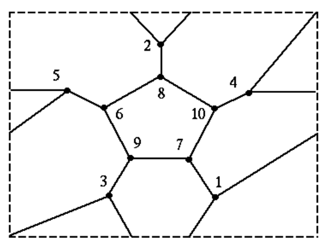
contains the other. Even though it might seem like a partial order doesn't provide much structure, we'll see that we can do quite a lot of magic with what seems to be very little!
Homework: Optional
Prerequisites: None.

Marisa's Classes

Crossing Numbers. (🌶🌶, Marisa, 2–4 days)
Here is the Petersen graph. My drawing (below) displays pretty symmetry, but has 6 edge crossings. I prefer to draw my graphs with as few edges crossing as possible, so by exhibiting different symmetries, we can draw the same graph with five crossings, or even three. (The best possible is actually two crossings; can you do it?)



But: if, instead of drawing on the plane, we drew the Peterson graph on the torus, we could do it with no edge crossings at all:



In this class, we'll discuss the minimum number of edge crossings for a graph drawn on the plane, the torus, and lots of other surfaces, and we'll find very quickly that most questions in crossing numbers are wide open. In fact, even a question as basic as the crossing number of $K_{m,n}$ on the plane is open! An answer to this question was conjectured—and a proof claimed—in the 1950s by Zarankiewicz, leading to a 1969 paper entitled "The decline and fall of Zarankiewicz's theorem."
Homework: Optional
Prerequisites: some exposure to Graph Theory; basic probability (like linearity of expectation) for the probabilistic proofs on the second day
Related to (but not required for): Coloring Maps (W2); Graphs on Surfaces (W3)

Latin Squares. (🌶–🌶🌶, Marisa, 1–4 days)
Do you remember Sudoku, the little popular logic puzzle? It went like this: take a 9×9 grid and fill it with the numbers 1 through 9 so that each digit appears once in every row and every column. (And there was some constraint about subsquares, which I will ignore.) If I give you two different Sudokus, with different fills of the 9×9 grid, you can superimpose one on top of the other and look at the pairs of symbols you've produced from the matching squares. Heres a deeper puzzle: can you find two Sudoku fills that are so different from each other that when you superimpose them, you get all 9^2 possible pairs? Great. Now can you find *eight* different Sudoku grids, with the property that *every* pair of grids produces all 9^2 possible pairs? No need to do it by hand — we can hit this with the hammer of modular arithmetic. The result is a set of *mutually orthogonal Latin Squares* (MOLS), a combinatorial object that turns out to be related to finite geometries, designing schedules, building codes, and all kinds of other combinatorial objects.

Homework: Optional

Prerequisites: None

Multitudes of Camper-Sarong Permutations. (🌶🌶, Marisa, 1 day)
If we have ten Mathcampers and ten sarongs, out of all of the 10! ways of matching the campers and the sarongs, what is the probability of matching them all wrong (so that nobody gets their own sarong back? Which is more likely — getting them all wrong, or getting at least one right?

This short class on the Hat Check Problem will use my favorite approach: counting the number of 1-factors in the graph $K_{n,n}$ minus a 1-factor by solving a recurrence relation.

Homework: Recommended

Prerequisites: Basic graph theory, plus either familiarity with the Taylor Series for e^x or willingness to take 5 minutes of calculus on faith.

Mark's Classes

Elliptic functions. (🌶🌶🌶, Mark, 4 days)
Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{k=1}^{n-1}\sigma_3(k)\sigma_3(n-k),$$

where $\sigma_i(k)$ is the sum of the i-th powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1] ,$$

which you are welcome to check if you run out of things to do.)

Homework: Recommended

Prerequisites: Functions of a complex variable, in particular Liouville's Theorem.

Finite Fields. (🌶🌶🌶, Mark, 1–2 days)
You may well know that the integers modulo p form a field (basically, a set in which all four arithmetic operations are possible, including division limited only by Rule 4) if and only if p is prime. However, there are other finite fields whose sizes are not prime. Finite fields have applications in areas such as coding theory and cryptography as well as in more abstract mathematics, and they are elegant algebraic objects that are well worth exploring. In this class we'll see how to construct finite fields, and if time permits, show that we have found them all.

Homework: Optional

Prerequisites: Some ring theory (in particular, polynomials, and rings modulo ideals) and a bit of linear algebra (the idea of dimension)

From Counting to a Theorem of Fermat. (🌶🌶, Mark, 1 day)
A standard theorem stated by Fermat (it's actually uncertain whether he had a proof) states that every prime p congruent to 1 modulo 4 is the sum of two squares. (On the other hand, if p is 3 modulo 4, it has *no* hope of being the sum of two squares.) There are many proofs of this theorem, but perhaps the weirdest one, due to Heath-Brown and simplified by Zagier, uses just counting — no "number theory" at all! In this class we'll see at least that proof, and maybe some others and/or related proofs of other things.

Homework: None

Prerequisites: None!

Multiplicative Functions. (🌶🌶–🌶🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such multiplicative functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.

Homework: Optional

Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)

Primitive roots. (🌶🌶, Mark, 1 day)
Suppose you start with 1 and keep multiplying by a modulo n, where a and n are relatively prime positive integers. As it turns out, you will always get back to 1. But will you have seen all the integers k with $\gcd(k, n) = 1$ by then, as part of the "number wheel" you just made? In this class we'll explore when (that is, for what values of n) you can find an a such that every integer modulo n that's relatively prime to n shows up on that single wheel (such an a is called a *primitive root* mod n). We may not get much beyond the case that n is prime, but even in that case the analysis is interesting. In particular, we'll be able to show that a exists in that case without having any idea of how to find a, other than the flat-footed method of trying $2, 3, \dots, n-1$ until we find a number that works.

Homework: None

Prerequisites: A little elementary number theory.

Quadratic Reciprocity. (🌶🌶–🌶🌶🌶, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

Q1: "Is q a square modulo p?"
Q2: "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square mod 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Homework: Optional

Prerequisites: Number theory through Fermat's little theorem.

