# CLASS DESCRIPTIONS—WEEK 5, MATHCAMP 2016

## Contents


**9:10 Classes** 2
- Building Groups out of Other Groups 2
- More Problem Solving: Polynomials 2
- The Hairy and Still Lonely Torus 2
- Turning Points in the History of Mathematics 4
- Why Aren't We Learning This: Fun Stuff in Statistics 4

**10:10 Classes** 4
- A Very Difficult Definite Integral 4
- Homotopy (Co)limits 5
- Scandalous Curves 5
- Special Relativity 5
- String Theory 6
- The Magic of Determinants 6
- The Mathematical ABCs 7
- The Stable Marriage Problem 7
- Quadratic Reciprocity 7

**11:10 Classes** 8
- Benford's Law 8
- Egyptian Fractions 8
- Gambling with Nathan Ramesh 8
- Hyper-Dimensional Geometry 8
- IOL 2016 8
- Many Cells Separating Points 9
- Period Three Implies Chaos 9
- Problem Solving: Tetrahedra 9
- The Brauer Group 9
- The BSD conjecture 10
- The Cayley–Hamilton Theorem 10
- The "Free Will" Theorem 10
- The Yoneda Lemma 10
- Twisty Puzzles are Easy 11
- Wedderburn's Theorem 11
- Wythoff's Game 11

**1:10 Classes** 11
- Big Numbers 11
- Graph Polynomials 12
- Hard Problems That Are Almost Easy 12
- Hyperbolic Geometry 12
- Latin Squares and Finite Geometries 13

Multiplicative Functions 13
Permuting Conditionally Convergent Series 13
The Cap Set Problem 13

**2:10 Classes** 14
Computer-Aided Design 14
Factoring with Elliptic Curves 14
Math and Literature 14
More Group Theory! 15
Paradoxes in Probability 15
The Hales–Jewett Theorem 15

## 9:10 Classes

**Building Groups out of Other Groups.** (🌶🌶🌶, Don, Wednesday–Friday)
In group theory, everybody learns how to take a group, and shrink it down by taking a quotient by a normal subgroup—but what they don't tell you is that you can totally go the other direction. It's a little harder, and a little messier, because while there's only one way to take a quotient, there's lots of ways to take a product. So, let's take a look at a few of these ways, from the clean, sterile direct product, to the messy, dangerous free product, and even the beautiful, mighty semi-direct product.

*Homework:* Recommended.

*Prerequisites:* Group Theory.

**More Problem Solving: Polynomials.** (🌶🌶🌶🌶, Pesto, Wednesday–Friday)
Polynomials are a frequent topic of Olympiad-style competitions, since there are many and interesting problems using them. For instance, if $a_1, \ldots, a_n$ are distinct real numbers, find a closed-form expression for

$$\sum_{1\le i\le n} \prod_{1\le j\le n, j\ne i} \frac{a_i + a_j}{a_i - a_j}.$$

Don't see the polynomials? Come to class and find them.

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to Olympiad-style problems you will have solved as homework the previous day.

Disjoint from 2013's (but not 2014's) polynomials problem solving class. Disjoint from the week 4 polynomials problem solving class.

*Homework:* Required.

*Prerequisites:* Linear algebra: Understand what a "basis for polynomials of degree at most 2 in two variables" is.

**The Hairy and Still Lonely Torus.** (🌶, Assaf, Wednesday–Friday)
You've heard of spiky hedgehogs and their associated theorems, but have you heard of the amazing and totally easy-to-prove Hopf–Poincaré index theorem?

It turns out that the hairy ball theorem generalizes to other orientable surfaces, and links their topology (Euler characteristic) with what vector fields are allowed on them.

In this class, we will present a shockingly simple proof of the Hairy ball theorem, using the Hopf–Poincaré index theorem for surfaces, and the invariancy of the Euler characteristic. If there is time, we will also discuss a bit about liquid crystals on spheres.

This class can be thought of as a sequel to The Lonely Torus in the sense that we will prove that the Torus is the only comb-able (closed orientable) surface.

*Homework:* Recommended.

*Prerequisites:* None.

