WEEK 5 CLASS PROPOSALS, MATHCAMP 2016

To vote on classes, go to `appsys.mathcamp.org`. Voting closes at 11:59 PM on Wednesday.

Contents

Alfonso's Classes

Arrow's Impossibility Theorem. (🌶, Alfonso, 1 day)
Arrow's Impossibility Theorem is often quoted as saying "The only fair voting system is a dictatorship." The actual statement is a bit more elaborate. The proof is not easy, but it is long and messy.

In this one hour lecture I will present you with the precise statement and will try to make the proof understandable.

Homework: None.
Prerequisites: None.

Fractal Dimensions. (🌶, Alfonso, 1 day)
A line has dimension 1, a plane has dimension 2, and the space we live in has dimension 3. Can you think of something of dimension 1.5? What does it mean to have dimension 1.5? Actually, what does it even mean to have dimension 2?

In this class, I will give you one possible definition of dimension and we will compute the dimension of a few objects, including some with non-integer dimensions.

Homework: Optional.
Prerequisites: You need to be comfortable with logarithms and limits.

Regions on a Circle. (🌶🌶, Alfonso, 1 day)
Draw N points on a circumference at random positions. Join every pair of points. The inside of the circle is now divided into R_N regions. The sequence of values of R_N begins as

$$1, 2, 4, 8, 16, 31, \ldots$$

Yes, that 31 is not a typo. There is a beautiful explanation behind this sequence, and there is also an explanation for why the sequence looks like powers of two, but really isn't.

In this class you will work through guided problems to figure out by yourselves the actual formula and the proof for this combinatorial problem.

(*Note: this class covers the same material as the first day of Kevin's Hyperplane Arrangements. Feel free to vote for either or both!*)

Homework: None.
Prerequisites: None.

Wythoff's Game. (🌶🌶, Alfonso, 1 day)
Let's play a game. We have a plate with blueberries and strawberries in front of us. We take turns eating them. In your turn, you may eat as many berries as you want as long as they are all of the same kind (at least one) or exactly the same number of berries of both kinds (at least one). Then it is my turn. The player who eats the last berry wins. Will you beat me?

This game with simple rules has a surprising strategy with a game that will blow your mind!

Homework: None.
Prerequisites: None.

ASSAF'S CLASSES

Benford's Law. (🌶, Assaf, 1 day)
Benford's Law was discovered when an astronomer was sifting through log tables (the old kind) and noticed that the pages containing logarithms of numbers whose first digit is 1 were more worn out than others.

It turns out that approximately $\frac{1}{3}$ of the numbers found in real life begin with a 1. In this class we will explain this phenomenon, and see where it comes from and how it can be applied for practical purposes.

Homework: Recommended.
Prerequisites: None.

Classification of Surfaces. (🌶️🌶️ → 🌶️🌶️🌶️, Assaf, 3–4 days)
For millennia, humans and dolphins debated what the surface of the Earth looked like topologically. The dolphins, orientable in nature, proposed a peculiar shape composed of a bunch of tori smushed together. This idea preoccupied the marine intelligentsia, and prevented any further advances in the subject. This was, until the dolphins were extinct.

Then came the age of man, where more and more surfaces were discovered. This culminated in the early 1800's, when Marco Polo sailed around the world in order to disprove the lies of Fermat, who said that the world was a projective plane.

In this class, we will review the crowning achievement of humanity over the dolphins—the classification of surfaces.

Homework: Recommended.

Prerequisites: None.

Lie Groups. (🌶️🌶️🌶️🌶️, Assaf, 4 days)
In this class we will talk about differential manifolds.

Homework: Recommended.

Prerequisites: Topology, linear algebra (weeks 1 and 2), group theory, calculus (preferably multivariable).

$\mathrm{SL}_2(\mathbb{Z})$**.** (🌶️🌶️ → 🌶️🌶️🌶️, Assaf, 3 days)
In this class, we will talk about the group $\mathrm{SL}_2(\mathbb{Z})$. We will talk about it in the context of Möbius transformations and the stretchings of a torus of unit area.

Homework: Recommended.

Prerequisites: Linear algebra (week 1).

The Hairy and Still Lonely Torus. (🌶️, Assaf, 1–3 days)
You've heard of spiky hedgehogs and their associated theorems, but have you heard of the amazing and totally easy-to-prove Hopf–Poincaré index theorem?

It turns out that the hairy ball theorem generalizes to other orientable surfaces, and links their topology (Euler characteristic) with what vector fields are allowed on them.

In this class, we will present a shockingly simple proof of the Hairy ball theorem, using the Hopf–Poincaré index theorem for surfaces, and the invariancy of the Euler characteristic. If there is time, we will also discuss a bit about liquid crystals on spheres.

This class can be thought of as a sequel to The Lonely Torus in the sense that we will prove that the Torus is the only comb-able (closed orientable) surface.

Homework: Recommended.

Prerequisites: None.

The Pie of Power. (🌶️🌶️ → 🌶️🌶️🌶️, Assaf, 2 days)
In this class we will model the Israeli Knesset. The way the Knesset works is by creating coalitions of representatives, which then have total power together. If no party gains a majority of the representatives, then they must negotiate with other parties to create a coalition.

Under this setup, how big of a slice should each party get from the pie of power?

Homework: Recommended.

Prerequisites: A bit of linear algebra.

Chris's Classes

Abstract Nonsense. (🌶🌶🌶🌶, Chris, 2–3 days)
Topological spaces are nice! But sometimes (*cough* algebraic geometry *cough*) the topology really does not resemble the geometry that you have in mind at all. So what's a topology anyways? Can we generalize this to get topologies that are like open sets but have nothing to do with open sets?

Yes, we can! Such a thing is called a topos and we can define in on any category! That's right, we can turn any category in a topological space. This is not only fun and weird, but also solves some of the issues that the weird topologies in algebraic geometry have.

Homework: Recommended.

Prerequisites: Category theory (definitions).

Homological Algebra. (🌶🌶🌶🌶, Chris, 3–4 days)
You love abelian groups? You love vector spaces? You love modules over a ring? You really enjoy forming kernels, images, and quotients. You might even have tried your hand in defining homologies and chain complexes? So you wonder if there's a more general environment where you can do all these things?

Then an abelian category is the place you want to live in. Not only are all the above things examples of one, but if we think about abelian categories correctly, we can free ourselves from the shackles of thinking about elements. Just as Jessie J, "it's not about the objects, objects, objects, we just want to make the morphism, forget about the object."

Homework: Recommended.

Prerequisites: Category theory (definitions, some universal properties).

Homotopy (Co)limits. (🌶🌶🌶🌶, Chris, 3–4 days)
Imagine you are back in your childhood and you're playing with some clay. You have 3 pieces of clay and want to assemble them according to some instructions your friend sent you. Unfortunately, your evil sibling took your clay and completely deformed your pieces. Nevertheless you go ahead and assemble it. The next day you and your friend realize that you both got completely different structures. Yours even has a hole where theirs doesn't!

What you just discovered is what topologists refer to as the fact that "the colimit of a diagram is not homotopy invariant." In this class, we will explore this phenomenon (possibly with or without clay), and hopefully we will be able to answer what the true and real and homotopy-invariant way of assembling your clay should be!

Homework: Optional

Prerequisites: Category theory (basic definitions, some of the following universal properties: (co)products, push-outs, pull-backs), topology (continuity).

Homotopy Theory. (🌶🌶🌶🌶, Chris, 4 days)
In topology, two spaces are (homotopy) equivalent if they can be "continuously transformed into each other": a cube is the same as a sphere, but not the same as a torus (donut). Deciding if two spaces are homotopy equivalent is very hard and we often have to turn to invariants. The first such invariants are the homotopy groups of a space X: for $i \in \mathbb{N}_0$ the sets $\pi_i(X)$ are the (homotopy) equivalence classes of maps from the i dimensional sphere S^i to X: If $i > 0$ this is a group and if $i \geq 2$ it's even abelian.

It is easy to see that two homotopy equivalent spaces have the same homotopy groups, so calculating those groups is a good way of deciding if spaces are not homotopy equivalent. There also is a huge class of spaces (so called CW-complexes) for which having matching homotopy groups immediately implies homotopy equivalence.

So these homotopy groups are easy to define and tell us a whole lot about the space, but there is one big problem: they are incredibly hard to calculate. So hard that after over 80 years we still have not been able to compute all the homotopy groups of the 2-sphere! Calculating those homotopy groups of spheres has been driving modern mathematics for over half a century and these efforts have produced a large amount of new ideas, techniques and theories, and have shaped the way we think about mathematics in many areas such as topology, algebra and geometry. In this course we will introduce the homotopy groups and make our way to the long exact sequence of the homotopy groups of a fibration, the easiest (and most important) tool used to calculate them.