The Magic of Determinants. (🌶🌶, Mark, 3 days)
Determinants, which are numbers associated to square matrices, have many useful properties; for example, they are needed to give a general formula for the inverse of a matrix. Unfortunately, determinants are often defined in a very *ad hoc* way (using Laplace expansion) which may obscure what is really happening. This class will give a "better" theoretical framework as well as some geometric intuition, and I'll try to at least give an outline of the proofs of all the main computational properties of determinants, such as the Laplace expansion.

Homework: Optional

Prerequisites: Matrix multiplication, and the idea of a linear transformation.

The Pruefer Correspondence. (🌶🌶, Mark, 1 day)
Suppose you have n points around a circle, with every pair of points connected by a line segment. (If you like, you have the complete graph K_n). Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments (without changing direction except at the points on the circle), but in only one way. (This tree will be a spanning tree for K_n.) How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll go through a combinatorial proof that is especially cool.

Homework: None

Prerequisites: None!

Wedderburn's Theorem. (🌶🌶🌶, Mark, 1 day)
Have you seen the quaternions? They form an example of a division ring that isn't a field. (A *division ring* is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j.$$

Have you seen any examples of *finite* division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Homework: None

Prerequisites: Some group theory and some ring theory; familiarity with complex roots of unity would help.

Milica's Classes

Cyclotomic Polynomials and Extensions. (🌶🌶🌶–🌶🌶🌶🌶, Milica, 1 day)
Definition: The n-th cyclotomic polynomial is the polynomial whose roots are the primitive n-th roots of unity.

Theorem: The regular n-gon can be constructed by straightedge and compass if and only if n is the product of a power of 2 and distinct Fermat primes.

Find out how these two things are related!

Homework: Recommended

Prerequisites: Familiarity with field extensions and with complex roots of unity.

Mira's Classes

Abel's Theorem. (🌶🌶🌶, Mira, 4 days)
Let's prove this thing!

Homework: Required

Prerequisites: Abel's Theorem, Weeks 1-4

A game you can't play (but would win if you could). (🌶🌶, Mira, 1 day)
Once upon an infinity, in the Kingdom of Aleph, King Alephonso decided to put his 100 advisors to a test. He had 100 identical rooms constructed in his palace. In each room the king placed an infinite sequence of boxes; in each box he put a real number. The sequence of numbers was exactly the same in each room, but otherwise completely arbitrary.

The king told his advisors that when they were ready, each of them would be locked in one of the 100 rooms. Each of them would be allowed to open all but one of the boxes in the room. (This, of course, would take an infinite amount of time, but in the Kingdom of Aleph, theyre pretty cavalier about infinity.) Finally, each advisor would be required to name the number in the box that he or she did not open. If more than one advisor names the wrong number, they would lose their jobs and their lives.

There is no reason for anyone to hurry in the Kingdom of Aleph, and the king gives his advisors an infinite amount of time to work out a strategy. Do they have any hope of making it through his cruel test alive? What should they do?

Homework: None

Prerequisites: None

The mathematics of polygamy (and bankruptcy). (🌶, Mira, 1 day)
Here is a passage from the *Mishnah*, the 2nd century codex of Jewish law:

A man has three wives; he dies owing one of them 100 [silver pieces], one of them 200, and one of them 300.

If his total estate is 100, they split it equally.

If the estate is 200, then the first wife gets 50 and the other two get 75 each.

If the estate is 300, then the first wife gets 50, the second one 100, and the third one 150.

Similarly, any joint investment with three unequal initial contributions should be divided up in the same way.

For 1800 years, this passage had baffled scholars: what could possibly be the logic behind the Mishnah's totally different ways of distributing the estate in the three cases? Then, in 1985, a pair of mathematical economists produced a beautifully simple explanation based on ideas from game theory. They showed that for any number of creditors and for any estate size, there is a unique distribution that satisfies certain criteria, and it turns out to be exactly the distribution proposed in the Mishnah. The proof is very cool, based on an analogy with a simple physical system! See if you can figure out this ancient puzzle for yourself, or come to class and find out.

Homework: None

Prerequisites: None

Two games, one perfect strategy. (🌶🌶, Mira, 1 day)
Question: What do the following two games have in common?

(1) You think of a number between 0 and 15. I ask you seven yes/no questions about. You are allowed to tell at most one lie (or, if you prefer, to answer truthfully throughout). At the end, I'll tell you if and when you lied, and then I'll guess your number! (We'll actually play this in class.)
(2) You and six friends are playing a cooperative game. You are each given a black or white hat at random. As usual, each person can see the color of everyone else's hats and has to guess the color of his or her own. Each of you writes your guess — either "BLACK" or "WHITE" — on a piece of paper, without showing it to anyone else. If a player doesn't want to guess, they also have the option of writing "PASS". Then everyone holds up their papers simultaneously. Your team wins if, among you, you have *at least one correct guess and no incorrect guesses.* ("Passes" don't make you lose, but they don't help you win either.) You are allowed to agree on a strategy before the hats are passed out, but no communication is allowed afterwards. What strategy will maximize your probability of winning, and what is the best probability you can achieve? (We'll play this in class if we have time.)

Answer: The same strategy, based on a perfect code!
Come and find out what that means.

Homework: None

Prerequisites: None

Related to (but not required for): Error-Correcting Codes (W3)

Misha's Classes

Knight's Tours. (🌶🌶, Misha, 1 day)
A well-known puzzle asks if a knight on an 8×8 chessboard can make 64 moves in a way that visits each square of the chessboard exactly once and then returns to the starting square.

You probably have already seen the answer, so here's a different question to think about: can you do the same in 16 moves on a 4×4 chessboard?

Here's a third question which the mathematician Sir William Hamilton tried to market as a puzzle: can you find a path along the edges of a dodecahedron that visits each vertex exactly once and returns to the starting point? (The puzzle did not sell: it was too easy.)