**Turning Points in the History of Mathematics.** (🌶️🌶️, Sam, Friday)
This class will focus on a few "turning points" in the history of mathematics: moments where the way we think about mathematics radically shifted. For example, the invention of symbolic notation (imagine how different your experience of math would be if you couldn't use variables to, e.g., cleanly state the quadratic formula).

Depending on the number of days and interest, we'll probably focus on some subset of: rigor in Greek mathematics, the use of symbolic notation, and analytic geometry (giving the correspondence between equations and curves).

*Homework:* None.

*Prerequisites:* None.

**Why Aren't We Learning This: Fun Stuff in Statistics.** (🌶️🌶️, Sam, Wednesday–Thursday)
In this class we'll talk about some fun topics in statistics that aren't often taught in the undergraduate statistics curriculum: classification techniques and/or multiple testing.

Classification techniques are used for when you want to classify stuff (e.g., to try and predict whether or not a camper will be eaten by a bear based on factors like the classes they took). There are some surprisingly beautiful ideas that go into classifiers, and we'll talk through a few of the most robust from an intuitive perspective. Multiple testing procedures help you do statistics right when you have tons and tons of simultaneous tests (like looking at whether or not each flavor of jelly beans cures cancer, a la XKCD).

Credit goes to Vaughan for the gimmicky title. This course is entirely unrelated to my week 2 course.

*Homework:* Optional.

*Prerequisites:* Know how to compute expected values and probabilities!

**10:10 Classes**

**A Very Difficult Definite Integral.** (🌶️🌶️🌶️🌶️, Kevin, Tuesday)
In this class, we will show

$$\int_0^1 = \frac{\log(1+x^{2+\sqrt{3}})}{1+x}\,dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3}).$$

We'll start by turning this Very Difficult Definite Integral into a Very Difficult Series. Then we'll sum it!

We'll need an unhealthy dose of clever tricks involving some heavy-duty algebraic number theory. It will be Very Difficult.

(This class is based on a StackExchange post by David Speyer.)

*Homework:* None.

*Prerequisites:* A ton of algebraic number theory. David's algebraic number theory marathon *might* suffice. A willingness to believe the facts we'll use is fine instead. Familiarity with rings and ideals is probably necessary regardless.

**Homotopy (Co)limits.** (🌶🌶🌶🌶, Chris, Wednesday–Friday)
Imagine you are back in your childhood and you're playing with some clay. You have 3 pieces of clay and want to assemble them according to some instructions your friend sent you. Unfortunately, your evil sibling took your clay and completely deformed your pieces. Nevertheless you go ahead and assemble it. The next day you and your friend realize that you both got completely different structures. Yours even has a hole where theirs doesn't!

What you just discovered is what topologists refer to as the fact that "the colimit of a diagram is not homotopy invariant." In this class, we will explore this phenomenon (possibly with or without clay), and hopefully we will be able to answer what the true and real and homotopy-invariant way of assembling your clay should be!

*Homework:* Optional.

*Prerequisites:* Category theory (basic definitions, some of the following universal properties: (co)products, push-outs, pull-backs), topology (continuity).

**Scandalous Curves.** (🌶🌶🌶, Jeff, Tuesday–Friday)
Sometimes you see a function, and you are comfortable looking at it. Functions like $f(x) = \sin(x)$ and $g(x) = x^2 + 1$. But sometimes you see a function and it makes you squirm a little bit.

- (Merely Misbehaving) A function like $\frac{1}{x}$ or $|x|$ has some bad points, but is mostly good.
- (Bad) A continuous function which has $f'(x) = 0$ almost everywhere, but increases monotonically from 0 to 1 would be bad (think about the fundamental theorem of calculus), but nothing too terrible to look at.
- (Sinful) A continuous function from the real numbers to the unit square that covers every point in the unit square would make most people uncomfortable.
- (Scandalous) If you are comfortable with a continuous function that is differentiable at no point, then you have a serious problem, and should probably talk to an analyst.

While it may seem unusual to find a scandalous function, we will show in this class that "most" functions are actually scandalous, and you can't spend your whole life ignoring them. Besides constructing the above functions, we will also explore convergence for functions, metric spaces of functions, random functions and what it means for a function to be "generic".

*Homework:* Recommended.

*Prerequisites:* You should be able to prove that a differentiable function is continuous using an epsilon-delta argument.