Homework: Recommended

Prerequisites: Group theory (quotient groups), some notion of continuity.

David's Classes

Binary Quadratic Forms. (🌶🌶 → 🌶🌶🌶, David, 1–4 days)
A binary quadratic form has the form $Ax^2 + Bxy + Cy^2$. We will be looking at the integers that can be represented by such a form, using a tool known as "Conway's topograph." These diagrams are a lot of fun to play around with, and reveal a lot about the hidden structure of quadratic forms!

Homework: Optional.

Prerequisites: A little linear algebra (bases of two-dimensional vector spaces).

Factoring with Elliptic Curves. (🌶🌶🌶, David, 4 days)
Elliptic curves are defined by equations of the form

$$y^2 = x^3 + ax + b.$$

One of their amazing properties is that the set of points (x, y) satisfying this equation forms a group. This even holds when you consider x and y modulo p. We will use these curves to find a method for factoring huuuge integers, which is used in practice.

Homework: Recommended.

Prerequisites: Elementary number theory, group theory.

Sato–Tate Distributions. (🌶🌶🌶, David, 1–2 days)
Suppose you have an equation such as $y^2 = x^3 + x + 1$. As p varies, how many solutions do you expect to find? The first part of the answer was proven by Hasse in the 1930's. The second part was conjectured independently by Sato and by Tate in the 1960's and proven in 2006. We will discuss these results and their generalizations to higher degree equations (including pretty pictures!).

Homework: None.

Prerequisites: Group theory, elementary number theory.

The BSD conjecture. (🌶🌶🌶🌶, David, 1–2 days)
The Birch and Swinnerton-Dyer conjecture is one of the biggest conjectures in modern mathematics (it is one of the seven million-dollar Clay Math problems). We'll go on a whirlwind tour of the objects involved (L-functions and Mordell–Weil groups of elliptic curves) and give a statement of the conjecture.

Homework: Optional.

Prerequisites: Group Theory, analytic number theory.

DON'S CLASSES

1-Dimensional Manifolds. (🌶🌶🌶, Don, 1 day)
Manifolds are at the heart of modern topology; structured enough that we know what they look like locally, yet loose enough to cover a wide variety of applications. So, like many things in modern math, they're really hard to define.

That is, unless you restrict your attention to the one dimensional case. Then, the definition is pretty easy to work with, and we'll even have time to classify them all—a lovely result used every day by low dimensional topologists.

Homework: Recommended.

Prerequisites: Calculus, Topology.

Building Groups out of Other Groups. (🌶🌶 → 🌶🌶🌶, Don, 1–3 days)
In group theory, everybody learns how to take a group, and shrink it down by taking a quotient by a normal subgroup—but what they don't tell you is that you can totally go the other direction. It's a little harder, and a little messier, because while there's only one way to take a quotient, there's lots of ways to take a product. So, let's take a look at a few of these ways, from the clean, sterile direct product, to the messy, dangerous free product, and even the beautiful, mighty semi-direct product.

Homework: Recommended.

Prerequisites: Group Theory.

Category Theory, Just for Fun. (🌶 → 🌶🌶, Don, 1–2 days)
The language of category theory has been a useful tool in mathematics over the past 70 years, facilitating results in algebraic topology, homological algebra, and myriad other mind-boggling areas of math. At its core, though, category theory is a beautifully simple language, one that can be used to describe as many weird and wacky topics as it can difficult and profound. For every Homotopy Type Theory, there is the Category of Foods, for every Simplicial Set, there is a category of little stuff. I want to talk about the latter, and see just how category theory can give us a better understanding of the little things in life.

Homework: Recommended.

Prerequisites: Never having learned category theory before.

Not Quite Group Theory. (🌶🌶 → 🌶🌶🌶, Don, 1–2 days)
Silly mathematicians - studying group theory. Every element has inverse? Such strong assumptions make students weak. With proper definitions, all that works in group theory works just as well in semigroup theory. Well, not all—but all that was done in week 1 group theory class. So with semigroups, we have same theorems, but even better and sillier examples!

Homework: Recommended.

Prerequisites: Group Theory.

The Brauer Group. (🌶🌶🌶🌶, Don, 1–3 days)
The Brauer Group of a field is a group whose elements are central simple algebras over the field, and whose operation is tensor product.

The Brauer Group is always torsion.

The Brauer Group is the right way to talk about class field theory.

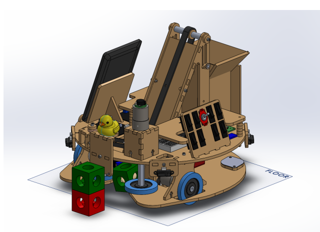
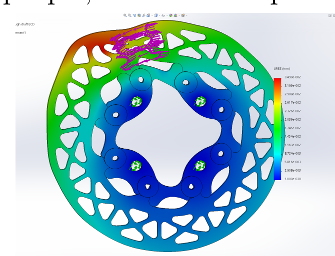
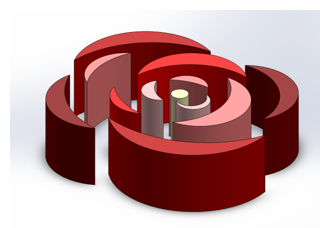
The Brauer Group classifies projective varieties that become isomorphic to projective space over an algebraic closure.

The Brauer Group is terrifying, and beautiful. In this class, we'll define it, and see that, surprisingly, it's actually a group.
Homework: Recommended.
Prerequisites: Ring Theory, Group Theory, a class where you defined a tensor product.

The "Free Will" Theorem. (🌶️🌶️🌶️ → 🌶️🌶️🌶️🌶️, Don, 1 day)
The so-called "Free Will Theorem" of Conway and Kochen takes, as givens a few (relatively) "uncontroversial" statements from physics, and then, using some slick 3 dimensional geometry, proves that if "humans" have "free will" then "particles" do too. In this class, we'll prove the theorem, unravel the relevant definitions of all the things I put in quotes in this blurb, and see that, while it's a little disappointing as a statement about free will, this theorem has some quite interesting implications for the ways quantum mechanics can and cannot work.
Homework: Recommended.
Prerequisites: None.

Elizabeth's Classes

Computer Aided Design. (🌶️, Elizabeth, 2–4 days)
Computers are awesome! They can do so many cool things! In particular, if you can imagine some shape or machine, you can make a computer draw it in 3D. Once the computer knows what it is, then it can show you what it would like from any angle, and you can tweak it without having to redraw the whole thing. You can also turn it colors and zoom in on small details. Basically, anything you can do in your head, you can show to other people, with the computer.

In addition, once you have told the computer about it, the computer can print out pictures or files that let machinists or machines make the part in real life. Computer aided design is useful for all kinds of things, from making robots to racecars to mathematical shapes.
Homework: Optional.
Prerequisites: None.

Differential Equations. (🌶️, Elizabeth, 1 day)
Differential equations show up in lots of places, notably in practical applications of engineering and physics. Differential equations are simply equations that include the derivative as a part of the equation. In this class, we will learn how to solve some of the simpler kinds of them.
Homework: None.
Prerequisites: Calculus (specifically derivatives and integrals).

Difficulties in Communication. (🌶, Elizabeth, 1 day)
Suppose you're named Alice, and you have a friend named Bob, and that you both have a bit, which is either 0 or 1. And you want to know if they're both 1. This problem is equivalent to something you may be more familiar with, the problem of deciding whether you and someone else have a crush on each other.

Suppose you want to prove something to someone else (mathematicians seem to do this a lot). But you're mean, so you don't want to tell them all your secrets, since they might publish before you. How do you prove to them that you really can do the thing you say you can?

Both of these are totally doable, and this is what the class will be about!

Homework: None.

Prerequisites: None.

Jane's Classes

Egyptian Fractions. (🌶, Jane, 1 day)
The ancient Egyptians only had notation to write fractions of the form $\frac{1}{n}$. If they wanted to write any other positive rational number, they had to write it as a sum of distinct terms of this form.

This sounds incredibly restrictive, but actually every rational number can be written as an Egyptian fraction. In this class, we'll take the idea of an Egyptian fraction and run with it. It turns out to generate quite a lot of interesting math. For example, how do we find Egyptian fraction decompositions? Can we find the shortest decompositions? As we discover things about Egyptian fractions, we'll also see how quickly we run into questions about them that are still open.

Homework: None.

Prerequisites: None.