Here is a fourth question (resolved only a few years ago): can you make 43 252 003 274 489 856 000 twists of a Rubik's cube that put it in each possible state exactly once before returning to the solved position? I won't tell you why the answer is what it is, but I'll explain why we should have seen it coming.

Homework: None

Prerequisites: None.

Problem Solving: Linear Algebra. (🌶🌶🌶, Misha, 1–2 days)
Most high school math contests (the IMO included) do not use any topic considered to be too advanced for high school, such as linear algebra. This is a shame, because there have been many beautiful problems about linear algebra in undergraduate contests such as the Putnam Math Competition.

In this class, we will look at linear algebra from a new perspective and use it to solve olympiad problems.

Homework: Recommended

Prerequisites: Linear algebra

Problem Solving: Tetrahedra. (🌶🌶🌶🌶, Misha, 1 day)
In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

Homework: Optional

Prerequisites: None.

Random Regular Graphs. (🌶🌶🌶, Misha, 2 days)
Let's say you want to pick a random graph. Sure, you could take n vertices and then flip a coin for each of the potential $\binom{n}{2}$ edges. I claim that the graph you'd get is boring.

Instead, suppose you want a graph in which every vertex has exactly d neighbors (d is a small constant, like 3). What does such a graph look like? How do you even pick a d-regular graph at random? Do you just keep flipping coins and hope that one of your attempts will work out?

If all goes well, after answering these questions, we will prove that almost all 3-regular graphs are Hamiltonian by using a bizarre probabilistic technique.

Homework: Recommended

Prerequisites: graph theory, probability (you should be comfortable with random variables and expected values).

Rapid Fire Problem Solving. (🌶, Misha, 1 day)
"Compute the ordered pair of integers (a, b) that minimizes $|a^4 + a^b - 2015|$."

This problem comes from the ARML 2015 Tiebreaker round. You can probably solve it. But can you solve it in less than a minute?

This class is about solving math problems very quickly. We will practice solving such problems and discuss tricks to speed up problem solving.

Homework: None

Prerequisites: None.

Nancy's Classes

Classification of Hyperbolic Isometries. (🌶–🌶🌶, Nancy, 1 day)
If we take only the top half of the Complex plane and put a strange distance function on it, we get the Hyperbolic plane. This plane is really cool and gives rise to lots of fun pictures. An isometry is a function that preserves distance. There is a way to let 2×2 matrices act as isometries on the plane and it turns out they can only work in 1 of 3 ways. We will learn why!

Homework: Optional

Prerequisites: understand basic matrix multiplication and complex number arithemtic

Related to (but not required for): DIY Hyperbolic Geometry (W4)

Complex roots of unity. (🌶, Nancy, 1 day)
In the complex numbers, there is a strange phenomenon that there are lots of numbers α with $\alpha \neq \pm 1$ but there exists n such that $\alpha^n = 1$. This is due to the fact that $x^n - 1$ has n roots in $\mathbb{C}$. These numbers are called roots of unity and they have lots of fun properties. Let's learn about them!

Homework: Optional

Prerequisites: None.

More Knots and Braids. (🌶–🌶🌶, Nancy, 1–3 days)
Wanna learn MORE about knots and braids?!?

Homework: Optional

Prerequisites: None, but would be helpful if you took my knot theory and braid groups classes

Sylow Theorems. (🌶🌶🌶–🌶🌶🌶🌶, Nancy, 3–4 days)
Sylow Theorems $\subset$ Group theory $\cap$ Number theory

The Sylow Theorems form a fundamental part of finite group theory and have very important applications in the classification of finite simple groups.

Basically, here's what the Sylow theorems do:

Suppose you have a finite group G of size m. Depending on the prime decomposition of m, you can determine how many subgroups G will have of various sizes.

Homework: Recommended

Prerequisites: Group theory

Paweł Piwek's Classes

A tale of symmetries is in order! (🌶🌶🌶, *Paweł Piwek*, 1 day)
Don't you agree?

Let p be Zach's favorite prime (or yours). Take an integer $x \in \{0, 1, \dots, p-1\}$ and start computing its powers $x, x^2, x^3, \dots$ mod p. The minimal exponent d for which $x^d \equiv 1 \bmod p$ is called the *order* of x mod p (If there is no d with $x^d \equiv 1 \bmod p$, then you took $x = 0$; unlucky for you.)

Now suppose you *fix* d and add up (mod d) all the $x \in \{0, 1, \dots, p-1\}$ which have that order d. What will you get? And what does this computation mod p have to do with complex roots of unity? To find out, come to this class!

Homework: None

Prerequisites: Some experience with polynomials; Fermat's Little Theorem (or Lagrange's Theorem from group theory)

Pesto's Classes

Character Table Sudoku. (🌶, Pesto, 1 day)
A character table is a grid of numbers with some constraints; for instance, this is a character table:

1	1	1	1	1	1
1	-1	-1	-1	1	1
2	0	0	0	-1	-1

If you carefully choose a subset of the grid and delete the rest, filling it back in is a logic puzzle. For instance, can you infer enough of the constraints that define a character table from the previous example to fill in this one?

1	1	1	1	1	1	1	
1	1	1	1				
1	1			1	1		
1	1						
	-						

Character tables are important in *representation theory*, a combination of group theory and linear algebra. We won't do any representation theory and the only linear algebra we'll do will be hidden, but familiarity with character tables makes learning representation theory later a bit easier.

Homework: Optional

Prerequisites: Anti-prerequisite: if you've taken representation theory, the puzzles may be trivial.

Related to (but not required for): Linear Algebra (W1); Representation Theory of Finite Groups (Week 1 of 2) (W3)

Kuratowski's Theorem. (🌶🌶🌶, Pesto, 1 day)
Draw five points on a piece of paper. Try to connect every pair of them without any edges crossing. You can't do it.

