CLASS DESCRIPTIONS—MATHCAMP 2016

Classes

A Crash Course in Axiomatic Probability. (Sam)

This class will be a brisk walk through some of the most fundamental topics in probability. We will start from Kolmogorov's axioms and directly build our way through formal notions of independence, conditional probability, and random variables. By the end of the course, we'll have sufficient tools to prove some very high-level theory and asymptotic results! We will mostly focus on discrete probability—so if you haven't seen much calculus that's fine—but will touch briefly on continuous random variables.

Throughout the class, we'll look at a few fun "applications" of probability. Typically, these will be applications to other areas of mathematics (like Graph Theory!).

Prerequisites: You should know the following words and corresponding symbols: (finite and countable) union, intersection, complement, and partition. Calculus will help for 20 minutes of this course. We'll also do one or two examples from graph theory, where knowing very basic terminology will be helpful (edge, vertex, clique, independent set, complete graph), but these are just fun asides and can be safely ignored.

Advanced Topics in... Sorting? (Zach)

What seems at first a trivial algorithmic task—arranging a list of numbers/names/campers in order—is actually just the beginning of many beautiful, difficult, and even unsolved problems in theoretical computer science. This class is not a standard laundry list of classical sorting algorithms (bubble sort, merge sort, etc.), though we'll certainly find uses for these. Instead, we'll discuss more advanced, meatier questions about sorting or sorted data, such as: How can we search sorted data *faster* than binary search, and how does this speed up many geometric algorithms? (A topic near and dear to my research.) What gains can we get from many processors working together to sort the same data? What sorting methods are better for humans, even if they may be slower for computers? (This last one we'll investigate empirically, of course.)

This is a *theoretical* computer science class; no actual computer code will profane its purely mathematical discussions, nor is programming knowledge required or especially relevant.

Prerequisites: Knowledge of binary search is recommended (it's easy to look up or ask me about), as is knowledge of any sorting algorithm like bubble sort (ditto). Familiarity with "Big-O notation" is helpful but not required.

Algebraic Groups. (Don)

When algebraists first made serious attempts to study $GL_n(\mathbb{C})$, the group of invertible $n \times n$ matrices, they were struck by just how much algebra is actually taking place. First, it has a group structure. Second, it's a group of matrices, so it is central to linear algebra. Third, its definition depends on a ring, $\mathbb{C}$, in a way that is almost modular - you could replace it with another ring, and you'd have a new, somehow related group to study.

And so, using a bit of Galois theory and category theory that we'll develop along the way, that's exactly what they did! Instead of studying $GL_n(\mathbb{C})$, they studied GL_n, the abstract concept of a group of invertible $n \times n$ matrices - and other concepts like it.

In this marathon class (meaning that we'll meet for all but one class block, plus half of TAU, each day), we'll define algebraic groups, prove that every algebraic group is a subgroup of some GL_n, and ultimately classify semisimple algebraic groups, in a way intrinsically linked to the classification of finite simple groups.

Prerequisites: Group Theory, Ring Theory, Linear Algebra, Planning Meeting.

Algebraic Number Theory. (David)

We will explore number fields: their Galois groups, class groups, unit groups, and factorization into prime ideals. We will then proceed to a discussion of local and global class field theory.

You should have already talked to me if you want to take this class.

Prerequisites: Field Extensions and Galois Theory, Domains and Factorization: When Everything Goes Wrong.

Almost Planar. (Marisa)

Draw a network with 120 Mathcampers, and add an edge between two campers if they have ever taken a class together before. I would bet you $5 that the resulting graph is not planar (as in, can't be drawn without crossing some edges), but I'm so confident about winning that the question just isn't very interesting. So let's ask a more subtle question: how *far* is our graph from planar? In this class, we'll look at several different ways of measuring closeness to planarity, from the structural to the space-bending. The format will be inquiry-based, so you'll be discovering and proving results throughout the week.

Prerequisites: Intro Graph Theory or equivalent.

Analytic Number Theory. (Sachi)

Suppose I pick a very large integer n. What is the expected value of the number of divisors of n?

How quickly does the sum

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverge?

Are there infinitely many primes congruent to 1 mod 4?

Given two random integers a and b, what is the probability that a and b are relatively prime?

To answer these questions, we have to develop tools that can predict the average behavior of number-theoretic functions. We will study functions like the number of divisors function, the Euler totient function, and the prime counting function. Surprisingly, we can say a lot about the average behavior of these functions using techniques in analysis that estimate sums using integrals and by manipulating the order of summation.

Prerequisites: You should be comfortable with integral calculus and series.

Asymptotics of Generating Functions. (Kevin)

Generating functions often allow us to solve combinatorial problems in seemingly miraculous ways. We define a power series whose coefficients are a sequence we're interested in (even if we don't know more than the first few values explicitly!), and then we can pull a closed-form function for the power series out of a hat. If we can then extract the coefficients from this function, we've found a formula for the sequence!

Unfortunately, there are many, many closed-form functions where extracting coefficients exactly is impenetrable. The mathematician seeking an exact formula for the sequence is in this case out of luck. But this is where *we* are just getting started! We will discuss incredibly powerful, incredibly widely applicable strategies to study the asymptotics of generating functions, finding beautiful formulas for the nth term of our sequence that approximate the exact answer arbitrarily closely as $n \to \infty$. For example, we'll see how to get an arbitrarily close approximation to $n!$, going beyond the usual form of Stirling's formula. The strategies we use will have fancy names like singularity analysis and saddle-point asymptotics, and we'll see how amazingly far a few derivatives and integrals can take us.

If you have no experience with generating functions, that's OK. We'll be starting from the closed-form function (by which point the machinery of generating functions has already done its job), so no prior knowledge is necessary. On the flip side, if you absolutely love generating functions, be warned that this will be a very different, more analytic side of them, without the same sort of algebraic magic found in most generating functions classes.

Prerequisites: Single-variable calculus (derivatives, integrals, and power series).

A Tale of Combs and Hedgehogs. (Alfonso + Chris)

Once upon a time a princess was walking through the woods and came across a hedgehog. "I'm a beautiful prince" the hedgehog said, "all you need to do to free me from my curse is comb me nice and smooth." The princess was of course well-educated in mathematics, paid him no mind and moved on, for she knew that such a task was impossible and that he would be cursed forever.

Why was the princess so sure to move on? Because the hairy ball theorem states that no ball can be combed in such a way that no hair sticks up, and that there's no parting in the hair. In this class you will develop the proof for the hairy ball theorem on your own and on the way you will discover many important tools from algebraic topology!

This is a superclass that meets for two class hours a day (and possibly the first hour of TAU if we need it). You will be doing most of the work: we will provide worksheets with the right definitions and questions; you will spent a big chunk of the time working, alone or in groups, sometimes with our help, on all the steps of the proof. Some of the class time will be spent on presentation and discussion of your work.

This course is time-consuming, but all the work (homework included) is contained in the two to three daily hours.

Prerequisites: Some notion of continuity, basic group theory and linear algebra (images, kernels, and quotients).

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Prerequisites: None.

A Very Difficult Definite Integral. (Kevin)

In this class, we will show

$$\int_0^1 = \frac{\log(1+x^{2+\sqrt{3}})}{1+x}\,dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3}).$$

We'll start by turning this Very Difficult Definite Integral into a Very Difficult Series. Then we'll sum it!

We'll need an unhealthy dose of clever tricks involving some heavy-duty algebraic number theory. It will be Very Difficult.

(This class is based on a StackExchange post by David Speyer.)

Prerequisites: A ton of algebraic number theory. David's algebraic number theory marathon *might* suffice. A willingness to believe the facts we'll use is fine instead. Familiarity with rings and ideals is probably necessary regardless.

Benford's Law. (Assaf)

Benford's Law was discovered when an astronomer was sifting through log tables (the old kind) and noticed that the pages containing logarithms of numbers whose first digit is 1 were more worn out than others.

It turns out that approximately 1/3 of the numbers found in real life begin with a 1. In this class we will explain this phenomenon, and see where it comes from and how it can be applied for practical purposes.

Prerequisites: None.

Building Groups out of Other Groups. (Don)

In group theory, everybody learns how to take a group, and shrink it down by taking a quotient by a normal subgroup - but what they don't tell you is that you can totally go the other direction. It's a little harder, and a little messier, because while there's only one way to take a quotient, there's lots of ways to take a product. So, let's take a look at a few of these ways, from the clean, sterile direct product, to the messy, dangerous direct product, and even the beautiful, mighty semi-direct product.

Prerequisites: Group Theory.

Building Mathematical Structures. (Zach)

Come transform ordinary items into extraordinary geometric sculptures! In these intricate, large-scale, collaborative construction projects, we will work together to assemble room-filling rubber-band knot-webs, bouncy paper polyhedra, gorgeous drinking straw jumbles, and more! Browse `http://zacharyabel.com/sculpture/` for examples of the types of projects this course may feature. Assembling these mathematical creations requires scrutiny of their beautiful mathematical underpinnings from such areas as geometry, topology, and knot theory, so come prepared to learn, think, and build!

Prerequisites: None.

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Prerequisites: None.

Burnside's Lemma. (*Alfonso Gracia-Saz*)

(1) In how many different ways (up to rotation) can you colour a cube if each face can be coloured in blue, red, or green?
(2) How many different necklaces can you make with 6 pearls if you have unlimited black and white pearls?
(3) How many different necklaces can you make with $2N$ pearls which use exactly 2 pearls of each of N colours?

Burnside's Lemma is a cute, little gem that will allow you to solve the first two problems quickly and elegantly, without having to brute-force things or count lots of cases.

To answer the third question, though, you need the Redfield-Polya Theorem, which is like Burnside's Lemma on steroids.

You will learn both results in this class.

Prerequisites: Basic group theory.

Calculus on Young's Lattice. (Kevin)

Young's lattice is a beautiful poset capturing the structure of partitions, with connections to combinatorics, geometry, and representation theory. Young's lattice is almost unique among posets[1], with the remarkable property that we can do calculus on it. We'll discuss how, and we'll see how doing calculus on Young's lattice yields powerful enumerative results, including some famous identities involving Young tableaux. Plus, if you like exponentiating things in Assaf's class, we'll even get a chance to exponentiate xD!

Prerequisites: None.

Card Shuffling. (Zach)

The two days of this class do not depend on each other, so feel free to come to either (or both!).
Day 1: Seven Shuffles Aren't Enough! Conventional wisdom (namely, renowned mathematician Percy Diaconis) says that 7 shuffles are enough to thoroughly randomize a deck. Why, then, is there a simple experiment that works only 50% of the time on a truly randomized deck, but succeeds more than 80% percent of the time when only 7 shuffles are used?! We'll run this experiment in a few different incarnations (and lots of decks of cards), build a strong intuition for why it is so skewed, and discuss what this says about the "7 shuffles" wisdom.

Day 2: Perfect Shuffling. How do I make sure a deck of cards is perfectly randomized? This is not a practical question—I don't just want it "close enough" to random. I want a mathematically precise uniformly random deck, where each possible ordering has exactly a 1/52! chance of arising. Worse, what if all I have is a coin? How many flips will it take? What if I have a *weighted* coin with an unknown bias? We'll discover methods for all of these questions

Prerequisites: None.

Combinatorial Games. (Jane)

Let's play a game. We'll start out with two piles of pebbles, one with 8 pebbles and one with 11. We'll take turns making moves. On my turn, I'll remove all of the pebbles from one pile and then divide the other pile into two piles with at least one pebble each. Then, you'll move and do the same. The last person to make a move wins. If we both play optimally, who will win in this game? Does the result change if we change the number of pebbles in the starting piles?

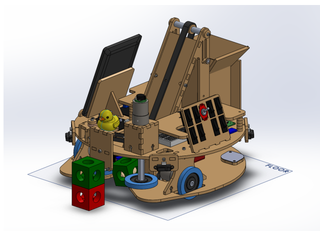
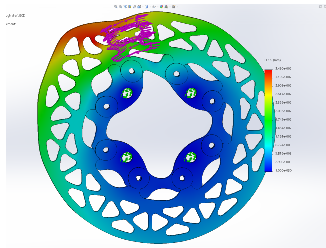
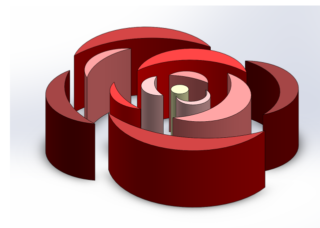
Combinatorial game theory is the study of games like the above empty and divide game, back and forth games of perfect information and no chance. In this class, we'll play and study various combinatorial games, and in the process learn some ways of analyzing the strategy of these games.

Prerequisites: None.

[1] Morally, there's only one other such poset, and no other lattices! Conjecturally, at least.

Computer-Aided Design. (Elizabeth)

Computers are awesome! They can do so many cool things! In particular, if you can imagine some shape or machine, you can make a computer draw it in 3D. Once the computer knows what it is, then it can show you what it would like from any angle, and you can tweak it without having to redraw the whole thing. You can also turn it colors and zoom in on small details. Basically, anything you can do in your head, you can show to other people, with the computer.

In addition, once you have told the computer about it, the computer can print out pictures or files that let machinists or machines make the part in real life. Computer aided design is useful for all kinds of things, from making robots to racecars to mathematical shapes.

Prerequisites: None.