Geometry and Numbers. (🌶🌶, Jane, 2–3 days)
In elementary school, you learned that the incorrect way to add numbers is by adding numerators to numerators and denominators to denominators. But what your teachers didn't tell you is that this is the correct way to create Farey sequences! The first Farey sequence is $\{\frac{0}{1}, \frac{1}{1}\}$, the second is $\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$, the third is $\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$, and each subsequent Farey sequence is obtained by including a new fraction between every two of the previous sequence by taking the bad fraction sum of the neighboring terms.

It turns out that this silly way of adding fractions leads to a lot of cool math. We'll see that the Farey sequence is related to lots of other topics in number theory like continued fractions and Pythagorean triples, and we'll use the geometry of circle inversions (and also secretly, hyperbolic geometry) to see all of that.

Homework: Optional.

Prerequisites: None.

How to Cut a Cake. (🌶🌶, Jane, 1 day)
Suppose that you and a friend are sharing a cake. How can you two divide up the cake fairly? Well, you could just cut it down the middle, and each person gets a half. That sounds fair.

But wait! What if there is frosting and such, and each person values different parts of the cake differently? Well, one person can cut the cake into two pieces and the other person can choose their piece. That way, both people are happy.

But wait! What if there are more people? Can we still achieve fair division? Come to this class and find out!

Homework: None.

Prerequisites: None.

Hyperbolic Geometry. (🌶🌶🌶, Jane, 2–4 days)
In the Euclidean plane, two lines that are not parallel will always intersect. This is the parallel postulate, one of the five postulates of Euclidean geometry, which date back to Euclid in 300 BC. For many hundreds of years afterward, mathematicians were convinced that the parallel postulate was a result of the other four, but proofs of this foiled them.

Then, in the 1800's, Gauss and others proved that non-Euclidean geometries exist. In entered the hyperbolic plane, a geometric space that satisfies the first four postulates of Euclidean geometry but not the parallel postulate. In this class, we'll explore the geometry of the hyperbolic plane and many of the interesting things that can happen there.

Homework: Optional.

Prerequisites: Calculus, group theory.

Paradoxes in Probability. (🌶, Jane, 1 day)
You're given a choice of two envelopes and are told that one has twice as much money as the other. After you choose one, you're given the option of switching to the other. Should you switch?

Now, you're playing a game where you repeatedly flip a coin. If the first instance of heads comes up on the first flip, you get two dollars. If the first instance is on the second flip, you get four dollars. In general, if the first instance is on the n^{th} flip, you get 2^n dollars. How much should you pay to play this game?

In this class, we'll test our intuition for probability by touring the world of probability paradoxes.

Homework: None.

Prerequisites: Comfort with basic probability and computing expectations.

The Gauss Circle Problem. (🌶🌶🌶🌶, Jane, 4 days)
In a circle of radius r centered at the origin in the plane, we expect about πr^2 integer lattice points. But how close is that estimate really? The Gauss circle problem asks for the size of that error, as a function of r. It is conjectured that the error is no more than $C_\epsilon r^{1/2+\epsilon}$ for any $\epsilon > 0$ and constant C_ϵ depending on ϵ.

We can get a step toward the conjecture by using Fourier analysis. Fourier analysis is the study of how functions can be decomposed into sums of trigonometric functions. In this class, we'll learn some of basic tools in Fourier analysis and use them to give a nice proof that the error in the Gauss circle problem is no bigger than $Cr^{2/3}$ for some constant C.

Homework: Recommended.

Prerequisites: Calculus.

The Mathematics of Juggling. (🌶, Jane, 1–2 days)
A juggler doing the standard three ball cascade juggle is very rhythmic. He alternates throwing balls with his left hand and right hand, and on each "beat" catches and throws one ball, which then stays in the air for three beats until it is caught and thrown again.

We can encode this data in the juggling sequence

$$3, 3, 3, 3, \ldots$$

In a juggling sequence, the first number encodes that the left hand throws a ball that then stays in the air for that number of beats, the second number encodes that the right hand throws a ball that stays in the air for that number of beats, and so on, alternating left-right-left-right. We can then ask what periodic sequences are juggle-able. For example, is

$$4, 4, 1, 3, 4, 4, 1, 3, \ldots$$

juggle-able? Given a juggle-able sequence, how many balls do we need to juggle it? In this class, we're explore these questions and more as we delve into the mathematics of juggling.

Homework: None.

Prerequisites: None.

Jeff's Classes

Extending Extending Inclusion-Exclusion. (🌶🌶🌶🌶, Jeff, 4 days)
Maybe you feel like you didn't get enough homological algebra throughout camp. Maybe you feel cheated from a lack of long exact sequences. Maybe you have a burning desire to make a pun about "Ext" functors and extending inclusion exclusion. In this class, we'll take the ideas of extending inclusion-exclusion, and formalize them into the language of derived functors.

Homework: Recommended.

Prerequisites: Extending Inclusion-Exclusion, Multilinear Algebra, Universal Properties.

Knotty Groups. (🌶🌶🌶, Jeff, 3 days)
Here are 3 different kinds of strings:

- A braid is a map from several strings to 3 dimensional space, where no string intersects eachother, and all the strings travel in the same direction.
- A knot is a map from a circle to 3-dimensional space, which does not intersect itself.
- A fundamental group is an algebraic object which understands the way that you can draw loops on a topological space.

In this class, we'll be using braids to describe how the fundamental group wraps around a link. If you've ever wanted to have strings describe the ways that you can string up your strings, boy, do I have a class for you!

Homework: Optional.

Prerequisites: You should know how to compute the fundamental group of a torus.

Scandalous Curves. (🌶🌶🌶, Jeff, 4 days)
Sometimes you see a function, and you are comfortable looking at it. Functions like $f(x) = \sin(x)$ and $g(x) = x^2 + 1$. But sometimes you see a function and it makes you squirm a little bit.

- (Merely Misbehaving) A function like $\frac{1}{x}$ or $|x|$ has some bad points, but is mostly good.
- (Bad) A continuous function which has $f'(x) = 0$ almost everywhere, but increases monotonically from 0 to 1 would be bad (think about the fundamental theorem of calculus), but nothing too terrible to look at.
- (Sinful) A continuous function from the real numbers to the unit square that covers every point in the unit square would make most people uncomfortable.
- (Scandalous) If you are comfortable with a continuous function that is differentiable at no point, then you have a serious problem, and should probably talk to an analyst.

While it may seem unusual to find a scandalous function, we will show in this class that "most" functions are actually scandalous, and you can't spend your whole life ignoring them. Besides constructing the above functions, we will also explore convergence for functions, metric spaces of functions, random functions and what it means for a function to be "generic".

Homework: Recommended.

Prerequisites: You should be able to prove that a differentiable function is continuous using an epsilon-delta argument.

Kevin's Classes

A Very Difficult Definite Integral. (🌶🌶🌶🌶, Kevin, 1 day)
In this class, we will show

$$\int_0^1 = \frac{\log(1+x^{2+\sqrt{3}})}{1+x}\,dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3}).$$

We'll start by turning this Very Difficult Definite Integral into a Very Difficult Series. Then we'll sum it!

We'll need an unhealthy dose of clever tricks involving some heavy-duty algebraic number theory. It will be Very Difficult.

(This class is based on a StackExchange post by David Speyer.)

Homework: None.

Prerequisites: A ton of algebraic number theory. David's algebraic number theory marathon *might* suffice. A willingness to believe the facts we'll use is fine instead. Familiarity with rings and ideals is probably necessary regardless.

Hyperplane Arrangements. (🌶🌶🌶, Kevin, 3–4 days)
Continue the sequence:

$$1, 2, 4, \ldots.$$

Did you say 8? Wrong! It's 7: it's the number of regions that n lines in general position in $\mathbb{R}^2$ determine.

Continue the sequence:

$$1, 2, 4, 8, \ldots.$$

Did you say 16? Wrong! It's 15: it's the number of regions that n planes in general position in $\mathbb{R}^3$ determine.

These are (the simplest examples of) *hyperplane arrangements*. Determining the number of regions in a hyperplane arrangement is a beautiful problem that ties together posets, Möbius inversion, and finite fields into an absolutely beautiful story!

(*Note: the first day of this class covers the same material as Alfonso's Regions on a Circle. Feel free to vote for either or both!*)

Homework: Optional.

Prerequisites: None.

More Group Theory! (🌶🌶, Kevin, 1–4 days)
Want more group theory? Let's continue where we left off in Week 1!

We'll start studying finite groups in detail, using our work with quotients and cosets to learn a surprising amount about a group just based on its order. If all goes well, we'll try to get through the Sylow Theorems, which tell us *a lot* about what the possible subgroups could be.

Homework: Recommended.

Prerequisites: Group Theory (Week 1) or equivalent.