Three utilities each need to connect to each of three houses. Then some two of the utility lines must cross.

Stated graph-theoretically, these results say that K_5 and $K_{3,3}$ aren't planar.

What's really surprising is that these are in some sense the *only* non-planar graphs: you can find one of them in every nonplanar graph if your vision's fuzzy enough that you can make a vertex of degree two look like it's just an edge. This is Kuratowski's Theorem, which we'll state and prove.

This is day two of Wagner's Theorem, and we'll also see that Wagner's Theorem is the right way to think about this sort of problem, but that Kuratowski's way can't be much worse.

Homework: Optional

Prerequisites: Wagner's Theorem.

Multi-Coefficient Solving of Problems. (🌶🌶🌶, Pesto, 2–4 days)
Try to find the product of the lengths of all the sides and diagonals of a regular n-gon of diameter 2. Don't see the polynomials? Come to class and find them.

If a polynomial f has a positive root, then it has a negative coefficient; if it has no positive root, then it doesn't necessarily have only positive coefficients, but some multiple of it does.

Come see techniques for these and other olympiad-style problems on polynomials.

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to Olympiad-style problems you will have solved as homework the previous day.

Disjoint from last year's polynomials problem solving class.

Homework: Required

Prerequisites: None.

Premodern Cryptography. (🌶, Pesto, 1–2 days)
UI UOHH JFHQI SGXEKFNGMT ELZZHIJ HOWI KRI FDIJ KRMK XFL TONRK AODP OD M DIUJEMEIG, AOGJK CX KGOMH MDP IGGFG, KRID CX KISRDOBLIJ KRMK YLJK IVKIG-TODMKI KRIT. Then we'll look at some of the premodern code-writers' attempts at strengthened versions, and see why they're still breakable.

(No relation to the real cryptography class.)

Homework: Optional
Prerequisites: None.
Related to (but not required for): Turing and his Work (W2)

Quantum factoring. (🌶🌶🌶🌶, Pesto, 1–2 days)
With the best currently known algorithms, all the world's supercomputers working together for a year couldn't factor a 4000-digit integer into prime factors. Even a small[2] computer that takes advantage of quantum mechanical phenomena that aren't well approximated by what classical computers do could factor such numbers using "Shor's algorithm". We'll race through a description of how we model quantum computation (without assuming any knowledge of physics), then see Shor's algorithm.
Homework: None
Prerequisites: Linear algebra (in some sense, almost everything we do in this class will be multiplication of matrices), Number Theory (Fermat's Little Theorem).

The Hoffmann Singleton Theorem. (🌶🌶🌶, Pesto, 1 day)
"I'm thinking of a set of four integers. Four of them are 2, 3, and 7. What's the other one?"

"How should I know? You could have picked any other integer."

"No, these are the answers to a perfectly natural mathematical question: as natural as "If you can get from any other vertex of a graph to any other vertex in at most two steps, but there are no triangles or squares, and every vertex has the same degree, what's that degree?" "

"Seems like a natural enough question. Maybe 5, like the first four primes?"

"Nope, 57."

"No way! How could 2, 3, 7, and 57 be the answer to *anything* that natural?"
Homework: Recommended
Prerequisites: Linear algebra (enough to understand the statement "if A is a matrix and $A^2 = I$, then every eigenvalue of A is 1 or -1" and either know its proof or be willing to accept it without proof).

Wagner's Theorem. (🌶🌶🌶, Pesto, 1 day)
Draw five points on a piece of paper. Try to connect every pair of them without any edges crossing. You can't do it.

Three utilities each need to connect to each of three houses. Then some two of the utility lines must cross.

Stated graph-theoretically, these results say that K_5 and $K_{3,3}$ aren't planar. (We'll prove so, quickly.)

What's really surprising is that these are in some sense the *only* non-planar graphs: you can find one of them in every nonplanar graph if your vision's fuzzy enough that you can make any connected set of vertices blur into one. This is Wagner's Theorem, which we'll state and prove.
Homework: Optional
Prerequisites: Graph theory: if G is a connected planar graph drawn with f faces, and G has v vertices and e edges, then $v - e + f = 2$.

Ruthi's Classes

Bayesian Statistics. (🌶🌶, Ruthi, 3–4 days)
Statistics is the science of analyzing data in the presence of uncertainty or with incomplete information. Since there is little in the world that is certain, and information is always scarce, we humans can't go

[2] "Small" according to some arbitrary attempt to compare quantum computer sizes to classical computer sizes, but still millions of times bigger than any currently-constructible quantum computer.

a day without doing some kind of statistics — in our routine cognitive functions, in science, in the political arena, etc.

When you start studying it in school, statistics at first looks a lot like math. Yet if you've ever taken a statistics class, you might have felt your inner mathematician getting increasingly disgruntled and annoyed — and with good reason! Many mathematicians shun statistics as not being a legitimate branch of mathematics. Among the general public too, statistics has a bad rep. Try to imagine the famous quote by Mark Twain ("There are three kinds of lies: lies, damned lies, and statistics") applied to mathematics. Impossible! So what's the difference?

The difference is that, unlike math, statistics (as it is usually done) appears to be just a scrap book of arbitrary tests and procedures, with no basic underlying principle. This attitude makes it easy to come up with a variety of "lies", through either negligence or malice. But there are serious problems with statistics even when it is done carefully and honestly. For instance, if you look at what some of the standard statistical tests are actually measuring, you will find that most of them are not asking the questions that they're supposed to be answering. Instead they're measuring something related, but different and much more convoluted.

In this class, we'll talk about some of the pitfalls statistics often leads us to fall into, how to think critically about the data we're given, and how Bayesian statistics can be a clean way of analyzing our intuitions.

Homework: Recommended

Prerequisites: None. Knowing basic probability and/or calculus will be helpful, but is not necessary.