Cut That Out! (Zach)

I hand you a square of paper and a pair of scissors, and your goal is to cut the square into some number of “puzzle pieces” that can be rearranged precisely into an equilateral triangle. Can you do it? Perhaps surprisingly this *is* possible, and even more surprisingly, you can do it with just 4 pieces. Try it! More generally, is it always possible to dissect any polygon into any other (assuming they have the same area)? What about a square and a circle? What about a cube and a tetrahedron? What if we allow infinitely many cuts, or “fractal” cuts, or…? In addition to discussing all of these questions and more, this class will feature a plethora of perplexing puzzles and a panoply of pretty pictures.

Prerequisites: None.

Cutting Surfaces into Silly Straws. (Assaf)

Sarah is an inventor, and has invented a remarkable product—the romantic straw. It is a straw with two ends, which forces couples to spend time together drinking as opposed to drinking apart. Sarah builds a factory that manufactures these romantic straws, but one day, the plastic injection machine fails, and instead outputs the genus g surface.

Sarah realizes that in order to recoup her losses, she can cut this surface and stretch it to get some straws. However, she doesn't want to waste any of the plastic. Her investors would like to know how many straws can she salvage?

In this course, we will define surfaces, and talk about something called the Euler Characteristic, which will help us solve Sarah's problem. If we have time, we will also discuss sillier straws, with an arbitrary number of holes in them.

Prerequisites: None.

dCalculus. (Jeff)

Can you guess the next number in the sequence?

$$1, 3, 6, 10, 15, ...$$
$$1, 1, 2, 3, 5, 8, ...$$
$$1, 2, 4, 8, 16, 32, ...$$
$$1, -1, 1, -1, 1, ...$$

I bet you solved all of these by recognizing that they were the solutions to discrete differential equations!

We'll take tools from standard single variable calculus and replace them with their discrete relatives, developing dDerivatives, dIntegrals, dFTOC, dTaylor Series and the dFourier transform. Then, we'll see what dProblems we can dSolve with our dTools, including why e is almost 2, and the hyperplane division problem.

Prerequisites: None.

de Bruijn Sequences. (Pesto)

de Bruijn sequences are sequences of 0s and 1s containing all possible subsequences of a given length exactly once: for instance, 0001110100 contains every possible sequence of 3 0s and 1s.

Days 1 and 2: For what lengths do such sequences exist, and how easily can we find them if they do?

Day 3: How were they used as an early error-correction code in Sanskrit poetry, 2500 years before error-correction codes even existed?

(Day 3 is and doesn't depend on Day 2 or any knowledge of Sanskrit.)

Prerequisites: None.

Divergent Series. (Sachi)

Shhh! We're going to sum series that the establishment doesn't want you to sum.

Have you ever wanted $1 - 1 + 1 - 1 + 1 - \cdots = 1/2$ to be true?

What should the value of $1 - 2 + 3 - 4 + 5 - \ldots$ be?

Abel said that "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever." Clearly he was trying to hide something.

Prerequisites: Calculus, knowledge of complex numbers and e to the i theta form.

Does ESP Exist? or, What's Wrong With Statistics? (*Mira Bernstein*)

In 2011, Daryl Bem, a professor of psychology at Cornell, published an unusual paper in the *Journal of Personality and Social Psychology*, titled "Feeling the Future: Experimental Evidence for Anomalous Retroactive Influences on Cognition and Affect". In this paper, Bem presents results from 9 experiments testing for "psi", aka ESP. For instance, in the first experiment, participants are instructed as follows:

> On each trial of the experiment, pictures of two curtains will appear on the screen side by side. One of them has a picture behind it; the other has a blank wall behind it. Your task is to click on the curtain that you feel has the picture behind it. The curtain will then open, permitting you to see if you selected the correct curtain.

Bem's analysis of the data shows that participants were able to predict the location of the picture at a rate significantly higher than chance, but only if the picture involved sex or violence. For other pictures, they did no better than chance. The follow-up experiments were designed to determine whether the psi effect was due to clairvoyance, psychokinesis, or retroactive influence in which "the direction of the causal arrow has been reversed".

Note: this is not just an individual loony. This is a highly respected professor of psychology, from one of the top departments in the country, writing for one of the top journals in his field, and being, if anything, more scrupulous in laying out his methodology than is standard practice. The journal editor wrote a preface to Bem's paper, stating that although he himself does not believe in the existence of psi, neither he nor the other reviewers could find a flaw in Bem's statistical analysis. Therefore, says the editor, scientific honesty compelled him to publish the paper, so that it could be discussed in a wider forum.

If you find all of this shocking, you are not alone. The entire psychology profession cringed in mortification and blamed the editor for not exercising better judgment.

Yet the real culprit here is not the author (who is free to pursue whatever research he wants, and whom no one suspects of violating scientific ethics) nor the editor (who really was just trying to adhere to the standard practice of his field), but the statistical methodology currently employed by most scientists. A famous 2005 paper published in a medical (!) journal states the problem very starkly in its title: "Why most published research findings are false".

In this course, we'll discuss the (many) problems with how statistics is currently done (not just in psychology). We'll talk about the historical reasons for these problems and present an alternative: Bayesian statistics—a methodology that is actually based on math and dates back to Gauss and Laplace. In particular, we'll apply a Bayesian analysis to Bem's results. (Spoiler alert: they give no evidence for the existence of ESP. Sorry to disappoint you.)

Prerequisites: Calculus and basic probability. If you don't know calculus, you will be able to follow some days but not others. No background in statistics required, though if you do have such a background, you may appreciate the de-brainwashing. .

Domains and factorization: when everything goes wrong. (*Alfonso Gracia-Saz*)

You know that every integer can be written as a product of primes in a unique way. But, are you sure this is true? It turns out that proving the uniqueness part is not easy at all, even though we all take this fact for granted since kindergarden!

In this class you learn how to prove this rigorously, and you will also study other "number systems" where the same result fails. Sometimes the uniqueness part fails. Sometimes some numbers cannot be written as product of primes at all! Pathological examples are delicious.

This class will alternate between mini-lectures and time for you to work things out in class. The homework won't be long, but I will assume you are finishing it every day.

Prerequisites: Ring theory. Specifically, you need to be comfortable with the concepts of ring, ideal, and "ideal generated by".

Dynamical Systems. (Jane)

Dynamical systems are spaces that evolve over time. Some examples are the movement of the planets, the bouncing of a ball around a billiard table, or the change in the population of rabbits from year to year. The study of dynamical systems is the study of how these spaces evolve, their long-term behavior, and how to predict the future of these systems. It is a subject that has many applications in the real world and also in other branches of mathematics.

In this class, we'll survey a variety of topics in dynamical systems, exploring both what is known and what is still open. Each day of class will focus on a different area of dynamical systems. We'll think about billiard trajectories on polygonal and non-polygonal tables. We'll learn about how fractals like the Mandelbrot set and Sierpinski's triangle arise from dynamics. We'll study cellular automata and how they can be used to model real-life situations. We'll also explore the connections between dynamical systems and number theory, such as what dynamics can tell us about the distribution of the leading digits of the powers or two.

Prerequisites: None.

Egyptian Fractions. (Jane)

The ancient Egyptians only had notation to write fractions of the form 1/n. If they wanted to write any other positive rational number, they had to write it as a sum of distinct terms of this form.

This sounds incredibly restrictive, but actually every rational number can be written as an Egyptian fraction. In this class, we'll take the idea of an Egyptian fraction and run with it. It turns out to generate quite a lot of interesting math. As we discover things about Egyptian fractions, we'll also see how quickly we run into questions about them that are still open.

Prerequisites: None.

Epsilons and Deltas. (Sam)

In the early 18th century, Bishop Berkeley was fed up with mathematicians calling religion irrational, and wrote the wonderfully titled *The Analyst, or a Discourse Addressed to an Infidel Mathematician.* Here he ripped into the philosophical foundations of Newton and Leibniz' calculus. Several mathematician's then tried to respond by putting calculus on a rigorous foundation, culminating in Cauchy's epsilon-delta formalization. More than that, Cauchy was a wonderful and colorful mathematician. He taught engineers at the Ecole Polytechnique, where he was notorious for forcing them to learn modern analysis instead of computational methods; he basically wrote the book on analysis; and he stuck to his principles (rightly or wrongly) at all costs.

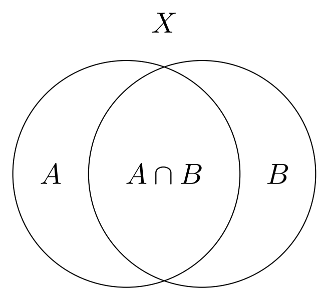
tl;dr: in my book, Cauchy is the quirky hero who saved calculus. Come to this class to find out why!

Prerequisites: Basic calculus, including the formal epsilon-delta definition of a limit and limit definition of a derivative will help.

Extending Inclusion-Exclusion. (Jeff)

A general principle in mathematics is that if you want understand some complicated mathematical object, you break it up into simpler objects, and the reassemble the larger object from the simple pieces, and see how those pieces fit together.

A simple example comes from combinatorics and sets. Let X be a set, and let $X = A \cup B$, like drawn in the following picture.



If we are trying to compute the size of X, we can instead compute it as

$$|X| = |A| + |B| - |A \cap B|$$

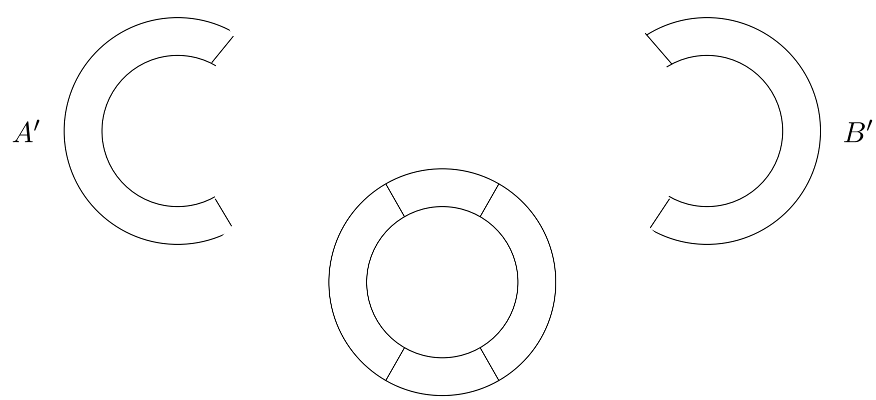
Notice that X is "glued" together from A and B, so we had to subtract the contribution from the intersection of $A \cap B$ when computing the size (or we would make the error of overcounting.)

Inclusion exclusion works well with some other quantities too– for instance, if we had replaced "cardinality" with "area", we would see the same principle holds.

There are a lot of places where the principle of Inclusion exclusion seem to hold, but are a little bit off. For instance, let $C(X)$ count the number of connected components of a shape. So, for our example, $C(X) = C(A) = C(B) = C(A \cap B) = 1$. Then we again have that

$$C(X) = C(A) + C(B) - C(A \cup B).$$

However, if we look at the shape drawn here



$Y = A' \cup B'$

it is no longer the case that we have this inclusion-exclusion principle. We get

$$C(Y) = C(A') + C(B') - C(A' \cap B') + 1.$$

However, the error to our principle measures the fact that Y has a hole punched in the middle of it. To get the full generalizations of this, we'll have to explore the relationships between combinatorics, linear algebra and topology.

Prerequisites: You should be able to explain that if $f : X \to Y$ is a linear map between vector spaces, why $\dim \ker(f) + \mathrm{im}(f) = \dim(X)$..

Factoring With Elliptic Curves. (David)

Elliptic curves are defined by equations of the form

$$y^2 = x^3 + ax + b.$$

One of their amazing properties is that the set of points (x, y) satisfying this equation forms a group. This even holds when you consider x and y modulo p. We will use these curves to find a method for factoring huuuge integers, which is used in practice.

Prerequisites: Elementary number theory, group theory.

Field Extensions and Galois Theory. (Mark)

We'll begin by defining what field extensions are, and seeing some of what they can be used for. As an example, if you were never comfortable with defining the complex numbers by postulating the existence of a square root of -1, we'll see early on how we can "make" a square root of -1 using polynomials. We'll also see why some ancient and (in)famous construction problems, such as squaring the circle, are provably impossible.

You may know the story of the brilliant mathematician and societal misfit Galois, who died tragically young in a duel after developing an exciting new area of mathematics that was beyond what most of his contemporaries could even follow. In this class we will build up to the fundamental theorem of Galois theory, which gives an unexpected and beautiful correspondence allowing us to find and describe field extensions in terms of subgroups of a certain group.

If time permits, we'll move on (perhaps with some side trips to scenic overlooks of other material) to sketch a proof that there is no general way of solving polynomial equations of degree 5 or more by radicals; that is, there is no analog of the quadratic formula for degree 5 and higher. (There are such analogs for degree 3 and 4.)

Prerequisites: Linear algebra, group theory, ring theory (familiarity with polynomial rings will be especially useful).

Finitely-Generated Algebras. (Susan)

Have you ever wanted to build an algebraic structure that does something *really* terrible? Sure, noncommutativity is cool and all, but how would you define a ring that has an element with a right inverse but no left inverse? Or maybe two nonzero elements a and b such that $ab = 0$, but ba is nonzero?In this class, we'll be talking about how to build rings with specific bad behaviors using polynomials as our building blocks. We'll talk about how to tell when two elements of these rings are "the same," and how we can measure the size of the objects we've built.

Prerequisites: I'll use the word "ring," so knowing the definition might be helpful, but probably not absolutely necessary.