Optimizing Neural Networks. (🌶🌶🌶, Kevin, 1–4 days)
In Week 2, we talked about the big ideas in neural networks, and we swept some details of computation and optimization under the rug.

Time to pull the rug out from under us!

In this class, we'll go through the calculus fully for training and computing with a neural network. Time permitting, we'll also study mathematical ways to optimize both the time it takes to train a neural network and its performance.

This will be very different from my Week 2 class—we will be spending most of our time deep in computation rather than taking it relatively easy with fun concepts and demos. So even if you liked that class, do think carefully about whether you want to go down this rabbit hole!

Homework: Optional.

Prerequisites: Taking or reading my notes from Week 2's Neural Nets. Additionally, we will need multivariable calculus, but single-variable calculus will do as a prerequisite: just spend a little time reading about partial derivatives and total derivatives. Finally, familiarity with vector and matrix notation, as well as a few basic words from statistics, will be useful.

Picard's Great Theorem. (🌶🌶🌶, Kevin, 1–2 days)
Picard's Great Theorem says that, near an essential singularity, any complex analytic function takes on every possible value infinitely often. With possibly one exception.

We'll define (but not prove) all the relevant details, and we'll see how this crazy sounding theorem actually has a lot of applications. For example, it's way overkill, but it certainly proves the Fundamental Theorem of Algebra for us!

Homework: None.

Prerequisites: Familiarity with high-school calculus might be helpful but is *not* required.

Marisa's Classes

Earth-Moon Map Coloring. (🌶 → 🌶🌶, Marisa, 2–4 days)
Say we have a planar graph of countries drawn on the earth, and another planar graph of corresponding colonies drawn on the moon (imagining that each earth nation has established its own lunar colony). You want to color the maps so that not only is each planet properly colored (two countries or colonies that share a boundary receive different colors), but they are color-coordinated: every country receives the same color as its lunar colony. What is the minimum number of colors necessary to color all earth-moon maps? What if we expand our question to consider not just the earth, but maps on any surface (say, the k-holed torus, or the projective plane), along with a corresponding map of colonies on the k-holed or projective moon?

Homework: Optional.

Prerequisites: Intro Graph Theory or equivalent.

Heawood's Conjecture, and Coloring the Klein Bottle. (🌶🌶, Marisa, 1 day)
The famous 4-Color Theorem says that we can properly color any planar graph using at most 4 colors. That number 4 doesn't depend on the graph we're coloring: it's an invariant of the plane itself. So if we wanted to properly color a graph that is drawn on, say, the projective plane, or the n-holed torus, or the Klein Bottle, how many colors do we need? This question was actually answered completely for every surface except the plane *before* the 4-Color Theorem was proved. It turns out that the answer is

$$\left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor .$$

In this class, we'll explore the proof of this statement, and see why the Klein Bottle is the only exception to that rule.

Homework: None.

Prerequisites: Intro Graph Theory, and familiarity with orientable and non-orientable closed surfaces.

Improbable Match-Ups. (🌶, Marisa, 1 day)
If we have ten Mathcampers and ten sarongs, out of all of the 10! ways of matching the campers and the sarongs, what is the probability of matching them all wrong (so that nobody gets their own sarong back)? Which is more likely—getting them all wrong, or getting at least one right? We'll answer this question with combinatorics and calculus.
Homework: None.
Prerequisites: Basic graph theory, plus either familiarity with the Taylor Series for e^x or willingness to take 5 minutes of calculus on faith.

Latin Squares and Finite Geometries. (🌶🌶, Marisa, 2–4 days)
Consider the 16 aces, kings, queens, and jacks from a regular 52 card deck of playing cards. Can the 16 cards be arranged in a 4×4 array so that no suit and no rank occurs twice in any row or column? To make it a little harder: is it possible to color the cards 4 different colors (say red, green, orange, blue) such that

(a) no two cards have the same color and suit,
(b) no two cards have the same color and rank, and
(c) no row or column has the same color twice?

Turns out the answer is yes, and also we can use that carefully-constructed array to produce finite affine and projective geometries (like the Fano plane).
Homework: Recommended.
Prerequisites: None.

Mark's Classes

Elliptic Functions. (🌶🌶🌶 → 🌶🌶🌶🌶, Mark, 3–4 days)
Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k),$$

where $\sigma_i(k)$ is the sum of the i^{th} powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1]\,,$$

which you are welcome to check if you run out of things to do.)
Homework: Optional.
Prerequisites: Some experience with functions of a complex variable, including Liouville's Theorem and the residue theorem.

From Counting to a Theorem of Fermat. (🌶 → 🌶🌶, Mark, 1 days)
A standard theorem stated by Fermat (it's actually uncertain whether he had a proof) states that every prime p congruent to 1 modulo 4 is the sum of two squares. (On the other hand, if p is 3 modulo 4, it has *no* hope of being the sum of two squares.) There are many proofs of this theorem, but perhaps the weirdest one, due to Heath-Brown and simplified by Zagier, uses just counting—no "number theory" at all! In this class we'll see at least that proof, and maybe some others and/or related proofs of other things.
Homework: None.
Prerequisites: None.

How to Behead x^2 and Find $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (🌶🌶🌶, Mark, 1 day)
The infinite list of functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$$

all have one thing in common: they have a period 2π. (Most of them have smaller periods, as well.) Suppose you have another continuous function with a period 2π. Might it have anything to do with the functions on the list? And what is the story with x^2, which is not periodic at all? Answering such questions will lead us to an evaluation of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

that is quite different (except for the answer, of course) from the one that was/will be presumably presented in Trail Mix on Thursday.
Homework: None.
Prerequisites: Integration by parts; some linear algebra would be helpful.

Integration by Parts, the Wallis Product, and the Delta "Function". (🌶🌶 → 🌶🌶🌶, Mark, 1–2 days)
Integration by parts is one of only two truly general techniques known for finding antiderivatives (the other is integration by substitution). In this class you'll first see (or review) this method, and two of its applications: how to extend the factorial function, so that there is actually something like $(\frac{1}{2})!$ (although the commonly used notation and terminology is a bit different), and how to derive the famous product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

which was first stated by John Wallis in 1655.

Finally, time permitting, we'll turn to the delta "function". In introductory books on quantum mechanics you can find "definitions" of a "function" called the delta function, which has two apparently contradictory properties: Its value is zero for all nonzero x, and yet its integral over any interval containing 0 is 1. This "function" was introduced by the theoretical physicist Dirac, and it turns out to be quite useful in physics, but how can we make any mathematical sense of such a creature?
Homework: None.
Prerequisites: Basic calculus (differentiation and integration). For the delta "function", the idea of a linear transformation will also be useful.

Multiplicative Functions. (🌶🌶 → 🌶🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which

makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.
Homework: Optional.
Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)

Quadratic Reciprocity. (🌶🌶 → 🌶🌶🌶, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is q a square modulo p?"
(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!
Homework: Optional.
Prerequisites: Some basic number theory (if you know Fermat's Little Theorem, you should be OK).

The Cayley–Hamilton Theorem. (🌶🌶🌶, Mark, 1 days)
Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.
Homework: None.
Prerequisites: Linear algebra, including a solid grasp of determinants (the "Magic of Determinants" class would definitely take care of that).

The Magic of Determinants. (🌶🌶🌶, Mark, 2 days)
This year's linear algebra class barely touched on determinants in general. If that left you feeling dissatisfied, either about not really having a good definition (by the way, using the Laplace expansion, while it is computationally often handy, leads to a miserable definition—with no intuitive basis at all) or about not having seen many of the properties that determinants have, this may be a good class for you. If all goes well, we'll give a definition of determinant that's both motivated and rigorous, and there will be proofs of all its main properties (such as Laplace expansion), as well as a few applications such as general formulas for the inverse of a matrix and for the solution of n linear equations in n unknowns ("Cramer's Rule").
Homework: Optional.
Prerequisites: Some linear algebra, including linear transformations, matrix multiplication, and determinants of 2×2 matrices.

Wedderburn's Theorem. (🌶🌶🌶, Mark, 1 day)
Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, ki = j, ik = -j, jk = i, kj = -i.$$

Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Homework: None.

Prerequisites: Some group theory and some ring theory; familiarity with complex roots of unity would help.

Misha's Classes

Mathematical Gambling. (🌶🌶, Misha, 1–2 days)
If you're betting at a roulette table, how should you minimize its unfairness, or to take advantage if it's skewed in your favor? What are the odds of this strategy working?

What if you have multiple betting games, and can choose to make progress playing each of them? How does your strategy change if you have a rich uncle sponsoring you? What if instead of playing roulette, you're writing a Ph.D. thesis—betting your academic future instead of money?