Chabauty Coleman. (🌶🌶🌶🌶, Ruthi, 1–4 days)
The premise of arithmetic geometry is this: we can turn questions from algebraic number theory into problems about geometry, and use our geometric knowledge to solve the problem there. The beautiful and surprising way these interplay changed the way people think about number theory and has formed modern research in the field.

One of the key theorems of arithmetic geometry is Faltings' Theorem (née the Mordell Conjecture), which states that no curve of genus 2 or larger has infinitely many rational points (the proof of which landed Faltings the Fields Medal in 1986). The problem with Faltings' proof is that it doesn't actually give a bound on how many rational points there actually are. A less strong result of Chabauty and Coleman gives a bound, using a beautiful approach involving integration over p-adic numbers.

Homework: Recommended

Prerequisites: Point-set topology. I will also use the words integral, p-adics, and abelian group without defining them, and I will define new things quickly.

Elliptic Curves Day 1. (🌶, Ruthi, 1 day)
Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful. They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

In this class, we will learn what the definition of an elliptic curve over the rationals is and how to add points on one.

Homework: Recommended

Prerequisites: None.

Elliptic Curves Day 2. (🌶🌶, Ruthi, 1 day)
Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful.

They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

In this class, we'll generalize our definition of elliptic curves, and discuss what their abelian group structure looks like. (In particular, we'll state Mordell-Weil Theorem.)

Homework: Recommended

Prerequisites: Elliptic Curves Day 1, know what an abelian group is

Elliptic Curves Day 3. (🌶🌶🌶, Ruthi, 1 day)
Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful. They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

In this class, we'll discuss heights on curves and prove Mordell's Theorem.

Homework: Recommended

Prerequisites: Elliptic Curves Day 2

Elliptic Curves Day 4. (🌶🌶🌶🌶, Ruthi, 1 day)
Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful. They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

In this class, I'll try to talk about some of modern questions of interest about elliptic curves, and recent research and results.

Homework: None

Prerequisites: Elliptic Curves Day 3

Profinite groups. (🌶🌶🌶🌶, Ruthi, 2–3 days)
Let's take a finite group and give it the discrete topology. We can prove all kinds of things about the topology of this group, that makes it kind of weird. But can we describe how to get all groups that have similar topology? It turns out, in some sense, we can build them all up from finite groups. In this class, we'll do that, and show what properties completely characterize them.

Homework: Optional

Prerequisites: point-set topology, group theory

(This topic is of particular interest to those who took Galois Cohomology or learned about p-adic integers in J-Lo's class, but you don't need to have been in that class to find this exciting!)

Related to (but not required for): Abelian Groups (W5)

Sieves. (🌶🌶, Ruthi, 1 day)
Maybe you hear the word sieve and you think, I should ask Steve about that. Maybe you think of the Sieve of Eratosthenes. It turns out that sieves are used all the time to study problems in number theory (including in my own research!). We'll talk about some more complicated problems that can be approached with sieve methods, and touch on how they fail at times as well.

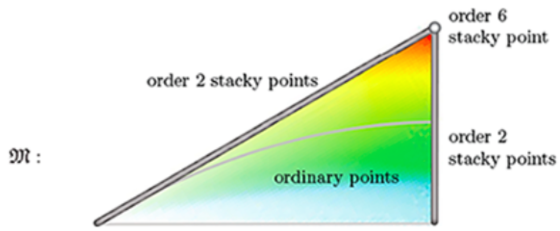
Homework: None

Prerequisites: feel comfortable with limits

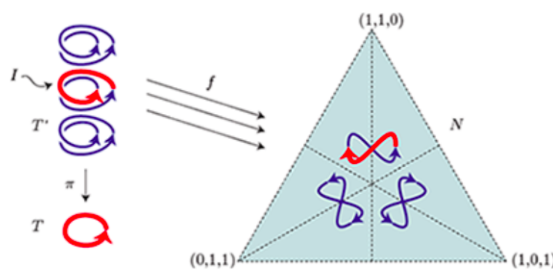
Sachi's Classes

Moduli Stack of Triangles. (🌶🌶🌶, Sachi, 4 days)
Stacks are really important objects that nobody I've asked on staff understands. Triangles are really important objects that everybody I've asked on staff understands.

This is the moduli stack of triangles:



This is why we use stacks:



Let's learn about stacks together.

Homework: Optional

Prerequisites: Be able to define and work with the symmetric group, know what a continuous map is, complex numbers and polar coordinates. You should know what a group action is, or ask someone to explain it to you before class.

You will enjoy this class more if you know some point set topology and even more if you know some covering space theory.

Related to (but not required for): Fundamental Group (W2)

Ring Theory, the Sequel. (🌶🌶, Sachi, 3–4 days)
Did you take ring theory and want to know more?

Have you heard about unique factorization, but never seen a rigorous treatment of exactly what it requires?

Are you curious about ascending and descending chains of ideals?

Did one of your Mathcamp classes gloss over a ring theory result that you want to know all of the details of?

Then this is the class for you! Time permitting, we will talk about Principal Ideal Domains, Euclidean Algorithms, Unique Factorization Domains, fields, Noetherianity, and at the very end get to some algebraic number theory through the topic of fractional ideals.

Homework: Recommended

Prerequisites: Ring theory week 1 or the equivalent

Related to (but not required for): Ring Theory (W1); Absolute Values: All the Other Ones (W3); The Factorial Function (W3); pNumber Theory (W4); Galois Theory (W4)

The Cake is a Lie. (🌶🌶, Sachi, 2 days)
It's my birthday, and I want to share cake with my friends.

Susan and Kevin are attending my party, but they're still having a feud, so I need to make sure I give them the same amount of cake. Unfortunately, Susan really like the pink frosted flowers, and Kevin likes the gel icing writing. So, cutting the cake in thirds will not look fair to them, depending on the distribution of the icing.

How can I cut the cake so that Kevin, Susan, and I each believe that we have at least 1/3 of the cake and would not like to trade with anyone else?

Homework: Recommended

Prerequisites: None.