**Special Relativity.** (🌶🌶, Nic, Tuesday–Friday)
Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but it forced physicists to come to grips with a strange new truth: there is no such thing as absolute space, and there is no such thing as absolute time. Depending on their relative velocities, different observers can disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

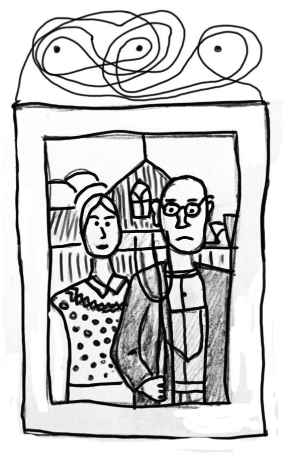
In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics has to be rebuilt to accommodate these observations. We will see how, as Minkowski said, "Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." Along the way, we will also have to revise the classical notions of momentum and energy, allowing us to see the context behind the famous relation $E = mc^2$.

This is a repeat of my special relativity class from Mathcamp 2015, so if you took that class you'll probably not be interested in this one.

*Homework:* Optional.

*Prerequisites:* Some high school physics—if you've seen the concepts of momentum and kinetic energy before, you're probably fine.

**String Theory.** (🌶🌶, Sachi, Thursday–Friday)
Let's say you want to hang a picture in your room, and you are worried that the 2000 fans you rented to beat the heat will blow it off the wall, so you want to hang it very securely. You start by wrapping string around some nails, like this:



You figure it will be super-duper secure, because you wrapped it around three nails a lot of times. Don, angry about the electricity bills from running all your fans, comes over and pulls one of your nails out of the wall... and your picture comes crashing to the ground! Actually, no matter which nail he removes, it falls! What happened?

In this class, you'll be playing with string to solve puzzles like this one, and we'll explore fundamental groups, homology, and monotone boolean functions.

*Homework:* None.

*Prerequisites:* None.

**The Magic of Determinants.** (🌶🌶🌶, Mark, Tuesday–Wednesday)
This year's linear algebra class barely touched on determinants in general. If that left you feeling dissatisfied, either about not really having a good definition (by the way, using the Laplace expansion, while it is computationally often handy, leads to a miserable definition—with no intuitive basis at all) or about not having seen many of the properties that determinants have, this may be a good class for you. If all goes well, we'll give a definition of determinant that's both motivated and rigorous, and there will be proofs of all its main properties (such as Laplace expansion), as well as a few applications such as general formulas for the inverse of a matrix and for the solution of $n$ linear equations in $n$ unknowns ("Cramer's Rule").

*Homework:* Optional.

*Prerequisites:* Some linear algebra, including linear transformations, matrix multiplication, and determinants of $2 \times 2$ matrices.

**The Mathematical ABCs.** (🌶, Susan, Wednesday)
Let's try an experiment. Consider the following expression: $x(f)$. I don't know about you, but that gives me a sort of nails-on-a-chalkboard sensation up my spine. But suppose we tweaked that $f$ just a little bit and wrote $x(t)$ instead. Now this makes perfect sense—it's just the $x$-coordinate of a parametrized curve.

As mathematicians, we sort of understand that the choices that we make in our notation are arbitrary… sort of. But there are definitely conventions and heuristics and best practices.

So! In this class we'll go through the entire alphabet, $a$ to $z$, and talk about how mathematicians use these letters.

*Homework:* Recommended.

*Prerequisites:* None.

**The Stable Marriage Problem.** (🌶, Alfonso + Marisa, Tuesday)
$N$ single men and $N$ single women want to pair up and get married. These are their names and preferences:

- Alfonso: Marisa > Susan > Jane > Vivian > Elizabeth
- David: Elizabeth > Marisa > Vivian > Susan > Jane
- Kevin: Elizabeth > Vivian > Susan > Marisa > Jane
- Nic: Jane > Susan > Vivian > Elizabeth > Marisa
- Pesto: Jane > Elizabeth > Marisa > Vivian > Susan

- Elizabeth: Alfonso > Nic > Pesto > Kevin > David
- Jane: Kevin > David > Alfonso > Pesto > Nic
- Marisa: Kevin > Nic > Pesto > David > Alfonso
- Susan: Kevin > Pesto > Alfonso > David > Nic
- Vivian: Kevin > Nic > David > Pesto > Alfonso

Is it possible to make everybody happy? Obviously not since almost everybody wants to marry Kevin. But is it possible to at least create a stable situation? For instance, it is a bad idea for Kevin to marry Susan and for Vivian to marry Nic, because then Kevin and Vivian would prefer each other rather than staying with their partners, so they will run away together. How can we at least avoid having a run-away couple? Is there more than one way to do it? What is the best way to do it? And what if Susan and Vivian decide that marrying each other is better than marrying Kevin?

