CLASS DESCRIPTIONS—WEEK 5, MATHCAMP 2017

Contents

9:10 Classes

Cap-Set Problem: Recent Progress. (🌶🌶🌶, Paweł Burzyński, Thursday–Friday)
Have you ever played the card game SET? If not, run to the game lounge and play it. The picture on every card has four different features (number, shape, color, filling), each of which can take three values. We call three cards a *SET* when they are all different or all the same in each feature. The goal of the game is to find a SET among 12 cards on the table. As it turns out, it may not always be possible. One may wonder: "How many cards without a SET can you have on the table?"

Figure 1. All 3 cards have different shapes, colors, and numbers and the same shading. Hence they form a SET.

A mathematician can imagine a SET card as a point in $\mathbb{F}_3^n$, n-dimensional space over $\{0, 1, 2\}$, for $n = 4$. Three points form a SET if they are collinear. The Cap-Set Problem asks, "What is the maximum number of points in $\mathbb{F}_3^n$ without a collinear triple?". For more than 30 years in the history of analytic combinatorics, nobody could even tell if the answer is of much smaller magnitude than 3^n. However, in 2016, Ellenberg and Gijswijt in their groundbreaking paper showed that indeed it is. What is more, in contrast to the previous Fourier Analysis approaches, they use only elementary methods.

In this class, I will show you the whole argument that every subset of $\mathbb{F}_3^n$ without a collinear triple is exponentially smaller than 3^n. The basic idea of this approach is a smart use of polynomials and ranks of matrices.

Homework: Optional

Prerequisites: Linear algebra (familiarity with vector space and its dimension and matrix and its rank). Understanding the meaning of $P(x)$, where P is a polynomial in n variables and x is a point in F_q^n.

Choosing Random Numbers. (🌶️, Misha, Wednesday)
If you want to pick a random number, but you don't care how, you might as well just roll a die. If you *do* care how, you had better do some math first. We'll do a few random things in this class, but the main goal is to see a very cute algorithm for doing what seems like magic.

You may know that factoring a 1000-digit number would take a computer thousands of years. But in this class, I will teach you how to factor a randomly chosen 1000-digit number in just a few minutes on a computer.

Homework: None

Prerequisites: Willingness to take on faith a few of my arbitrary claims about the values of infinite sums and products.

How Not To Prove That a Group Isn't Sofic. (🌶️🌶️🌶️🌶️, Viv, Thursday–Friday)
Cayley's Theorem tells us that finite groups are all subgroups of finite permutation groups. A sofic group is a possibly-infinite group that we sort of maybe want to have the same property roughly speaking. The group can't always be a subgroup of a finite permutation group, so instead we just require that all finite subsets of our sofic group act kind of like finite subsets of permutation groups. This ends up being a super-useful definition, but we're left with a burning question: do non-sofic groups exist? We don't actually know the answer to this one. We'll spend the class talking about a hopeful candidate for a non-sofic group and one great way not to prove that it isn't sofic.

Homework: Recommended

Prerequisites: Group Theory

Simplicity Itself: A_n and the "Other" A_n. (🌶️🌶️–🌶️🌶️🌶️, Mark, Thursday–Friday)
The monster group (of order roughly $8 \cdot 10^{53}$) gets a lot of "press", but it's not the largest finite simple group; it's the largest *exceptional* finite simple group. (Reminder: A simple group is one which has no normal subgroups other than the two "trivial" ones; by using homomorphisms, all finite groups can be "built up" from finite simple groups. The complete classification of finite simple groups was a monumental effort that was completed successfully not far into our new millennium.)

What about the unexceptional finite simple groups? They come in infinite families, and in this class we'll look in some detail at two of those families: the alternating groups A_n and one class of groups of "Lie type", related to matrices over finite fields. (If you haven't seen finite fields, think "integers mod p" for a prime p.) By the way, the simplicity of the alternating groups plays a crucial role in the proof that in general, polynomial equations of degree 5 and up cannot be solved by radicals (there is no "quintic formula").

We'll prove that A_n is indeed simple for $n \geq 5$, and we should be able to prove simplicity for the other class of groups also, at least for 2×2 matrices.

Homework: None

Prerequisites: Basic group theory and linear algebra; familiarity with finite fields would be helpful, but not really necessary.

Solving Cubic and Quartic Equations Without the Mess. (🌶️🌶️–🌶️🌶️🌶️, Aaron, Wednesday)
Probably you are well acquainted with the quadratic formula. It is similarly possible to solve cubic and quartic equations in terms of radicals, but unfortunately, these formulas are incredibly messy.

How might you come up with these formulas yourself? In this class, we'll explain how to solve cubic and quartic equations using radicals via pure thought, without any messy equations.

The method I'll describe is heavily motivated by Galois theory, though you don't need to know any Galois theory to appreciate it.