From Matrices to Representations. (*Noah Snyder*)

How do you classify matrices up to similarity? What about matrices satisfying some algebraic property, like that their 5th power is the identity? How about pairs of commuting matrices that both have their 5th power the identity? We'll answer these questions and some additional variations, and on the way we'll learn about some representation theory.

Prerequisites: Prereqs: Some linear algebra is necessary (vector spaces, spans, linear independence, basis). You should not take this class if you've already seen Jordan Normal Form, but the class should be interesting even if you've already learned the character-theoretic approach to representation theory.

Functional Programming. (Nic)

Functional programming languages are ones that encourage the programmer to focus not on which instructions a program is supposed to follow but on what the output is supposed to be. When writing in a functional style, variables can't be modified, data structures can't change their shape, and functions can't do anything with their input but use it to compute and return an output.

This might sound like coding with both hands tied behind your back, but once you get used to it, the functional paradigm can be incredibly powerful. By giving up side-effects, you gain the ability to reason more easily about what your program is doing. For example, here's a function which takes a binary tree with entries at each leaf and a function taking those entries to integers, then applies the function to all those entries and sums the results:

```
let rec sum_tree f t = match t with
| Leaf n -> f n
| Branch (l, r) -> sum_tree f l + sum_tree f r
```

In this class, we'll explore functional programming through the language OCaml. We'll meet for two hours a day, and most class time will be spent writing code to solve problems that will be provided. The problems will be on lots of different topics, and several will be built around writing your own interpreter for a simple programming language. You'll be encouraged to solve problems at your own pace; there should be more than enough to keep you occupied for the whole week, but if you finish I'll come up with more for you to do.

Prerequisites: Some programming experience, but not necessarily in a functional language.

"Gambling" Games. (Nathan, *camper teaching project*)

Suppose you have a dollar, but you need 5. There is a casino which allows you to give them a dollar for a chance (not necessarily $\frac{1}{2}$) to get your dollar back plus another dollar, but otherwise they keep your dollar. Should you play the game, and do you have enough time to play it? Come to this class to find out.

Prerequisites: You should understand linearity of expectation.

Generating Functions and Partitions. (Mark)

Generating functions provide a powerful technique, used by Euler and many later mathematicians, to analyze sequences of numbers; often, they also provide the fun of working with infinite series without having to worry about convergence.

As a “serious” example, we’ll use a generating function to get an expression in closed form for a famous and important sequence, that of the Catalan numbers. This sequence starts off 1, 2, 5, 14, 42, ..., and it comes up in the solution of many counting problems, involving, among other things, voting, lattice paths, and polygon dissection.

A partition of a positive integer n is a way to write n as a sum of one or more positive integers, say in nonincreasing order; for example, the seven partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, and 1 + 1 + 1 + 1 + 1. The number of such partitions is given by the partition function p(n); for example, p(5) = 7. Although an “explicit” formula for p(n) is known and we may even look at it, it’s quite complicated. In our class, we’ll combine generating functions and a famous combinatorial argument due to Franklin to find a beautiful recurrence relation for the (rapidly growing) partition function. This formula was used by MacMahon to make a table of values for p(n) through p(200) = 3972999029388, well before the advent of computers!

Note: If you were at Mathcamp 2015 and took a class with a similar name, this is indeed the same class, so you should probably look for something different.

Prerequisites: Summation notation; geometric series. Some experience with more general power series may help, but is not really needed. A bit of calculus may be useful.

Geometric Group Theory. (Susan)

Given a group with a finite set of generators, we can build a visualization called the Cayley graph. Unfortunately, the Cayley graph is not a group invariant—different choices of generators result in different graphs. But these different graphs have a certain same-ey-ness about them. For example, all Cayley graphs of the integers look long and thin. So what’s going on here?

In this class we’ll define the notion of a quasi-isometry, the formalization of this idea of same-ey-ness. We’ll explore this idea, and show that each group is associated to a particular metric space, and that this metric space is unique up to quasi-isometry. We’ll also play with group actions, and show that if a group has a particular kind of action on a particular kind of metric space, we can use that action to find a finite set of generators for the group.

Prerequisites: Group Theory.

Graph Colorings. (Mia)

Normal graph colorings are *so* cliche and old-school. Like, 1800s old-school. If you want to be modern, like bell bottoms, disco, and pet rocks of the 1970s, you might like list coloring. List coloring is a system in which each vertex comes with a list of acceptable colors and you may only color a vertex using a color from its list.

Or, if you prefer Rainbow Brite, Hacky-Sack, and John Hughes movies about angsty teenagers of the 1980s, you’ll be wiggin’ out over fractional coloring. Fractional coloring allows the graph theorist to color each vertex with multiple colors: for example, I could paint a certain vertex half red, a quarter blue, and a quarter pink.

And with all these new, rad coloring schemes, we get new, rad chromatic numbers. Will these chromatic numbers differ from our old-school $\chi(G)$? Can they differ by arbitrarily much? In this class we will investigate these questions and see some gnarly graphs.

Prerequisites: Intro to Graph Theory.

Graph Minors. (Pesto)

You may have seen two classic graph theory problems: "if you have five cities, can you route roads between each pair of them without crossings?" and "if you have three houses and three utilities, can you build utility lines from each utility to each house without crossings?"; that is, whether K_5 and $K_{3,3}$ are planar graphs. The answers are (spoiler alert!) both no.

In fact, those two graphs are *the* fundamental nonplanar graphs: every graph is planar unless and only unless it "contains" one of those two graphs, if we have the right definition of "contains".

With the right definition of "contains", we can prove a similar result for graphs drawn on any surface (say, on a torus instead of on the plane) and generalize the four-color theorem.

Prerequisites: Know what connected graphs, planar graphs, and Euler's formula are.

Graph Polynomials. (Jeff & Mia)

Are you one of millions of people who have a problematic graph G in your life? Do you lie awake and wonder:

- How will I count the number of spanning forests in G?
- What is the probability that G remains connected after randomly removing each edge based on a coin flip?
- How many ways are there to color G with my 256-color box of Crayola crayons?

Well, then we have a degree $|E| + |V|$ polynomial for you!
This 1954 masterpiece, defined by Tutte, computes exactly the properties that can be determined by edge contraction and deletion. No extra estimation or calculation needed. This *one* polynomial will solve all of your problems above (and many more). 100% recursively defined. Factorizable over connected components. Made in Canada.
If you order today, we'll even throw in two specialized polynomials, free of charge!

Prerequisites: Graph theory.

Group Actions. (Don)

Groups are great - they've got operations, they've got identities, they even have inverses. And let me tell you, I have the best groups. But you know what the groups of today don't have enough of? Applications! We need to put our groups back to work, and in this class, that's exactly what we're going to do.

The problem with everybody else's group actions is that they don't have enough freedom. That's not a problem for me - my actions are fantastic. They have the most freedom. If you want to make group theory great again by learning about affine spaces and applications to mathematical physics, come to this class!

Prerequisites: Group Theory.

Guarding an Art Gallery. (Jane)

Suppose that you're in charge of security for an art gallery. The gallery is a polygonal room, and you would like to station guards around the room. Each guard will stand in place, but can see infinitely far in all directions from where he is standing. What is the least number of guards that you need to place so that every spot in the gallery can be seen by at least one guard? In this short class, we'll examine this problem and some variations of it.

Prerequisites: None.

Hard problems that are almost easy. (Vivian)

Some problems are easy. Like, sorting a list of n numbers is not too bad; you can do it in time proportional to $n \log n$. And some problems are hard. For example, deciding if a graph with n vertices is k-colorable is a type of hard that is called NP-complete.

But how can you tell which ones are which? We're going to be looking at several problems that are NP-complete, but become polynomial-time solvable if you change a tiny piece of the phrasing. In addition we'll define what it means for a problem to be polynomial-time solvable, or NP-complete.

Prerequisites: None.

Harmonic Analysis on Abelian Groups. (*Michael Orrison*)

In this course, we will focus on how and why you might want to rewrite a complex-valued function defined on a finite abelian group as a linear combination of surprisingly simple functions called "characters". As we will see, doing so will quickly lead us to discussions of far-reaching algorithms and ideas in mathematics such as discrete Fourier transforms (DFTs), fast Fourier transforms (FFTs), random walks, and the Uncertainty Principle.

Prerequisites: Linear Algebra (bases, invertible matrices, eigenvalues, inner products, orthogonality), Group Theory (examples of finite abelian groups and their subgroups, cosets, homomorphisms), and the ability to add and multiply complex numbers. .

Harmonic Functions on Graphs. (Yuval)

Let's say you're a camper checked out to Walmart, and you have no idea how to get back to camp. You decide to walk home randomly: at every intersection, you roll a fair die to decide which path to take. Since you want to make sure you get back by the end of sign-in, you wonder: how long, on average, will your walk take?

This sounds like a hard question. However, it turns out that we can solve it—by turning all of the roads into rubber bands. Along the way we'll discover how to draw graphs, tile rectangles, and count trees.

Prerequisites: Introduction to Graph Theory.

History of Math. (*Moon Duchin*)

The two days of this class cover two topics.

Day 1: Math and the Body.

Math is done by people with bodies! And they use their bodies in various ways, like finger-reckoning, writing, gesture, and model-building. I've got some fun examples for us to think through, from Baghdad to Berkeley.

Day 2: Mind Reading with Artifacts.

By a happy fluke, the materials that some ancient Babylonians used for their math work were curiously permanent, and we've got a trove of clay tablets to work with. I'll talk about readings and mis-readings of artifactual evidence, and how to reconstruct a mathematical culture.

Prerequisites: None.

Homotopy (Co)limits. (Chris)

Imagine you are back in your childhood and your playing with some clay. You have 3 pieces of clay and want to assemble them according to some instructions your friend sent you. Unfortunately, your evil sibling took your clay and completely deformed your pieces. Nevertheless you go ahead and assemble it. The next day you and your friend realize that you both got completely different structures. Yours even has a hole where theirs doesn't!

What you just discovered is what topologists refer to as the fact that "the colimit of a diagram is not homotopy invariant." In this class, we will explore this phenomenon (possibly with or without clay), and hopefully we will be able to answer what the true and real and homotopy-invariant way of assembling your clay should be!

Prerequisites: Category theory (basic definitions, some of the following universal properties: (co)products, push-outs, pull-backs), topology (continuity).

How Not to Prove the Continuum Hypothesis. (Susan)

We know that the natural numbers are countable, and that the real numbers are uncountable, but what happens in between? What does it mean for a subset of the reals to be "small?" Is a small set necessarily countable? In this class we will explore different notions of smallness and largeness, touching on ideas from topology, measure theory, and logic. Especially logic! We'll learn the rules to several games played on infinite sets that can give us valuable information about how sets behave. One of these games will allow us to almost-but-not-quite prove the Continuum Hypothesis.

Prerequisites: Understand why $\mathbb{Q}$ is countable, and why $\mathbb{R}$ is not.

Huuuge Primes. (David)

If you want to check whether a 100-digit number is prime, trial division is infeasible: it would take trillions of years on our fastest supercomputer. But we'll see how to test such numbers for primality in milliseconds, and look at applications of such large primes to modern cryptography. Along the way we'll build up foundational ideas of number theory: modular arithmetic, the Euclidean algorithm, Fermat's theorem, Euler's theorem and the Chinese remainder theorem.

This class will be "inquiry-based:" you'll spend class time working in small groups at the board on problems exploring these ideas.

Prerequisites: None.

Hyperbolic Geometry. (Jane)

In the Euclidean plane, two lines that are not parallel will always intersect. This is the parallel postulate, one of the five postulates of Euclidean geometry, which date back to Euclid in 300 BC. For many hundreds of years afterward, mathematicians were convinced that the parallel postulate was a result of the other four, but proofs of this foiled them.

Then, in the 1800s, Gauss and others proved that non-Euclidean geometries exist. In entered the hyperbolic plane, a geometric space that satisfies the first four postulates of Euclidean geometry but not the parallel postulate. In this class, we'll explore the geometry of the hyperbolic plane and many of the interesting things that can happen there.

Prerequisites: Calculus, group theory .

Hyper-Dimensional Geometry. (Drake, *camper teaching project*)

You may have heard of the 5 Platonic solids. Why are there just 5? What happens if we try making something similar in 4 dimensions? 5? 6? What does it even mean to extend this to higher dimensions?

You've probably also seen the area of a circle, πr^2, and the volume of a sphere, $\frac{4}{3}\pi r^3$. How does this formula behave as we go into higher dimensions? How fast do those coefficients grow?

In this class we'll answer all of these questions in two parts: first, we'll talk about analogs of the platonic solids, and then we'll see how to compute the volume of an n-dimensional sphere. We'll also be able to apply our formula from the second part of the class to provide a counterintuitive solution to a problem that remained open for several decades!

Prerequisites: Calculus (for the second part).

Introduction to Graph Theory. (Marisa)

There is a theorem that says that for any map of, say, countries on your favorite continent, you can color the countries so that any two countries that share a border (not just meet at a point, but actually share some boundary) get different colors, and that the number of colors you will need is no more than 4. (Try inventing a complicated political landscape and coloring: no matter how crazy the scene, you'll always be able to color the map with four colors.)

Mathematicians have been pretty convinced about the truth of this Four Color Theorem since the late 1800s, but despite many false starts, no one gave a proof until 1976, when two mathematicians wrote a very good computer program to check 1,936 cases. (To this day, we have no human-checkable proof.)

In this class, we will definitely not prove the Four Color Theorem. You will, however, prove the Five Color Theorem, which is a whole lot shorter (and which was successfully proven by hand in 1890). Along the way, you'll meet many other cool concepts in Graph Theory.