Homework: Recommended.

Prerequisites: None.

Problem Solving: Tetrahedra. (🌶🌶🌶🌶, Misha, 1 day)
In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

(Note: this is the same class as the class with the same name that I taught in 2015.)

Homework: None.

Prerequisites: None.

The Hales–Jewett Theorem. (🌶🌶🌶🌶, Misha, 4 days)
Ramsey theory is a branch of combinatorics about proving that sufficiently large structures have ordered substructures.

Why is it called "Ramsey theory" and not "Ramsey collection of similarly flavored results"? Because of theorems such as the Hales–Jewett theorem that connect them, letting us prove many results from just one and giving a unified way to improve our bounds on "sufficiently large".

Imprecisely speaking, the Hales–Jewett theorem is about tic-tac-toe. If you color a $19 \times 19 \times \cdots \times 19$ grid with 37 colors, then in sufficiently many dimensions you are guaranteed to find 19 collinear points all of one color, and this is true for all values of 19 and 37.

We will give one or more proofs of this theorem, see some of the consequences, and survey mathematicians' attempts to make the upper bounds on the number of dimensions less than catastrophically large.

Homework: Recommended.

Prerequisites: None.

The Hanoi Graph. (🌶🌶🌶, Misha, 2 days)
A distance-uniform graph has the counter-intuitive property that there's some fixed constant we'll call "6" such that for every vertex v, 99% of all other vertices are exactly 6 steps away from v.

In the first day of this class, I'll explain why you might naively expect 6 to be rather small, and then use a solitaire puzzle game to show that actually, 6 is a big number. In the second day, I'll show that 6 doesn't get much bigger than big.

Time permitting, we'll see the connections to network creation games, and why the internet would have been much cheaper if 6 weren't a big number.
Homework: Recommended.
Prerequisites: Intuitive understanding of what a graph is (Mathcamp's introduction class is sufficient).

Nic's Classes

Calculus on the Hyperreals. (🌶🌶🌶, Nic, 4 days)
When Newton and Leibniz were inventing what we now know as calculus, they thought they were creating a system of rules for manipulating infinitely small quantities. For example, the derivative of the function x^2 might have been computed (using more modern notation) by adding an infinitely small quantity to x and then dividing by it to get

$$\frac{(x+dx)^2 - x^2}{dx} = \frac{x^2 + 2x \cdot dx + (dx)^2 - x^2}{dx} = 2x + dx.$$

Then, since dx is so small, we can forget about it and say that the final answer is $2x$.

Over the next couple centuries, mathematicians began to worry that it wasn't possible to make calculus mathematically rigorous—certainly there's something fishy about these infinitesimals that can turn into 0 whenever it's convenient—and a lot of effort was spent on developing rigorous foundations for calculus. By the late nineteenth century, this program had basically succeeded; the result was the approach that's usually taught today, with limits and epsilons and deltas.

While this approach is in fact perfectly rigorous, it lacks some of the intuitive clarity of the infinitesimal version. So it came as something of a surprise when, in the twentieth century, some logicians managed to show that it's possible to save the infinitesimals after all. They created a system of numbers called the *hyperreals*, which contains all the real numbers, but also infinitely small and infinitely large numbers. The resulting system is usually called *nonstandard analysis*, and it's what this class will be about.
Homework: Recommended.
Prerequisites: A decent understanding of basic calculus, at least up through derivatives.

Option Pricing and the Black–Scholes Equation. (🌶🌶🌶, Nic, 1–2 days)
An *option* is a financial instrument which gives the holder the right to buy or sell some chosen thing at a predetermined price at a predetermined time. For example, if I have an "August 7 blue tape roll 10 put", then I have the right to sell a roll of blue tape for $10 on the last day of Mathcamp. Whether I'm happy to have this option or not depends on whether I think blue tape will be selling for more or less than $10 that day.

It's always better to have an option than not to—you can always just decline to exercise it—so options ought to be worth some money. How much they should be worth, though, is a complicated question: it depends on how much you think the thing being bought or sold will be worth in the future.

Options existed for some time before anyone had a good model for how to price them in terms of quantities that people felt they could understand. In this short class we'll talk about the first successful option pricing model, called the *Black–Scholes model*, which was published in 1973. Even though it doesn't actually describe the real world perfectly (or even all that well) it had a huge influence on the way modern finance works.
Homework: None.
Prerequisites: Calculus, including derivatives and integrals. Some knowledge of probability will also be nice, but it can replaced with a willingness to believe me when I say things about probability, which will probably be necessary anyway.

Special Relativity. (🌶🌶, Nic, 3–4 days)
Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but it forced physicists to come to grips with a strange new truth: there is no such thing as absolute space, and there is no such thing as absolute time. Depending on their relative velocities, different observers can disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics has to be rebuilt to accommodate these observations. We will see how, as Minkowski said, "Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." Along the way, we will also have to revise the classical notions of momentum and energy, allowing us to see the context behind the famous relation $E = mc^2$.

This is a repeat of my special relativity class from Mathcamp 2015, so if you took that class you'll probably not be interested in this one.
Homework: Optional.
Prerequisites: Some high school physics—if you've seen the concepts of momentum and kinetic energy before, you're probably fine.

Pesto's Classes

Big Numbers. (🌶 → 🌶🌶🌶, Pesto, 1–2 days)
Write down the biggest number you can on a standard-sized sheet of paper, and submit it to Pesto. What you write for your number shouldn't rely on anything that's outside that sheet of paper (e.g. "The biggest number that someone else wrote, plus one").

In this class, Pesto will:

(1) Talk about why figuring out who submitted the biggest number is a hard problem and could in fact be impossible,
(2) Say who submitted the biggest number anyway,
(3) Run more variants of the above contest, and
(4) Talk about how to make really big numbers with combinatorics, computer science, or logic.

Homework: Recommended.
Prerequisites: None.

Character Table Sudoku. (🌶, Pesto, 1 day)
A character table is a grid of numbers with some constraints; for instance, this is a character table:

1	1	1	1	1	1
1	-1	-1	-1	1	1
2	0	0	0	-1	-1

If you carefully choose a subset of the grid and delete the rest, filling it back in is a logic puzzle. For instance, can you infer enough of the constraints that define a character table from the previous example to fill in this one?

1	1	1	1	1	1	1	
1	1	1	1				
1	1			1	1		
1	1						
	-						

Character tables are important in *representation theory*, a combination of group theory and linear algebra. We won't do any representation theory and the only linear algebra we'll do will be hidden, but familiarity with character tables makes learning representation theory later a bit easier.

Homework: None.

Prerequisites: Anti-prerequisite: if you've taken representation theory, the puzzles may be trivial.

IOL 2016. (🌶 → 🌶🌶🌶🌶, Pesto, 1 day)
Margarita and three other Mathcamp alums are currently competing at the International Linguistics Olympiad in Mysore, India. To celebrate their participation, Pesto will teach about whatever bits of linguistics are featured in this year's contest problems.

Disclaimer: Pesto has no idea what the content of this course will be or whether there'll be anything mathematical involved, since the problems won't be released until the start of week 5.

Homework: None.

Prerequisites: Who knows?

Line Graphs. (🌶🌶, Pesto, 1–2 days)
Each pilot for a certain airline regularly flies back and forth between two cities. Pilots can talk to each other if they're ever in the same city at the same time (and the airline's schedules are unreliable enough that any two pilots whose flights share a city will be able to talk to each other).

If G is the graph of cities connected by flights, then the graph of pilots connected by being able to talk to each other is called the *line graph* $L(G)$ of G.

What graphs are line graphs of some graph? What if the airline, in an efficiency measure, gets rid of all flights that could be replaced by two-flight trips?

Homework: Optional.

Prerequisites: Understand the blurb well enough to find a graph whose line graph is K_5.

Models of Computation Simpler than Programming: L, NL, coNL. (🌶🌶, Pesto, 2–3 days)
Almost all programming languages are equally powerful—anything one of them can do, they all can. We'll talk about three more weak classes of computation (called L, NL, and coNL) and how they relate to finite-state automata and context-free grammars.

This is a math class, not a programming one—we'll talk about clever proofs for what those models of computation can and can't do.
Homework: Recommended.
Prerequisites: None.

More Problem Solving: Polynomials. (🌶🌶🌶🌶, Pesto, 2–4 days)
Polynomials are a frequent topic of Olympiad-style competitions, since there are many and interesting problems using them. For instance, if $a_1, \ldots, a_n$ are distinct real numbers, find a closed-form expression for

$$\sum_{1\le i\le n} \prod_{1\le j\le n, j\ne i} \frac{a_i + a_j}{a_i - a_j}.$$

Don't see the polynomials? Come to class and find them.