Homework: None

Prerequisites: None.

Surreal Numbers. (🌶, Jalex, Wednesday)
Typical constructions of the reals go like this:

- Start with 0.
- Put in 1.
- Use addition to get $\mathbb{N}$
- Use subtraction to get $\mathbb{Z}$.
- Put in fractions to get $\mathbb{Q}$.
- Complete the Cauchy sequences to get $\mathbb{R}$.

That's a big number of steps! In this class, we'll start from the following definition:

- A Number is an ordered pair (L — R) where L and R are sets of numbers, and no number in L is bigger than any number in R,

and get not only the reals, but a much richer extension that has a proper subfield of every cardinality.

Homework: Recommended
Prerequisites: None.

The Baire Category Theorem. (🌶🌶, Lara, Thursday–Friday)
In this class we'll prove the Baire category theorem, which tells us that a countable intersection of dense open sets is dense and explore some of its amazing consequences. Examples of these are:

(1) Not only are the irrationals uncountable, but they can't even be written as the countable union of closed sets.
(2) There exists functions continuous at all the rationals but discontinuous at all the irrationals, but not vice versa.
(3) Any infinitely differentiable function on $[0, 1]$ that has some derivative vanish at each point of the interval must be a polynomial.

Homework: None
Prerequisites: You should know what a metric space is and be comfortable with the idea of open sets and closed sets and what the interior and closure of sets are.

The Kakeya Maximal Conjecture. (🌶🌶🌶🌶, Yuval, Wednesday)
Do you like the Kakeya Conjecture, but feel that it's too plain? If so, you might be interested in taking it… to the *max*.

In this class, we'll learn about the Kakeya Maximal Conjecture, which is an important strengthening of the Kakeya Conjecture. Much of the recent progress on the Kakeya Conjecture (including the proof of the conjecture for the case $n = 2$ that we saw on the last day of class in Week 1) has actually been progress on the Kakeya Maximal Conjecture.

In this class, we'll learn what the Maximal Conjecture is, and why it's such a useful framework for proving Kakeya-type results.
Homework: Optional
Prerequisites: The Kakeya Conjecture

10:10 Classes

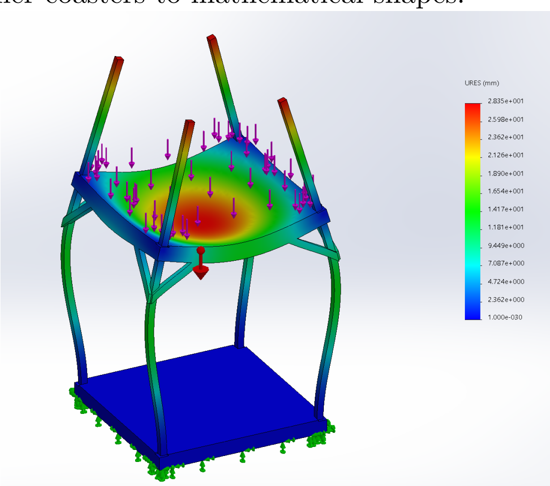
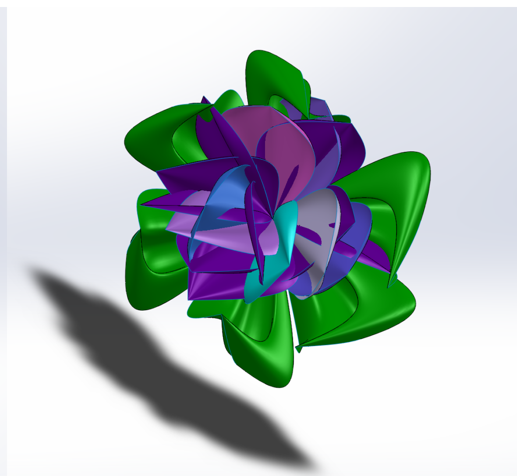
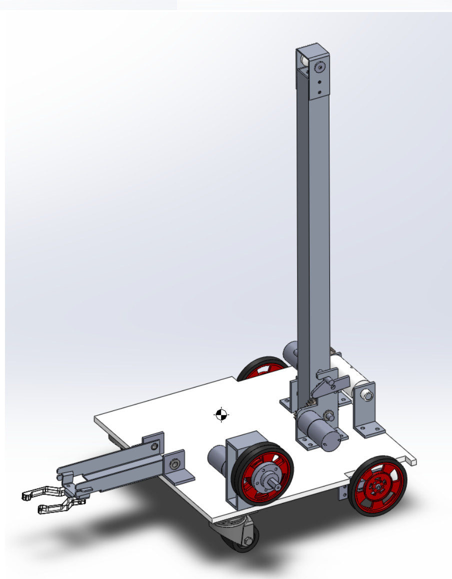
And That Is Why Birds Shouldn't Drink Alcohol. (🌶🌶, Beatriz, Thursday–Friday)
In the words of Kakutani, "A drunk man will eventually find his way home, but a drunk bird may get

lost forever." In this class we will study random walks on a d-dimensional lattice $\mathbb{Z}^d$. We will prove that for $d \leq 2$, the random walk on $\mathbb{Z}^d$ is recurrent (meaning the walker returns to its starting point with probability one), yet for $d \geq 3$, it is transient (meaning there is a positive probability that the walker may not return to its starting point).