The Projective Plane. (🌶🌶, Sachi, 1 day)
We can construct an object called the "projective plane" by taking a sphere and allowing teleportation between two antipodal points. Alternatively, we could wrap the normal plane $\mathbb{R}^2$ in a line "at infinity" and require that all parallel lines meet at one point. Or, we could take a Möbius strip and glue a disc to it. If we happened to already have a projective plane lying around, we could switch all its points with all its lines and get a projective plane back.

We'll talk about what kinds of geometry we can do with this weird shape, and why all these notions are the same.

Homework: Optional

Prerequisites: None.

Sam's Classes

Bad History: A Crash Course in Historiography and how not to do the History of Math. (🌶, Sam, 1 day)
It turns out that, when mathematicians try to do research in the history of mathematics, they sometimes do awful things. We'll talk about some of the common pitfalls (i.e., whig history) and how to avoid them. That way, if you want to ever think seriously about the history of math, you don't accidentally garner the scorn of the history of mathematics community! Yay!!

Homework: Recommended

Prerequisites: None.

Florence Nightingale and Effectively Communicating Data to Politicians. (🌶–🌶🌶, Sam, 1 day)
Lots of people know that Florence Nightingale did lots of cool stuff! Less people know about some of her contributions to statistics — being one of the first people to use visual representations of complex data to convince politicians to do important things.

In this class we'll look at what Florence Nightingale actually did in statistics. We'll also look at a few other cool examples of how graphs and charts can be used to effectively communicate content. And you'll get to watch the commercial that I find most infuriating!

Homework: None

Prerequisites: None.

Optimization Problems on Graphs. (🌶🌶, Sam, 2 days)
Suppose you were in charge of running the first chess-sprinting world championship. Because the sport is so popular, you have a ton of money to run the championship and, because you like the beach,

choose to host the tournament in Bora Bora. Nike has also sponsored the tournament, and has given you funding to fly competitors to Bora Bora (so long as everyone in the tournament wears Nike's chess-sprinting-shoe-line). All that remains is to book the flights.

Now suppose also that you like a challenge (that is, after all, how you got into the chess-sprinting business in the first place). You thus try to book flights without using any online tools: all you let yourself have is a map that shows, for each pair of airports, the cost of the cheapest flight between those airports (if no such flight is available, the cost is infinite). If you just had this map, how could you figure out what flights to purchase?

It turns out that this type of problem is ubiquitous, and actually occurs in significantly less contrived scenarios. More formally, suppose you have a graph G, and for each pair of vertices u, v, the edge between u and v is labeled with some sort of cost (say, corresponding to the cost to get from u to v). If there is no edge between u and v, we can represent that cost with a really large number. In the shortest path problem, we're interested in finding the path from u to v of minimum cost, where the cost of the path is defined as the sum of the costs along all edges in that path.

In this class, we'll introduce the shortest path problem then talk about a few algorithms for solving that problem (like Bellman-Ford and Dijkstra's). We'll also talk about a few related problems! We will not, alas, get to go to Bora Bora.

NOTE: this is going to be a course with a relatively practical flavor. My goal will be to introduce certain types of optimization problems that occur on graphs and then to show clever ways of solving those problems. I'll justify some of the techniques, but we won't necessarily be rigorously proving all of them. But, the techniques we will talk about are all pretty awesome, and they're great techniques to have in your toolkit!

Homework: Recommended

Prerequisites: A smattering of basic graph theory: know what a graph is and what a path between two vertices is.

Statistical Model Selection. (🌶🌶–🌶🌶🌶, Sam, 1–2 days)
Suppose you wanted to build a statistical model to predict the probability that a given camper would win the next chess-sprinting tournament. Since chess-sprinting is such a nuanced and refined sport, there are tons of predictors you'd want to consider, and you might end up with an overwhelming number of possible models. How might you select such a model?

If this is a 1 day course, we'll discuss strategies for model selection and a couple of the theoretical results for selecting models (and we'll spend a bit of time in the computer lab doing some real data analysis). If this is a 2 day course, on day 2 we'll talk about multiple testing: when we have lots of hypotheses we want to test, what do we do?

Homework: Recommended

Prerequisites: Statistical Modeling

The History of Calculus. (🌶🌶–🌶🌶🌶, Sam, 1–2 days)
Have you heard that Newton and Leibniz invented calculus? Have you heard anyone categorically reject that statement?

In this class we'll do a quick survey of the history of calculus. We'll talk about some of conceptual ideas that had occurred to the Ancient Greeks, we'll talk about what was known about calculus before Newton and Leibniz, we'll talk about what Newton and Leibniz actually did, and then we'll talk about why that wasn't enough. We'll also hear what Bishop Berkeley had to say, and how he used reason to try to shut down calculus*.

NOTE: This class can also be replaced with "The History of X," where X is another part of mathematics if there is sufficient demand.

BONUS: You'll learn (some of) why Cauchy is my favorite mathematician!

* We'll also qualify this sentence!
Homework: Optional
Prerequisites: Feeling good about calculus is necessary

Steve's Classes

Drawing {lines in the plane}. (🌶, Steve, 1 day)
What does the set $\mathcal{L}$ of lines in $\mathbb{R}^2$ look like? Individual lines are easy to draw, but it's hard to get a good sense for what the set of *all* lines looks like. In particular, there ought to be some geometry associated to $\mathcal{L}$ — for example, the line $y = x$ is intuitively closer to the line $y = \frac{11}{10}x$ than it is to $y = 1000000x + 32567$ — but how can we make this precise?

In this class I'll show how to describe the space of lines in the plane, by blowing stuff up. Towards the end, I'll say a few words about *moduli spaces*...
Homework: None
Prerequisites: None.

How hard is impossible? (🌶, Steve, 1 day)
Computers are great! And they can do a lot of math. But it turns out there are problems that *cannot* be solved by a computer, even in principle. The study of such non-computable problems is called *computability theory*, because logicians are bad people and shouldn't be allowed to name things.