*Homework:* None.

*Prerequisites:* None.

**Quadratic Reciprocity.** (🌶🌶🌶, Mark, Thursday–Friday)
Let $p$ and $q$ be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is $q$ a square modulo $p$?"
(2) "Is $p$ a square modulo $q$?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

*Homework:* Optional.

*Prerequisites:* Some basic number theory (if you know Fermat's Little Theorem, you should be OK).

## 11:10 Classes

**Benford's Law.** (🌶, Assaf, Tuesday)
Benford's Law was discovered when an astronomer was sifting through log tables (the old kind) and noticed that the pages containing logarithms of numbers whose first digit is 1 were more worn out than others.

It turns out that approximately $\frac{1}{3}$ of the numbers found in real life begin with a 1. In this class we will explain this phenomenon, and see where it comes from and how it can be applied for practical purposes.
*Homework:* Recommended.
*Prerequisites:* None.

**Egyptian Fractions.** (🌶, Jane, Thursday)
The ancient Egyptians only had notation to write fractions of the form $\frac{1}{n}$. If they wanted to write any other positive rational number, they had to write it as a sum of distinct terms of this form.

This sounds incredibly restrictive, but actually every rational number can be written as an Egyptian fraction. In this class, we'll take the idea of an Egyptian fraction and run with it. It turns out to generate quite a lot of interesting math. For example, how do we find Egyptian fraction decompositions? Can we find the shortest decompositions? As we discover things about Egyptian fractions, we'll also see how quickly we run into questions about them that are still open.
*Homework:* None.
*Prerequisites:* None.

**Gambling with Nathan Ramesh.** (🌶🌶, *Nathan Ramesh*, Friday)
Suppose you have a dollar, but you need 5. There is a casino which allows you to give them a dollar for a chance (not necessarily $\frac{1}{2}$) to get your dollar back plus another dollar, but otherwise they keep your dollar. Should you play the game, and do you have enough time to play it? Come to this class with Nathan Ramesh to find out.
*Homework:* None.
*Prerequisites:* You should understand linearity of expectation.

**Hyper-Dimensional Geometry.** (🌶🌶🌶, *Drake Thomas*, Wednesday)
You may have heard of the 5 Platonic solids. Why are there just 5? What happens if we try making something similar in 4 dimensions? 5? 6? What does it even mean to extend this to higher dimensions?

Youve probably also seen the area of a circle, $\pi r^2$, and the volume of a sphere, $\frac{4}{3}\pi r^3$. How does this formula behave as we go into higher dimensions? How fast do those coefficients grow?

In this class well answer all of these questions in two parts: first, well talk about analogs of the platonic solids, and then well see how to compute the volume of an $n$-dimensional sphere. Well also be able to apply our formula from the second part of the class to provide a counterintuitive solution to a problem that remained open for several decades!
*Homework:* None.
*Prerequisites:* Calculus (for the second part).

**IOL 2016.** (🌶–🌶🌶🌶🌶, Pesto, Tuesday)
Margarita and three other Mathcamp alums are currently competing at the International Linguistics Olympiad in Mysore, India. To celebrate their participation, Pesto will teach about whatever bits of linguistics are featured in this year's contest problems.

Disclaimer: Pesto has no idea what the content of this course will be or whether there'll be anything mathematical involved, since the problems won't be released until the start of week 5.

*Homework:* None.

*Prerequisites:* Who knows?

**Many Cells Separating Points.** (🌶🌶, *Nikhil Marda*, Thursday)
Consider a set of 7 points in the plane. Three lines can form 7 cells, so we wonder if we can draw the lines such that each cell has one point in it. You'll quickly realize that this isn't possible for all sets of 7 points, such as when the points lie on a circle. So what happens when we have more points and lines?