Homework: None

Prerequisites: Series

Computer Aided Design. (🌶, Elizabeth, Tuesday–Wednesday)
Computers are awesome! They can do so many cool things! In particular, if you can imagine some shape or machine, you can make a computer draw it in 3D. Once the computer knows what it is, then it can show you what it would like from any angle, and you can tweak it without having to redraw the whole thing. You can also turn it colors and zoom in on small details. Basically, anything you can do in your head, you can show to other people, with the computer. In addition, once you have told the computer about it, the computer can print out pictures or files that let machinists or machines make the part in real life. Computer aided design is useful for all kinds of things, from making robots to roller coasters to mathematical shapes.

If you took this class last year, days $n > 1$ will still be interesting, because they will be free time to work on CAD, but day 1 will probably not be, because it will be roughly the same tutorial as last year.
Homework: Optional
Prerequisites: None.

Galois Theory Crash Course. (🌶🌶🌶🌶, Mark, Tuesday–Friday)
In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully.

Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later). If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we just *might* be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class but which may be shown in a separate week 5 class.) Even if we don't get this far, the so-called Galois correspondence (which we should be able to get to, and prove) is well worth seeing!
Homework: Optional
Prerequisites: Group theory; linear algebra; some familiarity with fields and with polynomial rings.

Laws of Large Numbers. (🌶🌶, Lara, Tuesday–Wednesday)
In this class, for $X_1, \ldots, X_n$ independent and identically distributed random variables, we'll prove the convergence of expressions like $\frac{X_1+\cdots+X_n}{n}$ in different senses. We'll use these results to answer a few real-world problems such as:

(1) How long do we expect it to take us to collect an entire set of stamps if we acquire one stamp at random per time interval?
(2) How much should we pay to play a game where we win 2^j dollars if we get $j-1$ consecutive tails and it takes 2^j coin tosses to get heads for the first time?

Homework: None
Prerequisites: You should know what a random variable is, what it means to integrate a random variable and what it means for random variables to be independent.

Machine Learning (NO Neural Networks). (🌶🌶🌶, Linus, Thursday–Friday)
Emphasis on theory over practice. Possible topics:

- Learning a linear classifier (with noise)
- Models of learning: PAC, Mistake Bound, Bayesian
- VC dimension (a combinatorial property: "simple", "easy to learn" concepts tend to have low VC dimension)

Homework: Recommended
Prerequisites: Thinking in n-dimensional space
Related to (but not required for): Not Your Grandparents' Algorithms Class (W1)

Queueing Theory. (🌶🌶🌶, Misha, Tuesday–Wednesday)
We will look at continuous versions of Markov chains and their applications to queueing theory: the study of how long you will wait in line.
Homework: Recommended
Prerequisites: None—in particular, my class on Markov chains is *not* a prerequisite.

What is Homology? (🌶🌶🌶🌶, Apurva, Thursday–Friday)
Enough said.
Homework: None
Prerequisites: Linear algebra, and when I inadvertently utter the phrase 'topological space' in class you should be happy and not sad.

11:10 Classes

Algorithms to Generate Randomish Numbers. (🌶🌶, Sam, Wednesday)
Suppose you wanted to generate a truly random sequence of numbers. You sit down to start flipping a coin for forever, but shortly realize that this is boring. Instead, suppose you wanted to generate a sequence of numbers that looked random (so that you could fool your friend Tim!, say, into thinking that you were playing rock-paper-scissors randomly). How might you do this?

In this class, we'll introduce some of the methods for generating pseudorandom numbers. We'll also look at why some of these were terrible!
Homework: None
Prerequisites: Basics of modular arithmetic

Every Natural is "Fibonacci": Games, Miles and Kilometres. (🌶, Beatriz, Wednesday)
In this class we will prove Zeckendorf's theorem which states that every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This theorem has many very interesting application. In this class we will see how Fibonacci numbers can help us win the following game:

There is one pile of n stones. The first player may remove as many as they like on their first turn as long as they remove at least one and leave at least one. From then on, the next player may remove no more than their opponent did on their previous turn. The player who takes the last coin wins.