What are these unsolvable problems? Are any of them interesting? And are any of them "more unsolvable" than others? Come to this class and find out!
Homework: None
Prerequisites: None.

LARGE cardinals! (🌶🌶🌶🌶, Steve, 3–4 days)
Some numbers are so big we need special notation to describe them. Well, it turns out that some *sets* are so big, you need special *axioms* to even show they exist! These are called *large cardinals* and are a major focus of current research in set theory.

Large cardinals come in two flavors: the smaller(!) ones tend to arise from taking nice theorems in combinatorics and going "But what if INFINITY?!?!" The larger ones appear when we take the entire universe, stretch it, and put it inside itself for no obvious reason. Yay science!

Over time, set theorists have discovered many species of large cardinals; in this class, we'll tour the zoo.
Homework: Optional
Prerequisites: Familiarity with basic set theory: ordinals, cardinals, and the axiom of choice.
Related to (but not required for): Measure and Martin's Axiom (W1); Fun with Compactness in Logic (W1); Making Cantor Super Proud (W1); Ordinal Arithmetic (W3); Ultrafilters (W4); Category Theory in Sets (W4); Martin's Axiom and Ramsey Ultrafilters (W5); The Lowenheim-Skolem Theorem (W5)

The perfect set game. (🌶🌶🌶🌶, Steve, 1 day)
A *perfect set* is a closed set of real numbers with no isolated points. We can show that any perfect set has as many elements as the set of all real numbers. In my colloquium, I mentioned that "reasonable" sets of real numbers satisfy a strong version of the continuum hypothesis: they are either countable, or contain a perfect subset. In this class I'll show that this dichotomy is closely related to games: if you give me a set X, then I can give you a game G such that G has a winning strategy if and only if X is either countable or has a perfect subset.

Homework: None

Prerequisites: None.

Susan's Classes

Do The Hilbert Twist! (🌶🌶, Susan, 3 days)
You want noncommutative rings? I've got them! The Hilbert Twist construction does it all. Noncommuting polynomials. Noncommutative division rings. But wait! There's more! Assume x and y are ring elements with no relations between them. Try writing the element $(xy^{-1}x - yxy^{-1})^{-1}$ with no nested inverse signs. Having trouble? That's because it's impossible! Come and find out why!

Homework: Optional

Prerequisites: None.

Graphs and Wang Algebras. (🌶🌶, Susan, 1 day)
Want to see a gorgeous application of abstract algebra to graph theory? Suppose you have a particularly large and nasty graph, and you need a list of its spanning trees. By taking something called a Wang algebra—polynomials with the vertices as variables, and with $v^2 = 0$ for each vertex—we can construct a polynomial that spits out *exactly* the list of spanning trees.

Homework: Recommended

Prerequisites: None.

How To Make Rings That Do Terrible Things. (🌶🌶, Susan, 2 days)
Suppose I ask you to give me a ring that has a left zero divisor that is not a right zero divisor? Or a ring that has a left unit that is not a right unit?

A good first instinct is to try a quotient of $\mathbb{Z}\langle X\rangle$ for some appropriate collection of variables X. For example, in $\mathbb{Z}\langle X\rangle/(xy)$, we can see that x is a zero divisor that is not a right zero divisor.

But how can we be sure? How do we know that in the process of modding out by our ideal, we didn't accidentally set a bunch of elements equal to each other, thus collapsing the entire ring to zero, and along with it civilization as we know it?

Well, in certain cases, we have a result called the diamond lemma, which allows us to find a unique expression for each element, thus guaranteeing that our ring, and civilization, is safe.

Homework: Recommended

Prerequisites: None.

Malcev's Ring. (🌶🌶🌶, Susan, 1–2 days)
Fields are very nice rings. The multiplication is commutative, and every element has an inverse. Integral domains are fairly nice rings. The multiplication is still commutative, and there are no zero divisors. If we want multiplicative inverses, we can formally add them to our ring in a process called "localization". Noncommutative rings are…less nice. In particular, in 1937, Anatoly Malcev gave an example of a noncommutative ring in which there are no zero divisors, but which is not a subring of any division ring. In this class we'll see how this example works.

Homework: Optional

Prerequisites: None.

Stupid Games on Uncountable Sets. (🌶🌶, Susan, 2 days)
Let's play a game. You name a countable ordinal number. And then I name a bigger countable

ordinal number. We keep doing this forever. When we're done, we'll see who wins. In this class we'll be discussing strategies for winning an infinite game played on ω_1. In particular, we'll talk about how to set up the game so that at any point, *neither* player has a winning strategy.

Homework: Optional

Prerequisites: None.

The Littlewood-Offord Problem. (, Susan, 4 days)

Suppose I give you seven differently-colored fuzzy-wuzzies, each worth a positive real number of points. How many different teams of fuzzy-wuzzies can you construct that are worth exactly 100 points? What if you're the one assigning the points? How many different 100-point teams of fuzzy-wuzzies you can create?

We will begin by finding a solution to this problem? But what if we want to aim for fuzzy-wuzzy teams worth r points for an arbitrary real number r? What if we have n fuzzy-wuzzies. What if we allow fuzzy-wuzzies to be worth not just a positive number of points, but any non-zero real number of points? What if instead of a real number of points, we assign each fuzzy-wuzzy a vector in d-dimensional space, and want teams of fuzzy-wuzzies that add up to the vector $(100, 100, \ldots, 100)$?

Finding upper bounds for these problems is not hard, but proving that these bounds are optimal is tricky. These questions about sums of real numbers quickly transform into questions about posets. We'll talk about chains, antichains, Dilworth's theorem, and more. If you love fuzzy-wuzzies, posets, or stunningly beautiful proofs, then this is the class for you!

Homework: Optional

Prerequisites: None.

What's Up With e? (, Susan, 1 day)

The continued fraction expansion of e is

$$1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots$$

What's up with that? Come find out!