Notice how I said "*you* will prove"? That's because the course will be inquiry-based: I won't be lecturing at all. You'll be working in small groups to discover and prove all of the results yourself!

Prerequisites: None.

Introduction to Group Theory. (Kevin)

Groups are a type of mathematical object which generalize the notions of operations like addition and multiplication to things that aren't numbers in a way that allows us to analyze symmetries and transformations of other objects. They are essential to the study of geometry, linear algebra, number theory, real and complex analysis, topology, and many, many other fields

This class is an introduction to the theory of groups. We'll define groups and discuss several related constructions and examples, like cyclic and dihedral groups, homomorphisms, subgroups, cosets, quotients, and, time permitting, group actions.

The point of this class is to be a first exposure to the definition and basic theory of groups to prepare students for a number of classes coming up later in camp. If you've studied groups before and have some experience working with them, this is probably not the class for you. On the other hand, if you're new to the topic, you'll probably get a lot out of it, and it should be great preparation for a wide variety of other areas of math.

Prerequisites: None.

Introduction to Ring Theory. (*Ari Nieh*)

The idea of a ring is simple: it's just an algebraic structure with two operations, addition and multiplication, made compatible by the distributive law. This framework, however, provides a way to talk about so many useful mathematical objects: not only numbers of many sorts, but also polynomials, matrices, and different flavors of functions. In fact, it's difficult to find an area of pure mathematics that doesn't have rings in it somewhere!

In this class, we'll look at basic notions of rings and subrings. We'll talk about how to construct new rings out of old ones. Then, we'll discuss maps of rings, ideals, quotient rings, and the relationship between these concepts. As time allows, we'll also discuss fields of fractions, localization, modules, and other assorted topics.

Prerequisites: None.

IOL 2016. (Pesto)

Margarita and three other Mathcamp alums are currently competing at the International Linguistics Olympiad in Mysore, India. To celebrate their participation, Pesto will teach about whatever bits of linguistics are featured in this year's contest problems.

Disclaimer: Pesto has no idea what the content of this course will be or whether there'll be anything mathematical involved, since the problems won't be released until the start of week 5.

Prerequisites: Who knows?

King Chicken Theorems. (Marisa)

Chickens are incredibly cruel creatures. Whenever you put a bunch of them together, they will form a pecking order. Perhaps "order" is an exaggeration: the chickens will go around pecking whichever chickens they deem to be weaker than themselves. Imagine you're a farmer, and you're mapping out the behavior of your chickens. You would like to assign blame to the meanest chicken. Is it always possible to pick out the meanest chicken? Can there be two equally mean chickens? Are there pecking orders in which all the chickens are equally mean?

Prerequisites: None.

Knot Theory. (Jeff)

In the 1860s, Lord kelvin developed the following theory of matter: that atoms, the indivisible particles that composed the universe, were actually tiny whirlwind vortices in the ether. The shape of these vortices were tiny knots, and you could make compounds out of these Knots by linking them together. Inspired by the quest to classify atoms, a mathematician named Tait made a list of all knots up to 10 crossings (no small feat, considering that there are a round 250 of them.)

Kelvin's theory turned out to be bunk (as both the idea of ether and tiny vortices were too crazy,) but mathematicians kept on thinking about knots, It took mathematicians nearly a hundred years to realize that Tait's list was wrong, and we still have a lot to learn about knots. We now study knots not because they represent atoms, but because the are some of the simplest objects a topologist can study: maps for a the circle to 3-dimensional space. And despite these objects being so fundamental, a classification of knots eludes mathematicians to this very day.

In this class, we'll take the first step to classifying knots, by describing invariants of the knot and giving a procedure to (non-uniquely) describe every Knot.

Prerequisites: None!

K-Theory. (Don)

Proving that two things aren't the same is really hard. In algebraic topology, there are homotopy and homology groups, which can let you do so for topological spaces, but they are often fiendishly difficult to compute. In ring theory, there are K-groups, which aren't just hard to compute - they were historically difficult to define.

In this class, we'll look at just the first few K-groups. $K_0(R)$ is defined in terms of all $R-$modules, so we'll study modules and resolutions to understand it. $K_1(R)$ is defined in terms of $GL_infty(R)$, so we'll need to brush up on colimits in order to define it. Finally, $K_2(R)$ is already too much unless R is a field - but we might be able to see how $K_2(\mathbb{Q})$ is inextricably linked to quadratic reciprocity.

Prerequisites: Ring Theory.

Latin Squares and Finite Geometries. (Marisa)

Consider the 16 aces, kings, queens, and jacks from a regular 52 card deck of playing cards. Can the 16 cards be arranged in a 4×4 array so that no suit and no rank occurs twice in any row or column? To make it a little harder: is it possible to color the cards 4 different colors (say red, green, orange, blue) such that (a) no two cards have the same color and suit, and (b) no two cards have the same color and rank, and (c) no row or column has the same color twice? Turns out the answer is yes, and also we can use that carefully-constructed array to produce finite affine and projective geometries (like the Fano plane).

Prerequisites: None.

Linear Algebra. (Mark)

You may have heard that linear algebra involves computations with matrices and vectors, and there is some truth to that. But this point of view makes it seem much less interesting than the subject really is; what's exciting about linear algebra is not those computations themselves, but (1) the conceptual ideas behind them, which are elegant and which crop up throughout mathematics, and (2) the many applications, both inside and outside mathematics. In this class we'll deal with questions such as: What is the real reason for the addition formulas for sin and cos? What happens to geometric concepts (such as lengths and angles) if you're not in the plane or 3-space, but in higher dimensions? What does "dimension" even mean, and if you're inside a space, how can you tell what its dimension is? What does rotating a vector, say around the origin, have in common with taking the derivative of a function? What happens to areas (in the plane), volumes (in 3-space), etc. when we carry out a linear change of coordinates? If after a sunny day the next day has an 80% probability of being sunny and a 20% probability of being rainy, while after a rainy day the next day has a 60% probability of being sunny and a 40% probability of being rainy, and if today is sunny, how can you (without taking 365 increasingly painful steps of computation) find the probability that it will be sunny exactly one year from now? If you are given the equation $8x^2 + 6xy + y^2 = 19$, how can you quickly tell whether this represents an ellipse, a hyperbola, or a parabola, and how can you then (without technology) get an accurate sketch of the curve? How do astronomers know the chemical composition of distant stars?

Note: Although there was a linear algebra course at Mathcamp last year which I taught also, this won't be the same class: It runs for two weeks instead of one, and it has an extra chili so the pace should be brisker - and we should get to cover a lot more material.

Prerequisites: Although the blurb refers to taking a derivative, you'll be able to get by if you don't know what that means. If you have no previous exposure to abstract concepts, you should at least take the Mathcamp Crash Course at the same time.

Many Cells Separating Points. (Nikhil, *camper teaching project*)

Consider a set of 7 points in the plane. Three lines can form 7 cells, so we wonder if we can draw the lines such that each cell has one point in it. You'll quickly realize that this isn't possible for all sets of 7 points, such as when the points lie on a circle. So what happens when we have more points and lines?

In this class, we will explore the concept of equal point separation, which will lead to a generalized notion of convexity and new classifications for geometric objects. We will also generalize the Erdős–Szekeres Conjecture, an open problem for 81 years in Ramsey theory which states that a set of $2^{n-2}+1$ points contains a convex n-gon, to our new class of objects based off new research from just April of this year.

Prerequisites: None.

Math and Literature. (Yuval)

Many Mathcampers (including me!) love reading, but we often think that reading and doing math are fundamentally different things. Though this is sometimes the case, there are many instances in which math and literature are inextricably related. In this class, we'll explore some incredible pieces of literature and discuss the math that went into their creation

Note: Though I will provide some of the literature for you to read, doing so is totally optional. In particular, feel free to come even if you aren't comfortable reading in English!

Prerequisites: None.

Mathcamp Crash Course. (*Nina White*)

This course covers fundamental mathematical concepts and tools that all other Mathcamp courses assume you already know: basic logic, basic set theory, notation, some proof techniques, how to define and write carefully and rigorously, and a few other tidbits. If you are new to advanced mathematics or just want to make sure that you have a firm foundation for the rest of your Mathcamp courses, then this course is *highly* recommended.

Here are some problems to test your knowledge:

(1) Negate the following sentence without using any negative words ("no", "not", etc.): *"If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."*
(2) Given two sets of real numbers A and B, we say that A *dominates* B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two, disjoint, non-empty sets A and B such that A dominates B and B dominates A.
(3) Prove that there are infinitely many prime numbers.
(4) Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
(5) Define rigorously what it means for a function to be increasing.
(6) Prove that addition modulo 2013 is well-defined.
(7) What is wrong with the following argument (aside from the fact that the claim is false)?

> **Claim:** On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.
>
> **Proof:** We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. QED.

(8) Explain what it means to say that the real numbers are uncountable. Then prove it.

If you are not 100% comfortable with most of these questions, then you can probably benefit from this crash course. If you found this list of questions intimidating, then you should *definitely* take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Prerequisites: None.

Mathcampers Show Presentations. (Sam + Chris)

Once upon a time, a mathcamper had to give a technical talk. She was surprisingly self-aware, and realized that giving a good technical talk takes lots of practice and effort. If you've ever been in – or hope to be in – the position of teaching someone math, presenting to a math circle, or sharing research, this class is for you. If you're not as self-aware as that camper, this class is definitely for you!

Parts of the class will involve discussion and activities to facilitate giving good talks, but the main point of this class is for every camper to give a clear and compelling five minute technical talk to the class on Friday or Saturday. You'll also get to hear lots of awesome short talks by other campers!

Warning: You'll be expected to prepare – and practice at least 3 times – a technical talk in the course of the week. This takes time and effort!

Prerequisites: None.

Models of Computation Simpler than Programming. (Pesto)

Almost all programming languages are equally powerful—anything one of them can do, they all can.

We'll talk about less powerful models of computation—ones that can't even, say, tell whether two numbers are equal. They'll nevertheless save the day if you have to search through 200MB of emails looking for something formatted like an address.[2]

This is a math class, not a programming one—we'll talk about clever proofs for what those models of computation can and can't do.

Prerequisites: None.

Model Theory. (*Steve Schweber*)

We talk about math a lot. But *how* do we talk about it? The formal language we use to describe mathematical structures such as groups, rings, and fields is interesting in itself—and, in fact, can be viewed as a mathematical structure! In this class we'll explore the mathematics of talking about mathematics:

- How can we describe various structures in math?
- When are two structures "indistinguishable" using a given language?
- How can we use the study of mathematical languages to help us understand the structures we care about?

We'll focus on the main formal language of mathematics—*first-order logic*—but we'll also see some of its close relatives, including (as time permits) equational logic, second-order logic, and infinitary logic.

Prerequisites: None.

[2] xkcd.com/208

More Group Theory! (Kevin)

Want more group theory? Let's continue where we left off in Week 1!

We'll start studying finite groups in detail, using our work with quotients and cosets to learn a surprising amount about a group just based on its order. If all goes well, we'll try to get through the Sylow Theorems, which tell us *a lot* about what the possible subgroups could be.

Prerequisites: Group Theory (Week 1) or equivalent.

More Problem Solving: Polynomials. (Pesto)

Polynomials are a frequent topic of Olympiad-style competitions, since there are many and interesting problems using them. For instance, if $a_1, \ldots, a_n$ are distinct real numbers, find a closed-form expression for

$$\sum_{1 \leq in} \prod_{1 \leq j \leq n, j \neq i} \frac{a_i + a_j}{a_i - a_j}.$$

Don't see the polynomials? Come to class and find them.

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to Olympiad-style problems you will have solved as homework the previous day.

Disjoint from 2013's (but not 2014's) polynomials problem solving class. Disjoint from the week 4 polynomials problem solving class.

Prerequisites: Linear algebra: Understand what a "basis for polynomials of degree at most 2 in two variables" is.

Multilinear Algebra. (Nic)

If you've seen a lot of linear algebra before, you've probably encountered the definition of an inner product. (Sometimes it's called a "dot product" or a "scalar product.") This is a way of taking two vectors and producing a number, but unlike a lot of the other functions you encounter in linear algebra, it's not a linear transformation. Instead, it's what's called a "bilinear map" — it's linear in each of its coordinates separately.

This class is an exploration of a construction called a tensor product which is critical to analyzing functions like this one. Along the way we'll also produce a very clean and well-motivated definition of the determinant, and introduce the notion of a "universal property," which is foundation of the modern, categorical approach to abstract algebra.

Prerequisites: Linear algebra, up through the definitions of linear independence, spanning, bases, dimension, and kernels, and the relationship between the dimensions of the image and kernel of a linear transformation. The first week of Linear Algebra should be enough, but you're encouraged to take the second week at the same time as this class.

Multiplicative Functions. (Mark)

Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.

Prerequisites: no fear of summation notation; a little bit of number theory. (group theory is *not* needed.) .

Natural Language Understanding. (*Greg Burnham*)

Human language is tantalizingly close to being a formal system. We certainly *feel* like there is a clear relationship between the words we express and the facts we mean to convey. If this intuition were true, however, then we should be able to write computer programs to perform linguistic tasks (e.g., read a document and answer questions about it). But 50 years of trying to write such programs have yielded mixed results.

This class will motivate why it is so difficult to write computer programs capable of performing linguistic tasks and then describe what tasks computers *can* currently perform, focusing on how recent algorithmic and technological progress has allowed for improved performance. We will conclude by noting that the big goals remain unsolved and speculating on what might be necessary for a solution.