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to Olympiad-style problems you will have solved as homework the previous day.

Disjoint from 2013's (but not 2014's) polynomials problem solving class. Disjoint from the week 4 polynomials problem solving class.
Homework: Required.
Prerequisites: Linear algebra: Understand what a "basis for polynomials of degree at most 2 in two variables" is.

Riley's Classes

Period Three Implies Chaos. (🌶🌶🌶–🌶🌶🌶🌶, Riley, 1–3 days)
Not just a Norwegian thrash band, "Period Three Implies Chaos" is a landmark paper in the theory of dynamical systems. (Or, if you prefer, dyn̈amical systëms).

In this class we'll prove the titular theorem and (in the 3 day version) its strictly more metal Soviet version, Sharkovskii's Theorem, which tell us when a map on the interval exhibits chaotic behaviour.
Homework: Recommended.
Prerequisites: Compactness (in some form).

Tilings and Dynamics. (🌶🌶🌶, Riley, 3–4 days)
Dynamical systems are strange and complicated objects. They often require very sophisticated tools to understand (some of which you might have learned in Jane's class).

Tilings are ways of cutting space into pieces, all of which are congruent to a small number of tiles.

It turns out tilings are weird. So weird that the best way to understand them turns out to be by turning them into dynamical systems.

In this class we'll learn things like why there are uncountably many Penrose tilings even though you couldn't distinguish any of them, and what they have to do with the Fibonacci numbers.
Homework: Recommended.
Prerequisites: The definition of a metric space, compactness in some form.

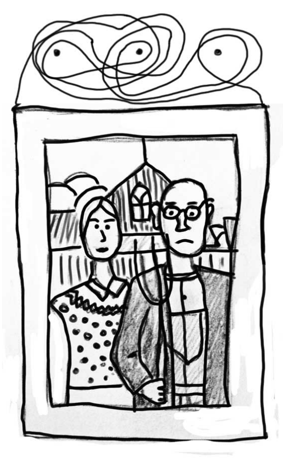
Sachi's Classes

Commutative Algebra. (🌶🌶🌶, Sachi, 3–4 days)
Commutative rings are the best rings. In fact, if you ask Nic, all rings are commutative. (Nic told me this when I was a camper.) In this class, we'll understand why commutative rings have so much more structure to them. We will prove theorems about prime ideals, maximal ideals, the nilradical, and the Jacobsson radical, and talk about the properties of being Noetherian and Artinian. This type of ring theory is what underlies algebraic geometry, as well as being a research area in and of itself, so if you want to see an introductory class in advanced algebra, this is a good fit for you.

Homework: Recommended.

Prerequisites: Ring Theory.

String Theory. (🌶🌶, Sachi, 1–2 days)
Let's say you want to hang a picture in your room, and you are worried that the 2000 fans you rented to beat the heat will blow it off the wall, so you want to hang it very securely. You start by wrapping string around some nails, like this:



You figure it will be super-duper secure, because you wrapped it around three nails a lot of times. Don, angry about the electricity bills from running all your fans, comes over and pulls one of your nails out of the wall... and your picture comes crashing to the ground! Actually, no matter which nail he removes, it falls! What happened?

In this class, you'll be playing with string to solve puzzles like this one, and we'll explore fundamental groups, homology, and monotone boolean functions.

Homework: None.

Prerequisites: None.

The Cross Ratio. (🌶🌶, Sachi, 1 day)

> "You might say it was a triumph of algebra to invent this quantity that turns out to be so valuable and could not be imagined geometrically. Or if you are a geometer at heart, you may say it is an invention of the devil and hate it all your life."
>
> Robert Hartshorne

The ratio Hartshorne is speaking of is the Cross Ratio, which is preserved under projection of lines. We will examine the properties of this evil cross ratio, see how it forces all sorts of magical coincidences

to occur, and quiz ourselves on whether we can see this quantity or whether its wholly invisible to the mathematical eye.
Homework: Optional.
Prerequisites: None.

The Stack of Triangles. (🌶🌶🌶🌶, Sachi, 2 days)
The coarse moduli space of triangles is a geometric space whose points represent isomorphism classes oft triangles. By studying this space, we can understand more generally what a moduli space is, and what goes wrong when we start taking families of triangles (or more generally, families of objects which might have symmetries.) In our quest for a geometric object which classifies triangles in a fine enough way (a fine moduli space), we will attempt to define the stack of triangles.
Homework: Optional.
Prerequisites: Point set topology and group theory.

The Yoneda Lemma. (🌶🌶🌶, Sachi, 1 day)
The Yoneda lemma is a ridiculous result, which makes a sweeping statement about all categories at once. It says that given a category C along with a functor F from C to Set, we can get this functor from a more familiar functor, namely the Hom functor.

The Hom functor is a really familiar functor: given an object a in my category, I can ask what is the set of C-morphisms from a to x. That set is called $\mathrm{Hom}(a, x)$. Then, what Yoneda says is that there is a one-to-one correspondence between natural transformations from $\mathrm{Hom}(a, -)$ to F and elements of Fa.

We'll discuss and prove the Yoneda lemma, so if you're curious about the proof, or want to see it again, this is a good class for you. At the beginning we will review the definition of natural transformation, Hom, functor, and category very quickly for people who need a refresher.
Homework: Optional.
Prerequisites: It would be very helpful to have seen the definition of category and functor already. It would also be helpful to know what a natural transformation is, though not strictly necessary. If not, maybe you know what a homotopy is?

Sam's Classes

Pattern Avoidance in Latin Squares. (🌶🌶, Sam, 1 day)
There's a small but vibrant community of combinatorialists interested in pattern avoidance. The most fundamental questions in the field ask about the number of permutations with some specific structure. We'll talk about one of the foundational results (that the Catalan numbers count the number of permutations avoiding a specific pattern), and then generalize that result to Latin squares. The class will end with a brief discussion of a whole bunch of open research questions related to pattern avoidance in Latin squares—my gut is that these are hard, but mostly because they require coming up with a new technique!
Homework: None.
Prerequisites: None.

Sam's Favorite Approximation Algorithm. (🌶🌶 → 🌶🌶🌶, Sam, 1 day)

Let's construct a graph: we'll have a vertex for each camper, and add an edge between a pair of campers that are currently paired to play each other in a chess sprinting match. We'll also allow campers to be signed up for multiple matches. A certain staff member arbitrarily decides to instead just host one major competition where the campers are divided into two teams, and to only allow matches if they involve one member from each team. (Again, a camper can play multiple games against people on the other team). That staff member likes chess sprinting, and doesn't really care if the teams are fair. Instead, he or she wants to maximize the number of matches.

That situation is a contrived example of the max cut problem: given a graph, find a subset of the vertices S so as to maximize the number of edges (u, v) with $u \in S$ and $v \notin S$. It turns out this problem is hard, but that there are a handful of beautiful fast approximation algorithms. We'll start with one fairly straightforward one, and then move onto sketching out a really cool one (it involves picking a hyperplane to divide a bunch of unit vectors, and an occurrence of arccos).

Homework: None.

Prerequisites: You should be very comfortable with taking expectations and using various properties of expectation. Other then that, you just need to know what a graph is.

The Math of (Online) Math Education. (🌶🌶, Sam, 1 day)

Have you ever tried to learn a foreign language using an app like DuoLingo? Memorize stuff using an "intelligent" flash card website? Take advantage of a forum while using a MOOC?

There's been a recent wave of online education tools, and this wave poses new questions that are inherently mathematical. How, for example, should an app optimally balance introducing new content and reviewing old content? How might that app dynamically adjust based on an user's history?

In this class we'll focus on at least one of these situations, and present one (or even two!) current mathematical models. We'll also prove at least a couple of fun combinatorial results along the way.

Fair Warning: you might find some of these models to be frustratingly simplistic. But, these are relatively new models. They (and potential improvements to them) leave several open questions!

Homework: None.

Prerequisites: None.

The St. Petersburg Paradox. (🌶, Sam, 1 day)

The St. Petersburg Paradox asks about the following game: I flip a coin, and the longer it takes until I flip the first heads, the more I pay you. If the first flip is heads, I pay you \$1. If the sequence is TH (tails followed by heads), I pay you \$2. IF it's TTH, you get \$4. This continues, and if it takes n flips until the first heads, I pay you 2^n dollars. The question: if I was standing at a carnival booth, how much should you pay to play this game?

The paradox was discovered just as probability was beginning to emerge and challenged the use of expectation as a fundamental tool. After it was discovered, a bunch of people came up with some resolutions that—to us today—might seem strange. We'll talk about some of these resolutions and go into more detail on the historical context of the paradox. (If you took my Development of Probability course last year, you should not take this class).