Homework: None

Prerequisites: Calculus (derivatives and integrals).

Yuval's Classes

The Fast More Four Do Over. (, Yuval, 1 day)

If I give you a real a and one more real b, how fast can you find out $a \cdot b$? We need to do this all the time, so we want to be able to do it fast. And if we try to do it the way we were told when we were kids, it will be too slow. Can we make it more fast?

The More Four Do Over is a math idea that does take some data and does turn it into a set of data in such a way that the new data has a lot of info on how the old data does vary. It's a tool that is very full of use, and we love it. Also, it lets us find out $a \cdot b$, and that is very good.

But wait! If we try to do the More Four Do Over, it is also too slow. We want to do it fast! Very very fast! What do we do? Well, luck is on our side. It does turn out that we can find a sly way to do the More Four Do Over that is way more fast than what we did in the past. And if we know how to do the More Four Do Over fast, then we can find out $a \cdot b$ fast too, and it all just gets more fast.

In this talk, we will find out how to do the Fast More Four Do Over and how to use that to find out $a \cdot b$. And we will do it all in the Game of Four.[3]

[3] If I give you two numbers a and b, how fast can you calculate their product $a \cdot b$? Since this is such an important operation, we want to be able to do it quickly, and the algorithm that we learned in school is actually too slow. Can we make it faster?

Homework: None

Prerequisites: None.

Related to (but not required for): Algorithms (W1); Algorithms in Number Theory (W1); Time-Frequency Analysis (W1); The Hidden Dance of Partial Differential Equations (W4)

The Kakeya Conjecture over finite fields. (, Yuval, 1 day)
The Kakeya conjecture asks about the minimal size of a set that contains a line in every direction. The finite fields Kakeya conjecture asks about the minimal size of a set that contains a "line" in every "direction".

In this class, we'll see how easy things get when we abandon our complicated Euclidean space for much simpler finite spaces.

Homework: None

Prerequisites: None.

Co-taught Classes

Martin's Axiom and Ramsey Ultrafilters. (, Steve + Susan, 5 days)
Lets play with the complete graph on $\mathbb{N}$![4] We like colors, so were going to color the edges — each edge gets colored either "red" or "blue"[5] It turns out that no matter how we do this, we can find an infinite set of vertices H (called a "homogeneous set") such that all the edges between vertices in H have the same color. For instance, maybe we color the edge between a and b red if and only if a and b have the same parity (even or odd); then we can take H to be the set of odd numbers.

But that's graph theory, and we want to do logic. Also, we're lazy; we want a machine that will magically find homogeneous sets for every coloring. Also also, we like ultrafilters. Can has useful ultrafilter please?

It turns out that, if we assume Martin's axiom[6], there is an ultrafilter $\mathcal{U}$ such that — for *any* coloring at all — there is a homogeneous set H for that coloring such that $H \in \mathcal{U}$. Time remaining, well show that *every* ultrafilter has an ultrafilter like $\mathcal{U}$ living inside it.[7]

Homework: Recommended

Prerequisites: Ultrafilters, Martin's Axiom

Related to (but not required for): LARGE cardinals! (W5)

Not Quite Group Theory. (, Don + Steve, 2 days)
A semigroup is like a group, only less so. In particular, elements don't have to have inverses; there doesn't even have to be an identity. Even despite these deficiencies, most of the basic results of group theory still hold; we'll see what needs to change, and what theorems stay the same.

On the second day, we'll look at symmetry semigroups, which allow us to tell the difference between an equilateral triangle and the Sierpinski triangle.

Homework: None

The (Discrete) Fourier Transform is an operation that converts a sequence of numbers into another sequence that remembers the structure of the original sequence. It's an extremely useful tool; in particular, it lets us calculate $a \cdot b$. However, there's a problem: if we try to calculate the Discrete Fourier Transform naively, then it's also too slow. But luckily, there is a fast algorithm for computing it, which also means that we can multiply quickly.

In this class, we'll learn about the Discrete Fourier Transform and about the Fast Fourier Transform algorithm, and we'll see how we can use this to multiply efficiently. And we will do it all in the Game of Four.

[4]Excitement, not factorial.

[5]We had more exciting colors, but Susan lost them.

[6]We had a great proof just from the axioms of ZFC, but Steve ate it, so now its independent of ZFC. Bad Steve.

[7]Yo dawg, we heard you like ultrafilters.

Prerequisites: Group Theory

Quaternions and rotations. (🌶🌶🌶, Alfonso + Chris, 2 days)
The composition of any number of rotations in $\mathbb{R}^3$ is a rotation. This is not trivial to prove! (For comparison, this is false in four dimensions, the composition of reflections in $\mathbb{R}^3$ is not a reflection.) In this class you will prove the result and you will use and learn a lot about quaternions to do so.

The quaternions are "a set of numbers" similar to the complex numbers, but where we have i, j, and k, instead of having only i. Some know them simply as a mathematical toy, but they have many applications. For example, they are the best way to represent rotations in $\mathbb{R}^3$, and the best way to do calculations with them.

You will be doing all the work yourselves during class time. Ask any student who took our Banach-Tarski class in Week 2 if you want to know more about the format.
Homework: Optional
Prerequisites: Linear algebra: you need to know how matrix multiplication works and how to write the matrix of a rotation.
Related to (but not required for): Linear Algebra (W1); The Banach–Tarski Paradox (W2)

The Löwenheim-Skolem Theorem. (🌶🌶🌶, Susan + Steve, 2 days)
Oh no! A ninja has snuck into the Museum of Real Numbers and stolen all but countably many of them! You, the curator, have a huge exhibition tomorrow. What are you going to do? Why, it's simple! You'll use the Löwenheim-Skolem theorem to build a countable model of set theory, complete with the real numbers. From inside the museum, no one will be able to tell that it's countable. To keep real number ninjas from interfering in your life, come to this class!
Homework: Optional
Prerequisites: None.
Related to (but not required for): LARGE cardinals! (W5)