As a teaser, here is my favorite short example motivating why computational language understanding is hard. Consider the following two sentences, which differ only in the last word:

"The cat caught the mouse because it was clever."

"The cat caught the mouse because it was careless."

Now, what does the pronoun "it" refer to in each sentence? Humans share a clear intuition about the right answer to this question. And yet, consider what it would take to write a computer program with this same capability. That's the problem in a nutshell.

Prerequisites: None.

Neural Networks. (Kevin)

Here's some math you won't learn at Mathcamp[3]:

The following lemma surjective restrocomposes of this implies that $\mathcal{F}_{x_0} = \mathcal{F}_{x_0} = \mathcal{F}_{\mathcal{X},\ldots,0}$.

Lemma 0.2. *Let X be a locally Noetherian scheme over S, $E = \mathcal{F}_{X/S}$. Set $\mathcal{I} = \mathcal{J}_1 \subset \mathcal{I}'_n$. Since $\mathcal{I}^n \subset \mathcal{I}^n$ are nonzero over $i_0 \leq \mathfrak{p}$ is a subset of $\mathcal{J}_{n,0} \circ \overline{A}_2$ works.*

Lemma 0.3. *In Situation ??. Hence we may assume $\mathfrak{q}' = 0$.*

Proof. We will use the property we see that $\mathfrak{p}$ is the mext functor (??). On the other hand, by Lemma ?? we see that

$$D(\mathcal{O}_{X'}) = \mathcal{O}_X(D)$$

where K is an F-algebra where δ_{n+1} is a scheme over S. □

[3]Taken from `http://cs.stanford.edu/people/jcjohns/fake-math/4.pdf`.)

In fact, you won't learn this math anywhere, because it's complete and utter nonsense. But it *looks* like math at a distance, and I bet you could convince a lot of non-mathematicians that it's real!

What twisted brain produced this abomination? An artificial one! Taking (loose) inspiration from our own brains, neural networks are more or less simply a jumble of interconnected nodes that fire or not based on the nodes connected to them. With proper training, this tangle can learn to do many things unreasonably effectively, from arithmetic to handwriting recognition to Shakespeare imitation to doing a remarkably good impression of a crackpot mathematician.

In this class, we'll see a lot of demos and we'll study the math and algorithms behind increasingly complex systems, from the lowly perceptron to the mighty recurrent neural network. This will *not* be a programming class, however—our goal is to focus on the ideas underpinning neural networks rather than details of implementation.

Prerequisites: None.

Nonzero-Sum Games. (Pesto)

Sarah wants to take the Combinatorial Games class and Stephanie wants to take the Board Games class, but they conflict. Even more than either of them wants to take their preferred class, though, they want to be in the same class, to confuse their teachers. Games like combinatorial games and board games, with only one winner, have optimal strategies, although they may be hard to find; also, since no part of gameplay is good strategy unless it increases your chance of winning at someone else's expense, you couldn't have a deal between all the players unless some player were making a mistake. In Sarah and Stephanie's class selection game, it's not clear what an optimal strategy means. We'll talk about and play games where everyone could win, or everyone could lose, or anything in between. We'll also discuss the game-theoretic implications of the golden rule, threats to shut down the government, and cloning.

Prerequisites: None.

Paradoxes in Probability. (Jane)

You're given a choice of two envelopes and are told that one has twice as much money as the other. After you choose one, you're given the option of switching to the other. Should you switch?

Now, you're playing a game where you repeatedly flip a coin. If the first instance of heads comes up on the first flip, you get one dollar. If the first instance is on the second flip, you get two dollars. In general, if the first instance if on the nth flip, you get 2^n dollars. How much should you pay to play this game?

In this class, we'll test our intuition for probability by touring the world of probability paradoxes.

Prerequisites: Comfort with basic probability and computing expectations.

Period Three Implies Chaos. (Riley)

Not just a Norwegian thrash band, "Period Three Implies Chaos" is a landmark paper in the theory of dynamical systems. (Or, if you prefer, dyn̈amical systëms).

In this class we'll prove the titular theorem and (in the 3 day version) it's strictly more metal Soviet version, Sharkovskii's Theorem, which tell us when a map on the interval exhibits chaotic behaviour.

Prerequisites: Compactness (in some form).

Permuting Conditionally Convergent Series. (Zach)

The startng point for this class is the well-known Riemann's Rearrangement Theorem: given any conditionally convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers—that is, the sum converges but $\sum_{n=1}^{\infty} |a_n|$ does not—you can permute the terms so that the new sum converges to *any* desired value. But what if the a_n are instead allowed to be *complex*? Can you permute to obtain any *complex* sum? Not always: for the sequence $a_n = (-1)^n \cdot i/n$, the achievable sums lie on the line $i \cdot \mathbb{R} \subset \mathbb{C}$. What other subsets of $\mathbb{C}$ arise in this way? And what about series in $\mathbb{R}^k$? We'll discuss and prove this beautiful classification.

We'll also consider the problem from the permutations' perspective: what are the permutations on $\mathbb{N}$ that always turn convergent series into different convergent series (not necessarily with the same sum)? Amazingly, there are some permutations that do better than $\mathbb{N}$ itself: they turn convergent series into convergent series, but they also transform some *divergent* series into convergent ones!

Prerequisites: Knowledge of the epsilon-delta delta definition of limits.

Point-Set Topology. (Chris)

Topology is the study of shapes and deformations "without ripping." But what does "ripping" mean mathematically? And when are two points "close together"? It might be clear how to define these notions inside of the real numbers (you probably can come up with it yourself), but can we generalize them to any set?

These questions of closeness and ripping are so fundamental that they are at the basis of many fields of modern mathematics. The answer to them is -you guessed it- point-set topology.

Prerequisites: None.

Ponzi Schemes in Infinite Groups. (*Fedya Manin*)

I asked Sachi and Susan to each lend me a dollar, but I don't want to pay them back. So instead I tell them that they should each ask two of their friends for a dollar, and then tell those friends to ask two of their friends for a dollar, and so on. That way everyone ends up with a dollar more than they started with! (Except me, I end up with two dollars.)

Unfortunately for me, in our finite world such a scheme always eventually fails. This is a theorem of Charles Ponzi, who went to prison in the 1920's for attempting a disproof. Things get more interesting when people live on an infinite graph that connects them to their friends. For example, there's no Ponzi scheme on an infinite Euclidean grid, but there is one on a grid in the hyperbolic plane.

My favorite infinite graphs are Cayley graphs of groups. It turns out that groups that admit Ponzi schemes on their Cayley graphs are exactly the "non-amenable" ones. We'll see at least one more of the 15 or so equivalent definitions of what this means, as well as various examples. On the way, we'll develop an extremely powerful and general tool called linear programming duality (nothing to do with programming.)

Prerequisites: None.

Probably Combinatorics. (*Po-Shen Loh*)

Some of the most interesting developments in the world come from the fusion of different ways of thinking. This applies in the mathematical world as well. In the middle of the last century, several incredibly creative researchers discovered surprising applications of Probability in Combinatorics, now known as the Probabilistic Method, popularized by Paul Erdős. In this class, we'll discuss a few applications of this technique, starting with a discussion of crossing numbers which were introduced earlier at Mathcamp.

Prerequisites: A solid understanding of probability is probably required.

Problem Solving: Induction. (Misha)

You probably first saw induction in the context of proving a result like

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}.$$

Such a proof is fairly straightforward and maybe your main worry was "Can my last sentence just be 'by induction, we're done' or do I need something fancier?"

In this class, we'll see how these proofs can get much more complicated. Our induction will start out strong, and on each day of class it will get stronger than all the previous days combined. You'll see examples of crazy induction in algebra, game theory, graph theory, number theory, and other theories. You'll learn how to use induction (and how *not* to use it) to solve problems of your own, olympiad and otherwise.

Prerequisites: None.

Problem Solving: Polynomials. (Pesto)

Polynomials are a frequent topic of Olympiad-style competitions, since there are many and interesting problems using them: for instance, to find the product of the lengths of all the sides and diagonals of a regular n-gon of diameter 2. Don't see the polynomials? Come to class and find them.

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to Olympiad-style problems you will have solved as homework the previous day.

Disjoint from 2014's polynomials problem solving class.

Prerequisites: None.

Problem Solving: Symmetry, Parity, and Invariants. (*Joshua Zucker*)

Parity is the simple idea that numbers are either odd or even. It's an example of an invariant, which is something that doesn't change when you do certain operations: the parity doesn't change if you only add and subtract even numbers from your starting number. Another invariant is the total of a list of numbers: if you rearrange the numbers, or even if you add up some pairs of numbers, the total stays the same. This is because addition has symmetry: the order of the inputs doesn't matter.

In this class we will look at a variety of operations and discover the invariants and symmetry that they have, and apply those to solve some easy problems, some hard problems, and some seemingly impossible problems. What happens if you combine numbers with the rule $xy + x + y$ instead of simple addition? How can you organize cups to be flipped right side up on a spinning table you can't even see?

Prerequisites: None.

Problem Solving: Tetrahedra. (Misha)

In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

(Note: this is the same class as the class with the same name that I taught in 2015.)

Prerequisites: None.

Problem Solving: Triangle Geometry. (Zach)

Come explore the rich, diverse, and endlessly surprising world of triangle geometry! Triangles have loads of named "center" points, and we'll venture well beyond the classical centroid and orthocenter into some lesser-known yet unreasonably beautiful ones. Why has the symmedian point been called "one of the crown jewels of modern geometry"? Why is the existence of Feuerbach's point even reasonable (I'm still not convinced...), and how might we approach its construction synthetically (i.e., without inversion)? What are the (literally!) more than $10,000$ triangle centers listed in the Encyclopedia of Triangle Centers, and how can this encyclopedia be interpreted?

This class is largely problem based: there will be some lecturing, but much of the time you will present your solutions to the previous day's olympiad-style homework problems.

Prerequisites: Some familiarity with synthetic geometry (similar triangles, cyclic quadrilaterals, etc.).

Projective Geometry. (Sachi)

When you look out along a pair of railroad tracks, they appear to meet at a point in the distance. Have you ever wondered what would happen if you decided to define such a point at infinity, where each pair of parallel lines met?

Sometimes I get sad thinking of all of those lonely parallel lines living their lonely Euclidean existence just passing each other by and never getting to say hi.

In fact, there is a happier universe out there, where all parallel lines meet at infinity, and in this universe, many interesting things happen. We can see the behavior of hyperbolas, parabolas, and cubics from the "other side", at infinity. We will see that for many of purposes, this is the right geometry to consider, and it makes many types of computation and visualization easier.

Prerequisites: None.

Pythagorean Triples, Diophantine Equations and Fermat's Last Theorem. (*John Mackey*)

Why should anyone spend time showing that the sum of the cubes of two positive integers is never the cube of an integer? Is it simply that we all enjoy a good challenge or want to be famous? Perhaps it is a case of the journey being more important than the destination.

Please consider joining us for the journey along topics related to Fermat's Last Theorem. We'll discuss Pythagorean Triples, and explore different ways of classifying them, before moving on to Fermat's problem and related variants. Along the way, we will encounter different types of math, and witness the power of co-mingling ideas.

Prerequisites: None.

Quadratic Reciprocity. (Mark)

Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

- 1: "Is q a square modulo p?"
- 2: "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square mod 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Prerequisites: Some basic number theory (if you know Fermat's Little Theorem, you should be OK).

Quantum Mechanics. (Nic)

Here are some things you might have heard about quantum mechanics:

- The thing that makes it so strange is the presence of probability.
- The thing that makes it so strange is that you can't measure something without changing the result.
- The thing that makes it so strange is that things can be two things at the same time, and that's what makes quantum computers work.
- Like, Schrödinger's cat, man: is it alive or dead?

In this class we'll talk about how probability actually behaves in quantum systems — I hope to convince you that none of these descriptions is anywhere near weird enough to get to the point of quantum mechanics! We'll start from an axiomatic perspective for the first couple days, focusing on the way in quantum states and observables behave in the abstract; we probably won't talk about position and momentum directly until near the end.

Prerequisites: Ideally you should know the material from both weeks of Linear Algebra. You should know the basic of basis, independence, and dimension, how inner products on finite-dimensional vector spaces work, and what eigenvalues and eigenvectors are. We'll be using linear algebra over the complex numbers in this course, but it's fine if you don't know much about that yet. Come talk to me if you have questions about this.

Random Graphs. (Misha)

Imagine taking a graph on 1000 vertices with no edges (a very boring object), and start putting down edges with randomly chosen endpoints one at a time. (You should avoid picking the same edge twice, although in the first few thousand steps this won't affect the result very much at all.)

Can we predict approximately how many edges the graph will have before it becomes connected? Before it contains your favorite subgraph? How will its chromatic number change with the number of edges?

In this class, we will study the evolution of the random graph, a beautiful story that will incidentally provide us with limitless examples and counterexamples for all sorts of problems in graph theory. Along the way, we will develop a number of techniques of the probabilistic method.

Prerequisites: Graph theory (I will assume familiarity with graph terminology, subgraphs, trees, chromatic number. If unsure, speak with me at ϵ-TAU.).

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Prerequisites: None.

Random Groups. (Assaf + Misha)

In the first half of this class, we will be talking about Gromov's model for random groups. Such a random model is useful for showing that there exist groups with certain properties without explicitly constructing them. We will present the model, prove that random groups of density $\geq 1/2$ are trivial, and state some recent results about random groups at other densities.