Homework: None.

Prerequisites: Knowing how to compute an expected value is helpful.

Turning Points in the History of Mathematics. (🌶 → 🌶🌶, Sam, 1–2 days)
This class will focus on a few "turning points" in the history of mathematics: moments where the way we think about mathematics radically shifted. For example, the invention of symbolic notation (imagine how different your experience of math would be if you couldn't use variables to, e.g., cleanly state the quadratic formula).

Depending on the number of days and interest, we'll probably focus on some subset of: rigor in Greek mathematics, the use of symbolic notation, and analytic geometry (giving the correspondence between equations and curves).
Homework: None.
Prerequisites: None.

Why Aren't We Learning This: Fun Stuff in Statistics. (🌶🌶, Sam, 1–2 days)
In this class we'll talk about some fun topics in statistics that aren't often taught in the undergraduate statistics curriculum: classification techniques and/or multiple testing.

Classification techniques are used for when you want to classify stuff (e.g., to try and predict whether or not a camper will be eaten by a bear based on factors like the classes they took). There are some surprisingly beautiful ideas that go into classifiers, and we'll talk through a few of the most robust from an intuitive perspective. Multiple testing procedures help you do statistics right when you have tons and tons of simultaneous tests (like looking at whether or not each flavor of jelly beans cures cancer, a la XKCD).

Credit goes to Vaughan for the gimmicky title. This course is entirely unrelated to my week 2 course.
Homework: Optional.
Prerequisites: Know how to compute expected values and probabilities!

Susan's Classes

Aronszajn Trees. (🌶🌶🌶, Susan, 2 days)
König's Infinity Lemma states that a tree with infinite height and finite levels must have a finite branch. But what if we step it up a cardinality? Is it necessarily the case that a tree with uncountable height and countable levels must have an uncountable branch?

The answer is no! In this class we will build something called an Aronszajn tree, which has uncountable height, countable levels, and somehow no uncountable branches.

Hope you like ordinals and posets!
Homework: Recommended.
Prerequisites: If you've never seen the ordinal numbers before, come chat with me for ten minutes.

Follow-up to Geometric Group Theory. (🌶🌶🌶, Susan, 2 days)
Hi! I'm Susan[1] and I promised my Geometric Group Theory students a follow-up in Week 5. This is that class.
Homework: Recommended.
Prerequisites: Geometric Group Theory.

[1] Actually, I am really Misha. Look again, this isn't Marty on the drums. But I'm just here to write the blurb—Susan'll be the only one teaching.

Graphs and Wang Algebras. (🌶, Susan, 1 day)
Want to see a gorgeous application of algebra to graph theory? Suppose you have an unusually large and nasty graph, and you need a list of its spanning trees. By taking something called the Wang algebra—polynomials with the vertices as variables, with $v^2 = 0$ for each vertex—we can construct a polynomial that spits out *exactly* the list of spanning trees.

Homework: None.

Prerequisites: None.

Malcev's Ring. (🌶🌶🌶, Susan, 1–2 days)
Fields are very nice rings. The multiplication is commutative, and every element has an inverse. Integral domains are fairly nice rings. The multiplication is still commutative, and there are no zero divisors. If we want multiplicative inverses, we can formally add them to our ring in a process called "localization." Noncommutative rings are... well... less nice. In particular, in 1937, Anatoly Malcev gave an example of a noncommutative ring in which there are no zero divisors, but which is not a subring of any division ring. In this class we'll see how this example works.

Homework: Recommended.

Prerequisites: Ring theory.

The Cap Set Problem. (🌶🌶🌶, Susan, 2 days)
Suppose you have a Set deck. You have four attributes: shape, color, fill, and number. Each attribute has three possibilities. The game is played by laying out twelve cards and searching for a "set," a collection of three cards in which each attribute is either all the same or all different. If there is no set out on the table, you add four additional cards. How long can this go on before you are guaranteed to have a set?

The generalization of this question is called the Cap Set Problem. In the Cap Set Problem, you have a Set deck with k attributes rather than four. How many cards do you need to have out on the table in order to be guaranteed a set?

Until recently, our best upper bound was polynomial. This on May 5 this year—less than three months ago—a paper by Croot, Lev, and Pach opened the door to a much better exponential bound. Come to this class if you would like to hear me talk in a wildly underinformed fashion about this exciting new mathematical result.

Homework: Recommended.

Prerequisites: None.

The Mathematical ABCs. (🌶, Susan, 1 day)
Let's try an experiment. Consider the following expression: $x(f)$. I don't know about you, but that gives me a sort of nails-on-a-chalkboard sensation up my spine. But suppose we tweaked that f just a little bit and wrote $x(t)$ instead. Now this makes perfect sense—it's just the x-coordinate of a parametrized curve.

As mathematicians, we sort of understand that the choices that we make in our notation are arbitrary... sort of. But there are definitely conventions and heuristics and best practices.

So! In this class we'll go through the entire alphabet, a to z, and talk about how mathematicians use these letters.

Homework: Recommended.

Prerequisites: None.

What's up with e? (🌶🌶🌶, Susan, 2–3 days)
The continued fraction expansion of e is

$$1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots$$

What's up with that? Come find out!

Homework: Recommended.

Prerequisites: Integration by parts.

Vivian's Classes

Hard Problems That Are Almost Easy. (🌶🌶, Vivian, 2 days)

Some problems are easy. Like, sorting a list of n numbers is not too bad; you can do it in time proportional to $n \log n$. And some problems are hard. For example, deciding if a graph with n vertices is k-colorable is a type of hard that is called NP-complete.

But how can you tell which ones are which? We're going to be looking at several problems that are NP-complete, but become polynomial-time solvable if you change a tiny piece of the phrasing. In addition we'll define what it means for a problem to be polynomial-time solvable, or NP-complete.

Homework: Optional.

Prerequisites: None.

Yuval's Classes

Math and Literature. (🌶, Yuval, 1–3 days)
Many Mathcampers (including me!) love reading, but we often think that reading and doing math are fundamentally different things. Though this is sometimes the case, there are many instances in which math and literature are inextricably related. In this class, we'll explore some incredible pieces of literature and discuss some of the math that went into their creation.

Note: Though I will provide some of the literature for you to read, doing so is totally optional. In particular, feel free to come even if you aren't comfortable reading in English!

Homework: Optional.

Prerequisites: None.

Probabilistic Coding Theory. (🌶🌶🌶 → 🌶🌶🌶🌶, Yuval, 2–4 days)
Did you know that one of Google's hard drives dies every minute? That means that every minute, several gigabytes of Google's data gets lost. Despite that, all of your emails and photos remain saved—in other words, Google is losing gigabytes of data every minute, without ever losing any data.

That's pretty weird. But this very weirdness makes the entire digital world possible (including the internet, cell phones, and file compression). In this class, we'll learn how this magic is done.

Also, we'll drill holes in some CDs.

Homework: Recommended.

Prerequisites: Some probability theory.

The Kakeya Conjecture over Finite Fields. (🌶🌶🌶 → 🌶🌶🌶🌶, Yuval, 2 days)
Suppose we have a set in $\mathbb{R}^n$ that contains a unit line segment in every direction. How small can it be? This turns out to be a really hard question, and only a few (extremely surprising) results are known.

Because of that, we won't think about this question—it's way too hard. Instead, we'll answer the discrete analogue of that question, which turns out to be much easier. We'll also learn about the so-called "polynomial method," an extremely powerful proof technique that uses only facts you learned in middle school.
Homework: Optional.
Prerequisites: Some linear algebra and ring theory will be helpful, but not necessary.

Zach's Classes

Circle Packing and Polyhedral Canonical Form. (, Zach, 1 day)
From non-overlapping disks in the plane we can define their *tangency graph*, with one node per disk and an edge whenever two disks are tangent. The field of circle packing arises from the beautiful observation that this correspondence goes both ways: any planar graph can be built as the tangency graph of a set of disks!

We'll prove this fact, and use it to prove that any convex polyhedron can be modified into a very strong *canonical form*—specifically, as a polyhedron P having planar faces with all edges tangent to the unit sphere. Additionally, the dual polyhedron P^*, also in canonical form, can be configured so that corresponding edges of P and P^* meet on the sphere at right angles.

If there's time, we'll sketch some deep connections between circle packings and complex analysis, conformal maps, and hyperbolic geometry.
Homework: None.
Prerequisites: None.