In this class, we will explore the concept of equal point separation, which will lead to a generalized notion of convexity and new classifications for geometric objects. We will also generalize the Erdős–Szekeres Conjecture, an open problem for 81 years in Ramsey theory which states that a set of $2^{n-2}+1$ points contains a convex $n$-gon, to our new class of objects based off new research from just April of this year.

*Homework:* None.

*Prerequisites:* None.

**Period Three Implies Chaos.** (🌶🌶🌶, Riley, Friday)
Not just a Norwegian thrash band, "Period Three Implies Chaos" is a landmark paper in the theory of dynamical systems. (Or, if you prefer, dyñamical systëms).

In this class we'll prove the titular theorem and (in the 3 day version) its strictly more metal Soviet version, Sharkovskii's Theorem, which tell us when a map on the interval exhibits chaotic behaviour.

*Homework:* Recommended.

*Prerequisites:* Compactness (in some form).

**Problem Solving: Tetrahedra.** (🌶🌶🌶🌶, Misha, Friday)
In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

(Note: this is the same class as the class with the same name that I taught in 2015.)

*Homework:* None.

*Prerequisites:* None.

**The Brauer Group.** (🌶🌶🌶🌶, Don, Tuesday)
The Brauer Group of a field is a group whose elements are central simple algebras over the field, and whose operation is tensor product.

The Brauer Group is always torsion.

The Brauer Group is the right way to talk about class field theory.

The Brauer Group classifies projective varieties that become isomorphic to projective space over an algebraic closure.

The Brauer Group is terrifying, and beautiful. In this class, we'll define it, and see that, surprisingly, it's actually a group.

*Homework:* Recommended.

*Prerequisites:* Ring Theory, Group Theory, a class where you defined a tensor product.

**The BSD conjecture.** (🌶🌶🌶🌶, David, Thursday)
The Birch and Swinnerton-Dyer conjecture is one of the biggest conjectures in modern mathematics (it is one of the seven million-dollar Clay Math problems). We'll go on a whirlwind tour of the objects involved ($L$-functions and Mordell–Weil groups of elliptic curves) and give a statement of the conjecture.
*Homework:* Optional.
*Prerequisites:* Group Theory, analytic number theory.

**The Cayley–Hamilton Theorem.** (🌶🌶🌶, Mark, Friday)
Take any square matrix $A$ and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of $A$). Now substitute $A$ into the polynomial; for example, if $A$ is a $4 \times 4$ matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.
*Homework:* None.
*Prerequisites:* Linear algebra, including a solid grasp of determinants (the "Magic of Determinants" class would definitely take care of that).

**The "Free Will" Theorem.** (🌶🌶🌶🌶, Don, Wednesday)
The so-called "Free Will Theorem" of Conway and Kochen takes, as givens a few (relatively) "uncontroversial" statements from physics, and then, using some slick 3 dimensional geometry, proves that if "humans" have "free will" then "particles" do too. In this class, we'll prove the theorem, unravel the relevant definitions of all the things I put in quotes in this blurb, and see that, while it's a little disappointing as a statement about free will, this theorem has some quite interesting implications for the ways quantum mechanics can and cannot work.
*Homework:* Recommended.
*Prerequisites:* None.

**The Yoneda Lemma.** (🌶🌶🌶, Sachi, Wednesday)
The Yoneda lemma is a ridiculous result, which makes a sweeping statement about all categories at once. It says that given a category $C$ along with a functor $F$ from $C$ to Set, we can get this functor from a more familiar functor, namely the Hom functor.

The Hom functor is a really familiar functor: given an object $a$ in my category, I can ask what is the set of $C$-morphisms from $a$ to $x$. That set is called $\mathrm{Hom}(a, x)$. Then, what Yoneda says is that there is a one-to-one correspondence between natural transformations from $\mathrm{Hom}(a, -)$ to $F$ and elements of $Fa$.

We'll discuss and prove the Yoneda lemma, so if you're curious about the proof, or want to see it again, this is a good class for you. At the beginning we will review the definition of natural transformation, Hom, functor, and category very quickly for people who need a refresher.
*Homework:* Optional.
*Prerequisites:* It would be very helpful to have seen the definition of category and functor already. It would also be helpful to know what a natural transformation is, though not strictly necessary. If not, maybe you know what a homotopy is?