In the second half, we'll talk about subgroups of S_n, the symmetric group, obtained by choosing permutations at random to serve as generators.

Prerequisites: None.

Representation Theory of Finite Groups. (Mark)

It turns out that you can learn a lot about a group by studying homomorphisms from it to groups of linear transformations (if you prefer, groups of matrices). Such a homomorphism is called a *representation* of the group; representations of groups have been used widely in areas ranging from quantum chemistry and particle physics to the famous classification of all finite simple groups. For example, Burnside, who was one of the pioneers in this area along with Frobenius and Schur, used representation theory to show that the order of any finite simple group that is not cyclic must have at least three distinct prime factors. (The smallest example of such a group, the alternating group A_5 of order $60 = 2^2 \cdot 3 \cdot 5$, is important in understanding the unsolvability of quintic equations by radicals.) We may not get that far, but you'll definitely see some unexpected, beautiful, and important facts about finite groups in this class, along with proofs of most or all of them. With any luck, the first week of the class will get you to the point of understanding character tables, which are relatively small, square tables of numbers that encode *all* the information about the representations of particular finite groups; these results are quite elegant and very worthwhile, even if you go no further. In the second week, the chili level may ramp up a bit (from about $\pi + \frac{1}{2}$ to 4) as we start introducing techniques from elsewhere in algebra (such as algebraic integers, tensor products, and possibly modules) to get more sophisticated information.

Prerequisites: Linear algebra, group theory, and general comfort with abstraction.

Scandalous Curves. (Jeff)

Sometimes you see a function, and you are comfortable looking at it. Functions like $f(x) = \sin(x)$ and $g(x) = x^2 + 1$. But sometimes you see a function and it makes you squirm a little bit.

- (Merely Misbehaving) A function like $\frac{1}{x}$ or $|x|$ has some bad points, but is mostly good.
- (Bad) A continuous function which has $f'(x) = 0$ almost everywhere, but increases monotonically from 0 to 1 would be bad (think about the fundamental theorem of calculus), but nothing too terrible to look at.
- (Sinful) A continuous function from the real numbers to the unit square that covers every point in the unit square would make most people uncomfortable.
- (Scandalous) If you are comfortable with a continuous function that is differentiable at no point, then you have a serious problem, and should probably talk to an analyst.

While it may seem unusual to find a scandalous function, we will show in this class that "most" functions are actually scandalous, and you can't spend your whole life ignoring them. Besides constructing the above functions, we will also explore convergence for functions, metric spaces of functions, random functions and what it means for a function to be "generic".

Prerequisites: You should be able to prove that a differentiable function is continuous using a $\epsilon - \delta$ argument. .

Special Relativity. (Nic)

Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but it forced physicists to come to grips with a strange new truth: there is no such thing as absolute space, and there is no such thing as absolute time. Depending on their relative velocities, different observers can disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics has to be rebuilt to accommodate these observations. We will see how, as Minkowski said, "space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." Along the way, we will also have to revise the classical notions of momentum and energy, allowing us to see the context behind the famous relation $E = mc^2$.

This is a repeat of my special relativity class from Mathcamp 2015, so if you took that class you'll probably not be interested in this one.

Prerequisites: Some high school physics — if you've seen the concepts of momentum and kinetic energy before, you're probably fine.

Spectral Graph Theory. (Sachi)

Here's something we know how to do really well: linear algebra. In comparison with many other areas of mathematics, matrices are really well understood.

On the other hand, I don't really particularly care about matrices for their own sake. One subject that I do find interesting to study in its own right is graph theory. So: here's what I propose. Let's apply linear algebra to graph theory, and see where it goes.

To each graph, we can associate a matrix, which is called its adjacency matrix. The construction works like this: put a 1 in position (i, j) if and only if vertex i is adjacent to vertex j, and otherwise fill it with a 0. Since adjacency is a symmetric relation, this yields a real symmetric matrix. So, an n vertex graph has a collection of n eigenvalues and n eigenvectors. In this class, we'll discover what the matrix, its eigenvectors, and eigenvalues can tell us about our graph.

This class will be about working on problem sets and learning through discovery. During the 2 hour superclass expect to be immersed in problem solving, discussing problems with a team of other mathcampers, and writing up solutions. There will be no homework to be done outside of class.

Prerequisites: Linear algebra, you should have some experience working with graphs (for example intro graph theory).

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Prerequisites: None.

Spin: Numbers as Rotations. (*J-Lo*)

Put a twist on the way you do geometry: every rotation is a number! Once we've let 2D rotations gyrate in our minds for a bit, we'll take our turn with 3D. In this class, we'll whirl through polar form of complex numbers, roots of unity, cyclotomic polynomials, quaternions, and Cayley-Dickson algebras, and wheel have a ball with applications revolving around geometry to round it all out. (I'm on a roll.)

A few words about class structure: We will start working with the arithmetic of complex numbers from scratch (so you're fine even if you've never seen it before), but we will go through it fairly quickly (so you shouldn't be too bored even if you have seen it before). Also, this is a very interactive course. Most of the time will be spent with *you* making conjectures, proving results, and even defining the right concepts, and you may be asked to share your work with the class.

Prerequisites: Basic trigonometry, polynomial division.

Statistical Modeling. (Sam)

Does LSD help you do math? Does the Amish population of a state predict whether it votes for democrats or republicans? Do SAT scores predict performance in college? To answer these questions, we need to do data analysis! (Except the first one. LSD does NOT improve your ability to do math.)

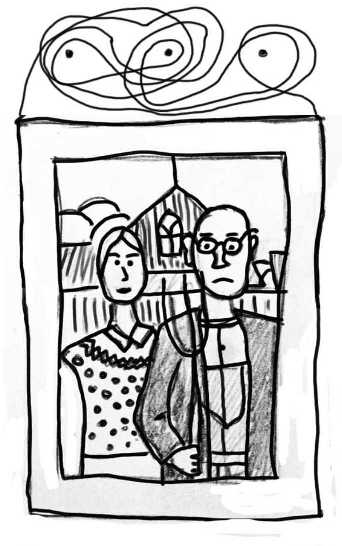
Serious blurb: this class is about using statistical techniques to draw meaningful conclusions from data. We'll start with one of the simplest tools, the linear model, and quickly extend it to handle more and more complicated (and entirely non-linear) situations. By the end the course, we'll be set to briefly discuss extensions of the linear model to machine learning. While we'll focus on one set of statistical tools, everything we do will be in terms of a general framework for statistical modelling! Along the way, we'll cover things you might have heard about like p-values and hypothesis testing; we'll highlight how to actually use those things without butchering statistical techniques and garnering the ill-will of the entire statistics community. Oh, and we'll see a few examples of some beautiful statistical theory!

Caveat Emptor: there will be 2 required practical homeworks where you'll get your hands dirty and use R to analyze data. Also, don't do LSD.

Prerequisites: Basic familiarity with probabilities and linear algebra (know what a transpose and inverse is). Ideally, you'd feel good about the sentence "X and Y are independent random variables, both having a normal distribution with mean zero and variance 1." If you're not 100% comfy with it, that's OK: the first lecture might seem more rough, but I'll hand out a probability review sheet one day 1.

String Theory. (Sachi)

Let's say you want to hang a picture in your room, and you are worried that the 2,000 fans you rented to beat the heat will blow it off the wall, so you want to hang it very securely. You start by wrapping string around some nails, like this:



You figure it will be super-duper secure, because you wrapped it around three nails a lot of times. Don, angry about the electricity bills from running all your fans, comes over and pulls one of your nails out of the wall . . . and your picture comes crashing to the ground! Actually, no matter which nail he removes, it falls! What happened?

In this class, you'll be playing with string to solve puzzles like this one, and we'll explore fundamental groups, homology, and monotone boolean functions.

Prerequisites: None.

Stupid Games on Infinite Sets. (Susan)

Let's play a game. You name a number. Then I name a bigger number. Then you name a number that's even bigger. We keep doing this forever, and then when we're done we check to see who wins.

This may sound like the worst game ever, but actually we can use this game to learn some cool things about the ordinal numbers. By the end of this class you'll be able to show that there exists a game in which neither player has a winning strategy. In fact, until the very end of the game, we will have no information about who is going to win.

No previous knowledge necessary—this class will include a nice introduction to the ordinal numbers.

Prerequisites: None.

Sum and Product Puzzles. (Don)

Sasha and Polina are both Mathcampers (and thus perfect logicians). I think of two numbers, X and Y, such that $1 < X < Y$, and $X + Y < 100$. I tell Sasha their sum, $X + Y$, and tell Polina their product, $X \cdot Y$. They then have the following exchange:

Polina: "I do not know the value of X."

Sasha: "I knew that you did not know X."

Polina: "I now know X."

Sasha: "Now I do too."

Not only is such an exchange possible, it is sufficient to inform an impartial observer (like you!) of the values of X and Y!

In class we'll look at this puzzle, and others like it (the recently popular Cheryl's Birthday puzzle is among them), where it is crucial that knowledge propagate at just the right speed. We'll work together to develop a general method for solving them, prove results about when solutions exist, and glimpse some tantalizing open conjectures.

Prerequisites: None.

Systems of Polynomial Equations. (Nic)

Suppose someone hands you a system of polynomial equations, that is, a bunch of equations with polynomials of several variables on both sides. There are tons of questions you could ask about it: Are there finitely many solutions? If so, what are they? If not, what does the space of solutions look like? Can you conclude that other polynomial equations also have to be true?

Remarkably, it's possible for many questions of this type to be answered completely systematically — you can even program a computer to do it! In this class we'll develop the main machine (called a Gröbner basis) that makes this go. Along the way, we'll also see some surprising applications to Euclidean geometry.

This class is built around a packet of problems that you'll work on with each other during class time and during the first hour of TAU; I plan to do almost no lecturing at all. This means that it'll be more work than most Mathcamp classes, but if you put in the time you'll learn it very well. The packet will be available on Monday night, and if you're planning to attend the first day you should spend a little time looking it over before Tuesday's class starts.

Prerequisites: None.

The Banach–Tarski Paradox. (Chris)

You have heard of the Banach–Tarski paradox: take a ball, break it into a few pieces, shuffle those pieces, glue them back in a different way, and now we have two balls of the same size as the original one! Nifty trick, but how does it work? In this course you get to develop the whole mathematical theory behind this construction and prove that it actually works!

This is a superclass that meets for two class hours a day (and possibly the first hour of TAU if we need it). You will be doing most of the work: we will provide worksheets with the right definitions and questions; you will spent a big chunk of the time working, alone or in groups, sometimes with our help, on all the steps of the construction. Some of the class time will be spent on presentation and discussion of your proofs.

This course is time-consuming, but all the work (homework included) is contained in the two to three daily hours.

Prerequisites: Basic group theory, linear algebra (matrix multiplication, and understand how a matrix represents a linear transformation). .

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Prerequisites: None.

The Brauer Group. (Don)

The Brauer Group of a field is a group whose elements are central simple algebras over the field, and whose operation is tensor product.

The Brauer Group is always torsion.

The Brauer Group is the right way to talk about class field theory.

The Brauer Group classifies projective varieties that become isomorphic to projective space over an algebraic closure.

The Brauer Group is terrifying, and beautiful. In this class, we'll define it, and see that, surprisingly, it's actually a group.

Prerequisites: Ring Theory, Group Theory, a class where you defined a Tensor Product.

The BSD conjecture. (David)

The Birch and Swinnerton-Dyer conjecture is one of the biggest conjectures in modern mathematics (it is one of the seven million-dollar Clay Math problems). We'll go on a whirlwind tour of the objects involved (L-functions and Mordell-Weil groups of elliptic curves) and give a statement of the conjecture.

Prerequisites: Group Theory, analytic number theory.

The Cayley-Hamilton Theorem. (Mark)

Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Prerequisites: Linear algebra, including a solid grasp of determinants (the "Magic of determinants" class would definitely take care of that).

The Chip-Firing Game. (*Sam Payne*)

Tropical geometry provides powerful combinatorial techniques for studying curves and surfaces and higher dimensional spaces defined by polynomial equations, by looking at piecewise linear solution sets over nonarchimedean fields, such as the p-adic numbers and Laurent series.

In the special case of curves, the piecewise linear shadows are graphs with finitely many vertices and finitely many edges. Algebraic geometers study curves in space by looking at configurations of points obtained by intersecting with a hyperplane, and examining how these configurations of points move as the hyperplane moves. When tropical geometers study the shadows of these configurations of points move, we see that they move according to the combinatorial rules of the chip-firing game.

My three day class will be an introduction to the combinatorics of the chip-firing game, highlighting relations to classical combinatorics (such as the matrix-tree theorem and Dhar's burning algorithm), and also with a view toward applications in algebraic geometry—we will go over most or all of the combinatorial arguments needed to prove the Brill–Noether theorem, which determines when every smooth projective curve of genus g admits a degree d embedding in projective space of dimension r—the statement goes back to the days of Riemann, but filling in the details of the proof took another hundred years.

Prerequisites: None.

The Democracy of Number Systems. (*Clifton Cunningham*)

This course begins by asking how 'real' the real numbers are. To explore this, we'll consider some famous divergent series and see how to interpret them by leaving the realm of real numbers entirely for the amazing p-adic numbers. After spending time time getting to know the p-adic numbers, we'll consider the democracy of number systems obtained by putting the real numbers and the p-adic numbers all in the same room. Then we'll bring this all back down to earth by using the democracy of number systems to answer a deceptively simple problem: Find all functions f from the rational numbers to complex units such that $f(x+y) = f(x)f(y)$.