Permuting Conditionally Convergent Series. (→ , Zach, 2–3 days)
The startng point for this class is the well-known Riemann's Rearrangement Theorem: given any conditionally convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers—that is, the sum converges but $\sum_{n=1}^{\infty} |a_n|$ does not—you can permute the terms so that the new sum converges to *any* desired value. But what if the a_n are instead allowed to be *complex*? Can you permute to obtain any *complex* sum? Not always: for the sequence $a_n = (-1)^n \cdot \frac{i}{n}$, the achievable sums lie on the line $i \cdot \mathbb{R} \subset \mathbb{C}$. What other subsets of $\mathbb{C}$ arise in this way? And what about series in $\mathbb{R}^k$? We'll discuss and prove this beautiful classification.

We'll also consider the problem from the permutations' perspective: what are the permutations on $\mathbb{N}$ that always turn convergent series into different convergent series (not necessarily with the same sum)? Amazingly, there are some permutations that do better than $\mathbb{N}$ itself: they turn convergent series into convergent series, but they also transform some *divergent* series into convergent ones!
Homework: Optional.
Prerequisites: Knowledge of the epsilon-delta definition of limits.

Stably Surrounding Spheres with Smallest String Size. (, Zach, 1 day)

> Once at Mathcamp's dinner table, while I pondered how much cable
> I would need to tie some netting tightly 'round my unit orb,
> While the sphere I thus tried trapping, suddenly a bound came, capping
> How much string is needed wrapping, wrapping round my precious orb.
> Merely 3π is required to accomplish such a chore!
> Only this? Well, just `\eps` more.

Ah, distinctly I remember, how I then wished to dismember,
Enough of the links that emb(e)raced that rare and radiant orb.
Seeking to dislodge my prize, I quickly saw, to my surprise, by
Choosing k strands to incise, my sphere from net could not be torn!
How much—how much length is needed?—tell me—tell me, I implore!
$(2k + 3)\pi$, and more.

Homework: None.
Prerequisites: None.

Summation by Parts on the Battlefield. (🌶🌶🌶, Zach, 1 day)
Integration by parts provides a handy method to integrate a product $f(x) \cdot g(x)$ by instead considering the integral of $f'(x) \cdot G(x)$, where f' is the derivative of f and G is an antiderivative of g. Its discrete analog, known as “Summation by Parts” or “Abel Summation”[2], deals instead with differences and partial sums, and it is a simple but unusually powerful weapon to have in your arsenal—especially, perhaps, when doing battle with some olympiad problems.

In this talk you will receive basic weapons training, learning how to use this innocuous-seeming hand grenade to blow some enemies (enemy sums?) into, well, parts. You will also learn to recognize when such a maneuver could be add-vantageous. Finally, we will apply this training in simulated combat against incomplete units of Egyptian factions—er, fractions.
Homework: Optional.
Prerequisites: None.

Twisty Puzzles are Easy. (🌶🌶 → 🌶🌶🌶, Zach, 1–2 days)
Algorithmically speaking, the Rubik’s Cube and many of its more complicated cousins are easy to solve: they can be expressed in terms of permutation groups acting on the pieces, and they therefore fall to an elegant and efficient algorithm due to Schreier and Sims. This algorithm is well-suited for many types of queries regarding permutation groups, such as testing group membership (can you turn a single corner in a $3 \times 3 \times 3$ cube? Swap two edges in a $4 \times 4 \times 4$ cube?), instantly computing the size of the group (there are precisely $43,252,003,274,489,856,000$ possible $3 \times 3 \times 3$ Rubik’s cube positions, and you can bet it didn’t enumerate them all individually!), and many others.

This is *not* a course on how to solve a Rubik’s Cube. We will deal more generally with computations in *any* permutation group (or group action), and the resulting algorithm can describe the group generated by *any* desired set of basic permutations. We will see how this can be applied to many types of puzzles, and we will also discuss some limitations of the approach.
Homework: Optional.
Prerequisites: Group Theory.

[2]Yes, it is really called “Abel Summation”. I did not just name it after myself. Some Norwegian mathematician named it after me instead…

Co-taught Classes

The Stable Marriage Problem. (, Alfonso + Marisa, 1 day)
N single men and N single women want to pair up and get married. These are their names and preferences:

- Alfonso: Marisa > Susan > Jane > Vivian > Elizabeth
- David: Elizabeth > Marisa > Vivian > Susan > Jane
- Kevin: Elizabeth > Vivian > Susan > Marisa > Jane
- Nic: Jane > Susan > Vivian > Elizabeth > Marisa
- Pesto: Jane > Elizabeth > Marisa > Vivian > Susan

- Elizabeth: Alfonso > Nic > Pesto > Kevin > David
- Jane: Kevin > David > Alfonso > Pesto > Nic
- Marisa: Kevin > Nic > Pesto > David > Alfonso
- Susan: Kevin > Pesto > Alfonso > David > Nic
- Vivian: Kevin > Nic > David > Pesto > Alfonso

Is it possible to make everybody happy? Obviously not since almost everybody wants to marry Kevin. But is it possible to at least create a stable situation? For instance, it is a bad idea for Kevin to marry Susan and for Vivian to marry Nic, because then Kevin and Vivian would prefer each other rather than staying with their partners, so they will run away together. How can we at least avoid having a run-away couple? Is there more than one way to do it? What is the best way to do it? And what if Susan and Vivian decide that marrying each other is better than marrying Kevin?

Homework: None.

Prerequisites: None.

Distributions and Fourier Transforms. (, Jeff + Nic, 4 days)
Imagine you're listening to a piece of music. When you hear the music, your brain doesn't interpret it as a wave of different air densities; instead it breaks the sound down into different frequency components, and processes those instead. This is also true with our vision: we don't see an oscillating amplitude of electromagnetic radiation. Rather, our eyes can transform the frequency data of light into three channels, which we experience as different colors.

The process of taking a function of time and interpreting its frequency information can be modeled mathematically by the *Fourier Transform.* It's not just a cool mathematical trick to taking some function and writing it down in a different way—it's a mathematically useful tool for solving differential equations, providing intuition in quantum mechanics, and understanding how physical processes like amplifiers work.

Unfortunately, most of the time in the real world, you are interested in functions whose Fourier transform cannot be modeled as a function. In fact, functions do not provide a good model for many physical (and mathematical) objects we want to describe:

- You may want to take the derivative of the step function
$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$
- You may want a function which is zero everywhere but at a single point, but whose integral is non-zero.
- A function f which describes the charge density of an electron, or the charge density of the dipole moment.
- You may be interested in a function which describes the frequencies of the sawtooth wave.

Unfortunately, you won't find any functions which fit the properties above, even though you may have some intuition of what these functions look like.

For about 100 years, mathematicians, physicists and engineers were content to use their intuition to understand these objects as some kind of "generalized function." In the mid-20th century, a group of analysts formalized these ideas into the notion of a *distribution*. Distributions have many of the same intuitive properties as functions (in fact, all smooth functions are distributions) but include some of the generalized notions of functions above.

In this class we'll give an introduction to Fourier analysis, and see some of the places where it suggests that we should generalize the notion of function. We'll use our intuition that we have from these tools to build up the theory of distributions, and apply these two tools to describe some physical phenomenon rigorously in a way that isn't possible by using ordinary calculus.

Homework: Recommended.

Prerequisites: None.

Graph Polynomials. (🌶🌶, Jeff + Mia, 2 days)

Are you one of millions of people who have a problematic graph G in your life? Do you lie awake and wonder:

- How will I count the number of spanning forests in G?
- What is the probability that G remains connected after randomly removing each edge based on a coin flip?
- How many ways are there to color G with my 256-color box of Crayola crayons?

Well, then we have a degree $|E| + |V|$ polynomial for you!

This 1954 masterpiece, defined by Tutte, computes exactly the properties that can be determined by edge contraction and deletion. No extra estimation or calculation needed. This *one* polynomial will solve all of your problems above (and many more). 100% recursively defined. Factorizable over connected components. Made in Canada.

If you order today, well even throw in two specialized polynomials, free of charge!

Homework: Optional.

Prerequisites: Graph theory.

Morse Homology. (🌶🌶🌶🌶, Jeff + Chris, 4 days)

So you learned what homology is? You even tried calculating it? Was it hard? Probably too hard? Do you wish there was an easier way of doing this whole thing? I mean, most of the time we're working with nice, beautiful spaces anyways. Shouldn't we be able to use that?

Stop wasting your time now. We, the disciples of Morse, have the answer! Not only can we define a homology that is easy to compute *and* easy to work with, we can even do all of this without appealing to any of those old hedonistic methods.

Homework: Recommended.

Prerequisites: Calculus, group theory (kernel, image, quotients), linear algebra (matrices and linear maps).