**Twisty Puzzles are Easy.** (🌶🌶🌶, Zach, Wednesday)
Algorithmically speaking, the Rubik's Cube and many of its more complicated cousins are easy to solve: they can be expressed in terms of permutation groups acting on the pieces, and they therefore fall to an elegant and efficient algorithm due to Schreier and Sims. This algorithm is well-suited for many types of queries regarding permutation groups, such as testing group membership (can you turn a single corner in a $3 \times 3 \times 3$ cube? Swap two edges in a $4 \times 4 \times 4$ cube?), instantly computing the size of the group (there are precisely $43,252,003,274,489,856,000$ possible $3 \times 3 \times 3$ Rubik's cube positions, and you can bet it didn't enumerate them all individually!), and many others.

This is *not* a course on how to solve a Rubik's Cube. We will deal more generally with computations in *any* permutation group (or group action), and the resulting algorithm can describe the group generated by *any* desired set of basic permutations. We will see how this can be applied to many types of puzzles, and we will also discuss some limitations of the approach.
*Homework:* Optional.
*Prerequisites:* Group Theory.

**Wedderburn's Theorem.** (🌶🌶🌶, Mark, Thursday)
Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, ki = j, ik = -j, jk = i, kj = -i.$$

Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).
*Homework:* None.
*Prerequisites:* Some group theory and some ring theory; familiarity with complex roots of unity would help.

**Wythoff's Game.** (🌶🌶, Alfonso, Tuesday)
Let's play a game. We have a plate with blueberries and strawberries in front of us. We take turns eating them. In your turn, you may eat as many berries as you want as long as they are all of the same kind (at least one) or exactly the same number of berries of both kinds (at least one). Then it is my turn. The player who eats the last berry wins. Will you beat me?

This game with simple rules has a surprising strategy with a game that will blow your mind!
*Homework:* None.
*Prerequisites:* None.

**1:10 Classes**

**Big Numbers.** (🌶🌶🌶, Pesto, Tuesday–Wednesday)
Write down the biggest number you can on a standard-sized sheet of paper, and submit it to Pesto or under the door of Woodman 164 by Sunday midnight. What you write for your number shouldn't rely on anything that's outside that sheet of paper (e.g. "The biggest number that someone else wrote, plus one").

In this class, Pesto will:

(1) Talk about why figuring out who submitted the biggest number is a hard problem and could in fact be impossible,
(2) Say who submitted the biggest number anyway,
(3) Run more variants of the above contest, and
(4) Talk about how to make really big numbers with combinatorics, computer science, or logic.

*Homework:* Recommended.

*Prerequisites:* None.

**Graph Polynomials.** (🌶🌶, Jeff + Mia, Thursday–Friday)
Are you one of millions of people who have a problematic graph $G$ in your life? Do you lie awake and wonder:

- How will I count the number of spanning forests in $G$?
- What is the probability that $G$ remains connected after randomly removing each edge based on a coin flip?
- How many ways are there to color $G$ with my 256-color box of Crayola crayons?

Well, then we have a degree $|E| + |V|$ polynomial for you!

This 1954 masterpiece, defined by Tutte, computes exactly the properties that can be determined by edge contraction and deletion. No extra estimation or calculation needed. This *one* polynomial will solve all of your problems above (and many more). 100% recursively defined. Factorizable over connected components. Made in Canada.

If you order today, well even throw in two specialized polynomials, free of charge!

*Homework:* Optional.

*Prerequisites:* Graph theory.

**Hard Problems That Are Almost Easy.** (🌶🌶, Vivian, Tuesday–Wednesday)
Some problems are easy. Like, sorting a list of $n$ numbers is not too bad; you can do it in time proportional to $n \log n$. And some problems are hard. For example, deciding if a graph with $n$ vertices is $k$-colorable is a type of hard that is called NP-complete.

But how can you tell which ones are which? We're going to be looking at several problems that are NP-complete, but become polynomial-time solvable if you change a tiny piece of the phrasing. In addition we'll define what it means for a problem to be polynomial-time solvable, or NP-complete.

*Homework:* Optional.