Prerequisites: None.

The "Free Will" Theorem. (Don)

The so-called "Free Will Theorem" of Conway and Kochen takes, as givens a few (relatively) "uncontroversial" statements from physics, and then, using some slick 3 dimensional geometry, proves that if "humans" have "free will" then "particles" do too. In this class, we'll prove the theorem, unravel the relevant definitions of all the things I put in quotes in this blurb, and see that, while it's a little disappointing as a statement about free will, this theorem has some quite interesting implications for the ways quantum mechanics can and cannot work.

Prerequisites: None.

The Fundamental Group. (Jane)

Topologists can't tell a donut and a coffee cup apart because they have the same topology. That is, we can deform the donut into a coffee cup without ripping it. This process of deformation is called a homotopy. But suppose that we wanted a way to prove that two spaces were topologically different, like the sphere and the torus. In enters the fundamental group, a super important tool in algebraic topology! The fundamental group is a topological invariant that records what loops we have in our space, up to homotopy.

In this class, we'll learn some methods of computing the fundamental groups of spaces, talk about covering spaces, and also see some applications of the fundamental group. As a preview, consider the following question: given a ham sandwich consisting of two possibly strangely shaped pieces of bread and one piece of ham, can we always split the sandwich evenly between two people with one cut (i.e. a plane) so that both people get equal amounts of each piece of bread and ham?

Prerequisites: Point-Set Topology, Group Theory.

The Hadwiger-Nelson Problem. (Riley)

Hadwiger and Nelson (independently) posed the following problem: Take the Euclidean plane and connect points with an edge exactly when they are at distance 1 from each other. What is the chromatic number of the resulting graph?

Currently, no answer is known. Better yet, there is some evidence it depends on the axioms of set theory.

This class will state what is known about the Hadwiger–Nelson problem and build some intuition about strange infinite graphs.

Prerequisites: None.

The Hairy and Still Lonely Torus. (Assaf)

You've heard of spiky hedgehogs and their associated theorems, but have you heard of the amazing and totally easy-to-prove Hopf-Poincare index theorem?

It turns out that the hairy ball theorem generalizes to other orientable surfaces, and links their topology (Euler characteristic) with what vector fields are allowed on them.

In this class, we will present a shockingly simple proof of the Hairy ball theorem, using the Hopf-Poincare index theorem for surfaces, and the invariancy of the Euler characteristic. If there is time, we will also discuss a bit about liquid crystals on spheres.

This class can be thought of as a sequel to The Lonely Torus in the sense that we will prove that the Torus is the only comb-able (closed orientable) surface.

Prerequisites: None.

The Hales–Jewett Theorem. (Misha)

Ramsey theory is a branch of combinatorics about proving that sufficiently large structures have ordered substructures.

Why is it called "Ramsey theory" and not "Ramsey collection of similarly flavored results"? Because of theorems such as the Hales–Jewett theorem that connect them, letting us prove many results from just one and giving a unified way to improve our bounds on "sufficiently large".

Imprecisely speaking, the Hales–Jewett theorem is about tic-tac-toe. If you color a $19 \times 19 \times \cdots \times 19$ grid with 37 colors, then in sufficiently many dimensions you are guaranteed to find 19 collinear points all of one color, and this is true for all values of 19 and 37.

We will give one or more proofs of this theorem, see some of the consequences, and survey mathematicians' attempts to make the upper bounds on the number of dimensions less than catastrophically large.

Prerequisites: None.

The Lonely Torus - A Lie Group Opera. (Assaf)

A Lie group G is a group that is also a topological manifold M, and the group operations are continuous. The group structure induces a nonvanishing vector field on M by push-forward, and since Lie groups are orientable, by the Hopf-Poincare Index Theorem, $\chi(M) = 0$. This proves that the torus is the only compact 2-dimensional Lie group.

If you want to understand and prove the above, come to this class.

Prerequisites: Group Theory, Linear Algebra, differential calculus, continuous maps and local homeomorphisms.

The Magic of Determinants. (Mark)

This year's linear algebra class barely touched on determinants in general. If that left you feeling dissatisfied, either about not really having a good definition (by the way, using the Laplace expansion, while it is computationally often handy, leads to a miserable definition - with no intuitive basis at all) or about not having seen many of the properties that determinants have, this may be a good class for you. If all goes well, we'll give a definition of determinant that's both motivated and rigorous, and there will be proofs of all its main properties (such as Laplace expansion), as well as a few applications such as general formulas for the inverse of a matrix and for the solution of n linear equations in n unknowns ("Cramer's Rule).

Prerequisites: Some linear algebra, including linear transformations, matrix multiplication, and determinants of 2×2 matrices.

The Mathematical ABCs. (Susan)

Let's try an experiment. Consider the following expression: $x(f)$. I don't know about you, but that gives me a sort of nails-on-a-chalkboard sensation up my spine. But suppose we tweaked that f just a little bit and wrote $x(t)$ instead. Now this makes perfect sense—it's just the x-coordinate of a parametrized curve.

As mathematicians, we sort of understand that the choices that we make in our notation are arbitrary…sort of. But there are definitely conventions and heuristics and best practices.

So! In this class we'll go through the entire alphabet, a to z, and talk about how mathematicians use these letters.

Prerequisites: None.

The Mathematics of Board Games. (Assaf)

Have you ever wanted to win a board game so much that you resorted to making probability calculations on a piece of paper under the desk? In this short course, we will discuss how this can be done. We will talk about probabilities in the context of board games, with specific examples. We will also show how some classical combinatorial game theory scenarios can be approximated in board games, and discuss the differences between combinatorial game theory and board game theory.

Note: this course will continue as an evening activity where we will analyze board games. It will also be a project of designing a board game.

Prerequisites: Played at least one game of Love Letter, Hanabi, Sushi-Go, and Dominion.

The stable marriage problem. (Alfonso & Marisa)

N single men and N single women want to pair up and get married. These are their names and preferences:

- Chris: Marisa > Susan > Jane > Vivian > Elizabeth
- David: Elizabeth > Marisa > Vivian > Susan > Jane
- Kevin: Elizabeth > Vivian > Susan > Marisa > Jane
- Nic: Jane > Susan > Vivian > Elizabeth > Marisa
- Pesto: Jane > Elizabeth > Marisa > Vivian > Susan

- Elizabeth: Chris > Nic > Pesto > Kevin > David
- Jane: Kevin > David > Chris > Pesto > Nic
- Marisa: Kevin > Nic > Pesto > David > Chris
- Susan: Kevin > Pesto > Chris > David > Nic
- Vivian: Kevin > Nic > David > Pesto > Chris

Is it possible to make everybody happy? Obviously not since almost everybody wants to marry Kevin. But is it possible to at least create a stable situation? For instance, it is a bad idea for Kevin to marry Susan and for Vivian to marry Nic, because then Kevin and Vivian would prefer each other rather than staying with their partners, so they will run away together. How can we at least avoid having a run-away couple? Is there more than one way to do it? What is the best way to do it? And what if Susan and Vivian decide that marrying each other is better than marrying Kevin?

Prerequisites: None.

The Topology and Geometry of Surfaces. (Jane)

A Square lives on a surface called Flatland, a two-dimensional world. For many years, most Flatlanders assumed that their world was a giant plane, extending infinitely in all directions. However, the curious and adventurous A Square wasn't satisfied by this assumption and wanted to test it. One day, he set out in a straight line direction and came back to his starting point, without changing direction. Another day, he took a long walk and came back to his starting position to find that his whole world was now mirrored. Certainly, A Square's experiments proved that Flatland wasn't an infinite plane, but what was it?

In this course, we'll explore the question of how to tell different surfaces (two-dimensional spaces that locally look like planes) apart with both topological and geometric tools. We'll take a hands-on approach to learn about what different surfaces there are and what geometries they can have. We'll also use this knowledge to help us understand the geometry and topology of three-dimensional space, and speculate as to the shape of our universe.

Prerequisites: None.

The Word Problem for Groups. (Assaf)

A group is a collection of reversible operations, with some rules about how they relate to each other. For example, we can think of the operation of rotating a circle by $\frac{1}{3}$. This operation has order 3, meaning that doing it 3 times is like doing nothing, which is codified as $(rotation)^3 = 1$. In this manner, we can write down all of our operations (generators) and what kinds of rules (relators) they follow in order to describe the group. This is called a group presentation. In some cases, writing down all the rules is redundant; for example, the rule: $(donothing)^2 = 1$ is a consequence of $(donothing) = 1$, since doing nothing twice is obviously the same as doing nothing once.

Now, given a group presentation with a finite set of generators and relators, can we always tell if some other sequence of operations will be the ”do nothing” operation? In other words, does there exist an algorithm which decides if a sequence of operations is the trivial operation? This is called the Word Problem, and it was posed by Max Dehn in 1911.

The surprising answer, shown by Pyotr Novikov in 1955, is no - there does not exist such an algorithm. In this course, we will prove this result by studying how we can embed "uncodable" sets in groups. This will give us a candidate for a "bad" group for which a solution to the word problem would contradict the "uncodablity" of the set.

Prerequisites: None.

The Yoneda Lemma. (Sachi)

The Yoneda lemma is a ridiculous result, which makes a sweeping statement about all categories at once. It says that given a category C along with a functor F from C to Set, we can get this functor from a more familiar functor, namely the Hom functor.

The Hom functor is a really familiar functor: given an object a in my category, I can ask what is the set of C-morphisms from a to x. That set is called $\text{Hom}(a, x)$. Then, what Yoneda says is that there is a one-to-one correspondence between natural transformations from $C(a, -)$ to F and elements of Fa.

We'll discuss and prove the Yoneda lemma, so if you're curious about the proof, or want to see it again, this is a good class for you. At the beginning we will review the definition of natural transformation, Hom, functor, and category very quickly for people who need a refresher.

Prerequisites: It would be very helpful to have seen the definition of category and functor already. It would also be helpful to know what a natural transformation is, though not strictly necessary. If not, maybe you know what a homotopy is?

Trail Mix. (Mark)

Is your mathematical hike getting a little too strenuous? Would you like to relax a bit with a class that offers an unrelated topic every day, so you can pick and choose which days to attend, and that does not expect you to do homework? If so, some Trail Mix may be just what you need to regain energy. Individual descriptions of the three topics follow.

Day 1 (1 chili, Tuesday): **Perfect Numbers**. Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called *perfect*, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes - a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers. (No prerequisites.)

Day 2 (2 chilis, Wednesday): **Intersection Madness**. When you intersect two ellipses, you can get four points, right? So why can't you get four points when you intersect two circles? Well, actually you can, and what's more, two of the four points *are always in the same places*! If this seems paradoxical (and, I hope, interesting), wait until we start intersecting two cubic curves (given by polynomial equations of degree 3). There's a "paradox" there too, first pointed out by the Swiss mathematician Cramer in a letter to Euler, and the resolution of that paradox leads to a "magic" property of the nine intersection points. If time permits, we'll see how that property (known as the Cayley-Bacharach theorem) gives elegant proofs of Pascal's hexagon theorem and of the existence of a group law on a cubic curve. (Prerequisites: A little bit of linear algebra would help; having attended Nick's colloquium would help, too. And we'll use complex numbers, but very lightly. There are no "serious" prerequisites.)

Day 3 (3 chilis, Thursday): **The Jacobian Determinant and** $\sum_{n=1}^{\infty} \frac{1}{n^2}$ How do you change variables in a multiple integral? For example, when you change to polar coordinates, a somewhat mysterious factor r is needed. This is a special case of an important general fact involving a determinant of partial derivatives. We'll see how and roughly why this works; then we'll use it to evaluate the famous sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (You may well know the answer, but do you know a proof? If so, do you know a proof that doesn't require Fourier series or complex analysis?) (Prerequisites: Some multivariable calculus, and a bit of experience with determinants - what was covered in week 2 of linear algebra this year should do.)

Prerequisites: See individual "mini-blurbs" above.

Turning Points in the History of Mathematics. (Sam Gutekunst)

This class will focus on a few "turning points" in the history of mathematics: moments where the way we think about mathematics radically shifted. For example, the invention of symbolic notation (imagine how different your experience of math would be if you couldn't use variables to, e.g., cleanly state the quadratic formula).

Depending on the number of days and interest, we'll probably focus on some subset of: Rigor in Greek mathematics, the use of symbolic notation, and analytic geometry (giving the correspondence between equations and curves).

Prerequisites: None.

Twisty Puzzles are Easy. (Zach)

Algorithmically speaking, the Rubik's Cube and many of its more complicated cousins are easy to solve: they can be expressed in terms of permutation groups acting on the pieces, and they therefore fall to an elegant and efficient algorithm due to Schreier and Sims. This algorithm is well-suited for many types of queries regarding permutation groups, such as testing group membership (can you turn a single corner in a $3 \times 3 \times 3$ cube? Swap two edges in a $4 \times 4 \times 4$ cube?), instantly computing the size of the group (there are precisely $43,252,003,274,489,856,000$ possible $3 \times 3 \times 3$ Rubik's cube positions, and you can bet it didn't enumerate them all individually!), and many others.

This is *not* a course on how to solve a Rubik's Cube. We will deal more generally with computations in *any* permutation group (or group action), and the resulting algorithm can describe the group generated by *any* desired set of basic permutations. We will see how this can be applied to many types of puzzles, and we will also discuss some limitations of the approach.

Prerequisites: Group Theory.