*Prerequisites:* None.

**Hyperbolic Geometry.** (🌶🌶🌶, Jane, Tuesday–Wednesday)
In the Euclidean plane, two lines that are not parallel will always intersect. This is the parallel postulate, one of the five postulates of Euclidean geometry, which date back to Euclid in 300 BC. For many hundreds of years afterward, mathematicians were convinced that the parallel postulate was a result of the other four, but proofs of this foiled them.

Then, in the 1800s, Gauss and others proved that non-Euclidean geometries exist. In entered the hyperbolic plane, a geometric space that satisfies the first four postulates of Euclidean geometry but not the parallel postulate. In this class, we'll explore the geometry of the hyperbolic plane and many of the interesting things that can happen there.

*Homework:* Optional.

*Prerequisites:* Calculus is a prerequisite. Linear algebra and knowing what a group is might be helpful, but aren't required.

**Latin Squares and Finite Geometries.** (🌶🌶, Marisa, Thursday–Friday)
Consider the 16 aces, kings, queens, and jacks from a regular 52 card deck of playing cards. Can the 16 cards be arranged in a $4 \times 4$ array so that no suit and no rank occurs twice in any row or column? To make it a little harder: is it possible to color the cards 4 different colors (say red, green, orange, blue) such that

(a) no two cards have the same color and suit,
(b) no two cards have the same color and rank, and
(c) no row or column has the same color twice?

Turns out the answer is yes, and also we can use that carefully-constructed array to produce finite affine and projective geometries (like the Fano plane).

*Homework:* Recommended.

*Prerequisites:* None.

**Multiplicative Functions.** (🌶🌶, Mark, Tuesday–Wednesday)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.

*Homework:* Optional.

*Prerequisites:* No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)

**Permuting Conditionally Convergent Series.** (🌶🌶🌶, Zach, Thursday–Friday)
The starting point for this class is the well-known Riemann's Rearrangement Theorem: given any conditionally convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers—that is, the sum converges but $\sum_{n=1}^{\infty} |a_n|$ does not—you can permute the terms so that the new sum converges to *any* desired value. But what if the $a_n$ are instead allowed to be *complex*? Can you permute to obtain any *complex* sum? Not always: for the sequence $a_n = (-1)^n \cdot \frac{i}{n}$, the achievable sums lie on the line $i \cdot \mathbb{R} \subset \mathbb{C}$. What other subsets of $\mathbb{C}$ arise in this way? And what about series in $\mathbb{R}^k$? We'll discuss and prove this beautiful classification.

We'll also consider the problem from the permutations' perspective: what are the permutations on $\mathbb{N}$ that always turn convergent series into different convergent series (not necessarily with the same sum)? Amazingly, there are some permutations that do better than $\mathbb{N}$ itself: they turn convergent series into convergent series, but they also transform some *divergent* series into convergent ones!

*Homework:* Optional.

*Prerequisites:* Knowledge of the epsilon-delta definition of limits.

**The Cap Set Problem.** (🌶🌶🌶, Susan, Thursday–Friday)
Suppose you have a Set deck. You have four attributes: shape, color, fill, and number. Each attribute has three possibilities. The game is played by laying out twelve cards and searching for a "set," a collection of three cards in which each attribute is either all the same or all different. If there is no set out on the table, you add four additional cards. How long can this go on before you are guaranteed to have a set?

The generalization of this question is called the Cap Set Problem. In the Cap Set Problem, you have a Set deck with $k$ attributes rather than four. How many cards do you need to have out on the table in order to be guaranteed a set?

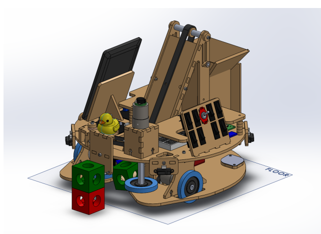
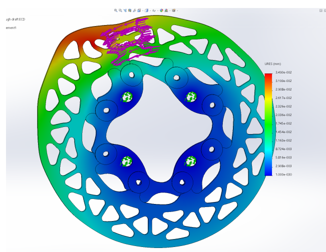
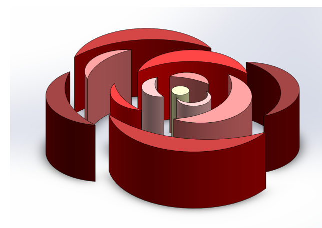
Until recently, our best upper bound was polynomial. This on May 5 this year—less than three months ago—a paper by Croot, Lev, and Pach opened the door to a much better exponential bound. Come to this class if you would like to hear me talk in a wildly underinformed fashion about this exciting new mathematical result.