Universal Properties. (Don)

The Cartesian product of sets, direct product of groups, the product topology on the cartesian product of topological spaces. If you've seen some of these before, you may have noticed some similarities between the different definitions - and that's because they're all objects with the same universal property.

Category theory is a language used to describe these kinds of properties. It's a framework that almost any kind of math fits into, and often by putting a subject into the language of category theory, interesting questions arise. Additionally, you can use category theory to prove many theorems at once — by proving them in the general case. In this class, we'll learn about categories, universal properties that can exist across categories, and functors, which let you compare different categories.

Prerequisites: Group theory, or ring theory, or linear algebra, or topology. The more, the better, though.

Wallis and His Product. (*Jon Tannenhauser*)

John Wallis (1616-1703) published what is essentially the infinite product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$$

in his 1656 treatise *Arithmetica infinitorum.* His work mixed suggestive analogy, intuitive leaps, and sheer moxie—and sparked reactions ranging from awe to skepticism to spluttering rage. We'll trace Wallis's argument (day 1; 2 chilis) and discuss how and why it touched off a battle in a long-running 17^{th}-century war over infinitesimals, which was actually—although no one knew it at the time—a struggle over the foundations of calculus (day 2; 1 chili). Then we'll look at a recent (October 2015) derivation of the Wallis product via the quantum mechanics of the hydrogen atom (day 3; 4 chilis)!

Prerequisites: None.

Wedderburn's Theorem. (Mark)

Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, ki = j, ik = -j, jk = i, kj = -i\,.$$

Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Prerequisites: Some group theory and some ring theory; familiarity with complex roots of unity would help.

What Can We Exponentiate? (Assaf)

e^x is an **awesome function. It's always defined, it's always continuous, always differentiable, and generally awesome.** e^x **can make any matrix invertible.** e^x **can make vector spaces into manifolds.** e^x **can turn anything into a group.** e^x **can solve differential equations.** e^x **can even travel through time itself.**

All of this is true, and we will prove it all.

Prerequisites: Linear algebra (weeks 1 and 2).

Why Aren't We Learning This: Fun Stuff in Statistics. (Sam)

In this class we'll talk about some fun topics in statistics that aren't often taught in the undergraduate statistics curriculum: classification techniques and/or multiple testing.

Classification techniques are used for when you want to classify stuff (e.g., to try and predict whether or not a camper will be eaten by a bear based on factors like the classes they took). There are some surprisingly beautiful ideas that go into classifiers, and we'll talk through a few of the most robust from an intuitive perspective. Multiple testing procedures help you do statistics right when you have tons and tons of simultaneous tests (like looking at whether or not each flavor of jelly beans cures cancer, a la XKCD).

Credit goes to Vaughan for the gimmicky title. This course is entirely unrelated to my week 2 course.

Prerequisites: Know how to compute expected values and probabilities!

Why Are We Learning This? A seminar on the history of math education in the U.S.. (Sam)

Have you ever sat in class and thought: who the @

This seminar will provide historical context for those types of thoughts. We'll try to understand that context for both the actual content in the curriculum and the way that content is communicated. Most importantly, we'll pay attention to how math education reform movements have succeeded – or failed. If you've ever wanted to try and fix math education, this class should provide the context to help. Oh, and we'll also see some fun and crazy examples of math problems that have plagued students for centuries.

Caveat Emptor: For this class, *seminar* and *homework required* mean that you'll have to read some history of math papers and look at parts of math textbooks from ages past. In class we'll devote a considerable amount of time to discussing that stuff, and you should expect approximately 45 minutes of reading for each day.

Prerequisites: None.

Wythoff's Game. (Alfonso)

Let's play a game. We have a plate with blueberries and strawberries in front of us. We take turns eating them. In your turn, you may eat as many berries as you want as long as they are all of the same kind (at least one) or exactly the same number of berries of both kinds (at least one). Then it is my turn. The player who eats the last berry wins. Will you beat me?

This game with simple rules has a surprising strategy with a game that will blow your mind!
Prerequisites: None.

Colloquia

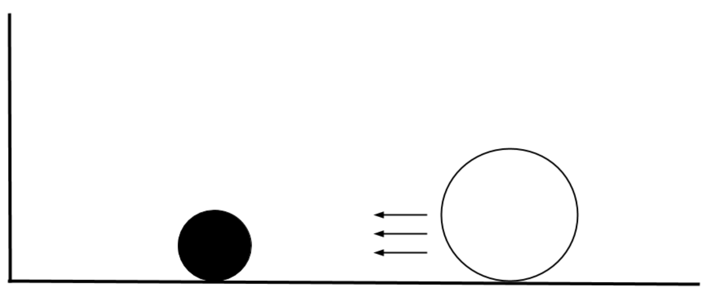
Algebraic Proof of Heron's formula. (*Noah Snyder*)

Heron's formula for triangular area says that the area of a triangle is given in terms of the three side lengths by the formula

$$\frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

I'll give a largely algebraic proof of this formula, which uses very little geometry. Roughly the idea of the proof is that this formula is the simplest possible formula that gives the right answer for triangles of zero area.

A simple question with a very curious answer. (*Mira Bernstein*)



In the diagram above, a white ball of mass $M \geq 1$ is rolling toward a stationary black ball of mass 1. Once they collide, the black ball will roll toward the wall, bounce off, and collide again with the white one. (Small print for the physicists: we assume all collisions are perfectly elastic.)

What happens next? If $M = 1$, the black ball stops moving and the white ball rolls away – so there are 3 collisions total. If $M > 1$, the black ball will bounce some more between the white ball and the wall, until eventually both balls roll away.

Our question is: how many collisions will there be, as a function of M? Come to colloquium to find out; I promise the answer will blow your mind!

(Note: this talk requires no physics background and only very elementary math. Everyone should be able to follow.)

Crossing Numbers of Graphs: Origins and Recent Progress. (*John Mackey*)

When Paul Turan was interned in a Nazi labor camp during World War II, he was assigned to a job which required that bricks be moved from kilns to storage depots. There were paths from each kiln to each storage depot, but the carts on which the bricks were moved had a tendency to overturn when crossing the path of other carts. Turan devised a system of paths which seemed to minimize the number of such crossings, but could not prove that his configuration was indeed best possible.

Zarankiewicz published a paper containing a flawed proof of the optimality of Turan's construction in the 1950s. Richard Guy exposed the flaw and introduced several generalizations of Turan's problem in 1969. All of the large conjectures from Guy's paper remain unsolved to this day, but recent progress (including some by CMU undergraduates) indicates that solutions may be near.

Elo × Education. (*Po-Shen Loh*)

When new mathematics enters an industry, it has the potential to transform the status quo. We'll introduce and motivate the Elo rating system (originally used for competitive Chess), and then talk about how to use this breakthrough to solve personalized Education on a global scale: using math to teach math, science, and more.

Intersecting Curves in the Plane. (Nic)

From any polynomial f in two variables, you can form a curve by looking at all the points (x, y) in the plane for which $f(x, y) = 0$. These are called "algebraic plane curves" — lines, ellipses, hyperbolas, and graphs of one-variable polynomials are all examples. In this colloquium, we're going to study Bézout's Theorem, a result from the eighteenth century which says that the number of intersection points of two algebraic plane curves is given by the product of their degrees.

We won't be able to prove this theorem in an hour, and for a very good reason: the way I've stated it, it's not true! But by looking more closely at the cases where it fails, I hope to convince you that a true statement is hiding underneath the surface; in our attempt to uncover it, we'll get a tour of some of the most beautiful foundational ideas in twentieth-century algebraic geometry.

Lattices, Knots and Chaos. (Jeff)

Here are a couple of different grids of dots:

[image with dots]

All of them give examples of lattices: discrete $\mathbb{Z}\times\mathbb{Z}$ subgroups of $\mathbb{R}^2$. These beasties make themselves present in number theory, algebraic geometry, topology—they're a piece of mathematics that shows up over and over again. However, it's not immediately clear how we can classify lattices; for instance, the first and third lattice in the above figure can be related to each other by a rotation.

We'll construct a topological space whose points parameterize lattices, and show how we can better understand operations on lattices through the geometry of this space. Then, we'll flip the relationship around, and restate a problem of the Lorenz attractor in terms of the geometry of lattices.

Oops! I Ran Out of Axioms. (*Steve Schweber*)

Is there a question which the axioms we use for math can't solve? It's reasonable to think that the answer is no; or at least, that if we do find such a question, that we can just add a couple new axioms and then *those* will be enough to answer everything. However, in 1931, a mathematician named Kurt Goedel surprised everyone by proving that we will never have a complete list of axioms: given any "reasonable" set of axioms, there will be true statements which are not provable from those axioms. Moreover, these statements aren't weird or vague—they are perfectly concrete statements about natural numbers, and over the next few decades Godel's result was improved to the following: given any "reasonable" set of axioms, there is a *polynomial* p with integer coefficients, which has no integer solutions, such that the axioms cannot prove that p has no integer solutions.

I'll talk about what exactly this means, and how it turns out to be true.

Ron Graham's Sequence. (*Joshua Zucker*)

It's hard to write about Ron Graham's sequence without giving too many spoilers. The surprises are a big part of the fun! So you may prefer to stop reading now.

Can you make a list of all the non-prime numbers? Sure, $1, 4, 6, 8, 9, \ldots$, no problem. OK, quick, what's the thousandth number in that list? Millionth? I wonder if there's an order we can put them in such that we could reasonably quickly find out the thousandth or the millionth number in the list, without having to find all the numbers before them. Well, Ron Graham found one! We'll see what his discovery was and how to prove that the sequence contains every non-prime number exactly once.

Should We Vote on How We Vote? (*Michael Orrison*)

Voting is something we do in a variety of settings, but how we vote is seldom questioned. In this talk, we'll explore a few different voting procedures from a mathematical perspective as we try to make sense of the paradoxical results that can occur when we vote in more than one way.

Spin! (*Clifton Cunningham*)

Rotation by 360 degrees brings you back to your starting point. But if your limbs are tied to the walls, you'll get tangled, with no way to untwist the ropes. Amazingly, if you continue your rotation to a full 720 degrees, there is a way to untangle yourself. We'll see a demo of this phenomenon in action, explain it in terms of stereographic projection, and describe the connection with spin 1/2 from quantum mechanics. Come get tangled up in the fundamental group of SO(3)!

The Littlewood-Offord Problem. (Susan)

Don is going to offer a small group of campers a fabulous investment opportunity. One lucky camper collective will have the opportunity to invest 100 Susan Bucks in the 5/3 National Bank of Don, at twice the usual interest rate! Here's the catch: this camper collective must have exactly 100 Suzin Bux between them.

To complicate matters, Vivian has decided that it would be fun if there were as much competition as possible for this fabulous investment opportunity. So she has been running around handing out extra Suzin Bux to campers, with the goal of creating as many 100-Suzin-Buk camper collectives as possible.

In this talk we will discuss the optimal strategy for Vivian. Note: this is not an endorsement of the Suzin Buk money system. Suzin Bux are not worth anything, and Susan has absolutely nothing to do with them.

The Music of Zeta. (*J-Lo*)

The Riemann Zeta function is defined quite simply as a sum over the natural numbers, all raised to a given exponent. But what exactly does it measure?

On an unrelated note, ever wondered why a musical scale (CDEFGAB) has seven notes, or why the chromatic scale has 12?

Oh wait, my bad, these aren't unrelated at all. Find out why, and you'll also discover how Pythagoras wrote the score for his own worst nightmare, why you sometimes need to listen to a function before you can define it (or more precisely, its so-called "analytic continuation"), and how you can earn a million dollars for transcribing a song played by the world's strangest keyboard.

There Are Eight Flavors of Three-Dimensional Geometry. (*Moon Duchin*)

Picture it: circa 1977. Hippies, disco, Elvis dies, Star Wars comes out... and some guy comes along and works out that for all the possible ways to build a three-dimensional shape, you only need EIGHT building blocks. Seven won't do, and there's no ninth one. I'll tour you through the eight 3D geometries and put this amazing fact in context.

Tree Trotting. (Zach)

As a criminal mathtermind on the run from the law (of large numbers?), you have constructed a complicated network of hideouts to navigate via underground tunnels. To throw the authorities off your (not-necessarily acyclic) path, you sporadically move from each hideout to one of its randomly chosen neighbors in your trusty van. As you do this, every time you visit a new hideout for the first (I mean zeroth) time, you mark the corridor you just traversed as that hideout's "escape route" to use in case of a surprise raid. Once you have visited all hideouts, your choices of escape routes form a smaller network for quickly driving to freedom—an emergency vanning tree spanning your hideouts. What is this tree likely to look like? Which kinds of trees are more likely to arise? Do answers to these questions in any way facilitate your daring escape?

Tropical Geometry. (*Sam Payne*)

Tropical geometry provides powerful combinatorial techniques for studying curves and surfaces and higher dimensional spaces defined by polynomial equations, by looking at piecewise linear solution sets over nonarchimedean fields, such as the p-adic numbers and Laurent series.

In the special case of curves, the piecewise linear shadows are graphs with finitely many vertices and finitely many edges. Algebraic geometers study curves in space by looking at configurations of points obtained by intersecting with a hyperplane, and examining how these configurations of points move as the hyperplane moves. When tropical geometers study the shadows of these configurations of points move, we see that they move according to the combinatorial rules of the chip-firing game.

In colloquium, I will try to give an overview of the basic notions of tropical geometry, and explain how the combinatorial chip-firing rules arise from the Poincare–Lelong formula, governing solutions to a certain partial differential equation over nonarchimedean fields.