*Homework:* Recommended.

*Prerequisites:* None.

**2:10 Classes**

**Computer-Aided Design.** (🌶, Elizabeth, Tuesday–Wednesday)
Computers are awesome! They can do so many cool things! In particular, if you can imagine some shape or machine, you can make a computer draw it in 3D. Once the computer knows what it is, then it can show you what it would like from any angle, and you can tweak it without having to redraw the whole thing. You can also turn it colors and zoom in on small details. Basically, anything you can do in your head, you can show to other people, with the computer.

In addition, once you have told the computer about it, the computer can print out pictures or files that let machinists or machines make the part in real life. Computer aided design is useful for all kinds of things, from making robots to racecars to mathematical shapes.

*Homework:* Optional.

*Prerequisites:* None.

**Factoring with Elliptic Curves.** (🌶🌶🌶, David, Tuesday–Friday)
Elliptic curves are defined by equations of the form

$$y^2 = x^3 + ax + b.$$

One of their amazing properties is that the set of points $(x, y)$ satisfying this equation forms a group. This even holds when you consider $x$ and $y$ modulo $p$. We will use these curves to find a method for factoring huuuge integers, which is used in practice.

*Homework:* Recommended.

*Prerequisites:* Elementary number theory, group theory.

**Math and Literature.** (🌶, Yuval, Thursday)
Many Mathcampers (including me!) love reading, but we often think that reading and doing math are fundamentally different things. Though this is sometimes the case, there are many instances in which math and literature are inextricably related. In this class, we'll explore some incredible pieces of literature and discuss some of the math that went into their creation.

**Note:** Though I will provide some of the literature for you to read, doing so is totally optional. In particular, feel free to come even if you aren't comfortable reading in English!

*Homework:* Optional.

*Prerequisites:* None.

**More Group Theory!** (🌶🌶, Kevin, Tuesday–Friday)
Want more group theory? Let's continue where we left off in Week 1!

We'll start studying finite groups in detail, using our work with quotients and cosets to learn a surprising amount about a group just based on its order. If all goes well, we'll try to get through the Sylow Theorems, which tell us *a lot* about what the possible subgroups could be.
*Homework:* Recommended.
*Prerequisites:* Group Theory (Week 1) or equivalent.

**Paradoxes in Probability.** (🌶, Jane, Friday)
You're given a choice of two envelopes and are told that one has twice as much money as the other. After you choose one, you're given the option of switching to the other. Should you switch?

Now, you're playing a game where you repeatedly flip a coin. If the first instance of heads comes up on the first flip, you get two dollars. If the first instance is on the second flip, you get four dollars. In general, if the first instance is on the $n^{\text{th}}$ flip, you get $2^n$ dollars. How much should you pay to play this game?

In this class, we'll test our intuition for probability by touring the world of probability paradoxes.
*Homework:* None.
*Prerequisites:* Comfort with basic probability and computing expectations.

**The Hales–Jewett Theorem.** (🌶🌶🌶🌶, Misha, Tuesday–Friday)
Ramsey theory is a branch of combinatorics about proving that sufficiently large structures have ordered substructures.

Why is it called "Ramsey theory" and not "Ramsey collection of similarly flavored results"? Because of theorems such as the Hales–Jewett theorem that connect them, letting us prove many results from just one and giving a unified way to improve our bounds on "sufficiently large".

Imprecisely speaking, the Hales–Jewett theorem is about tic-tac-toe. If you color a $19 \times 19 \times \cdots \times 19$ grid with 37 colors, then in sufficiently many dimensions you are guaranteed to find 19 collinear points all of one color, and this is true for all values of 19 and 37.

We will give one or more proofs of this theorem, see some of the consequences, and survey mathematicians' attempts to make the upper bounds on the number of dimensions less than catastrophically large.
*Homework:* Recommended.
*Prerequisites:* None.