Also, because Mathcamp is so international, and I want to help you understand each other, we'll learn how to use Fibonacci numbers to easily convert from miles to kilometres and vice versa.
Homework: None
Prerequisites: None

Exclusion-Inclusion. (🌶🌶, *Po-Shen Loh*, Tuesday)
We all know that $|A \cup B| = |A| + |B| - |A \cap B|$, and there's some generalization of this to n sets. So we will not discuss that at all, and talk about other things, like spheres and the exclusive-or operator. We will discover that lots of ideas are connected.
Homework: None
Prerequisites: $|A \cup B| = |A| + |B| - |A \cap B|$

How to "Divide". (🌶🌶🌶, Don, Thursday–Friday)
Adding two sets together is pretty easy - we just take their union, with an extra copy of anything they share.

If $A + B$ is the same size as $A + C$, and A is finite, it's possible, but not obvious, to show that B is the same size as C — that is, to subtract!

Multiplying two sets is also pretty easy — we take their cartesian product.

But, how do we divide? It turns out, entirely without a need for the axiom of choice, as long as we're clever enough. Instead, we get by with one secret ingredient — love[1].

Homework: None

Prerequisites: None

Intersecting Polynomials. (🌶🌶, Tim!, Friday)
You might think that everything there is to know about one-variable real polynomials has been known for hundreds of years. Except, in 2009, while bored at a faculty meeting, Kontsevich scribbled down a brand new fact about polynomials. You'll discover it.

Homework: None

Prerequisites: None.

Mathematics in Crisis: The St. Petersburg Paradox. (🌶🌶, Sam, Tuesday)
By the early 18th century, probability was starting to develop as a field of its own right. Core to this development was the concept of *expectation* (expected value). The hope was that expectation would become the tool for making rational decisions: the decisions a mathematician would make. In 1713, however, Nicolas Bernoulli came along and proposed what is now known as the St. Petersburg paradox: a game that no mathematician would pay even $100 to play, but for which mathematics dictated that anyone should jump at the chance to pay any amount of money to play. Many of the initial resolutions to this paradox (posed by serious mathematicians) might seem crazy today. For example, an event having different **mathematical** and **moral** probabilities.

In this class, we'll briefly trace out the early history of probability, leading up to the crisis that was the St. Petersburg Paradox. We'll then spend the rest of the class talking about the historical resolutions proposed.

Homework: None

Prerequisites: You should know what expected value is.

My Favorite Magic Trick. (🌶🌶, Don, Tuesday)
I've always liked magic, especially mathematical magic — a good mathematical magic trick is essentially a live-action puzzle, with particularly good flavortext. There are a number of tricks that I hold dear to my heart, but this class is about my favorite one: the one where you do all the work. Come perform this trick, and then figure out how it works!

Homework: Optional

Prerequisites: None

Non-Classical Logic. (🌶🌶, Joshua Schluter, Wednesday)
Mathematics has this long-standing tradition of creating new systems. When the reals weren't enough, we invented the complex numbers. When geometry was too strict, we invented topology. But what happens if we want to change the very laws of logic? In this class, we'll do just that.

For the most part, we'll stick to a type of logics called modal logics. These logics can help us make sense of statements like "Misha must be the crowman," among other statements. We might also touch on other interesting types of logics near the end of class.

Homework: None

[1]Where by love, I may or may not mean topological cobordism theory

Prerequisites: No Prior knowledge necessary.

Penrose Tilings. (🌶🌶, Steve, Tuesday)
It's easy to tile the plane with squares; it's also pretty boring. More complicated tiles lead to more interesting tilings—tiling with hexagons is pretty neat—but of particular interest are those tilings where *every part looks unique*: although they tile the plane with few shapes, the resulting tiling has few or no symmetries at all. It is not obvious that (interesting) tilings of this type exist, but they do. One particularly famous one is the *Penrose tiling*, which is a tiling (or rather, class of tilings) with very little actual symmetry but lots of tantalizing structure; if you haven't seen this before, it's worth googling (it's really quite pretty). We'll look at a couple simple examples of "symmetry-less" tilings, and then dive into a mathematical explanation of the Penrose tilings and their various interpretations. Come for the tilings, stay for the projections of five-dimensional cubes!

Homework: Recommended

Prerequisites: None

Problem Solving: Linear Algebra. (🌶🌶🌶, Misha, Thursday–Friday)
Most high school math contests (the IMO included) do not use any topic considered to be too advanced for high school, such as linear algebra. This is a shame, because there have been many beautiful problems about linear algebra in undergraduate contests such as the Putnam Math Competition.

In this class, we will look at linear algebra from a new perspective and use it to solve olympiad problems.

Homework: Recommended

Prerequisites: Linear algebra. In particular, you should know what eigenvalues are.

Problem Solving: the "Just Do It" method. (🌶🌶, Linus, Wednesday)
Example Problem: Is there a sequence of integers $a_1, a_2, a_3, \ldots$ which contains every integer exactly once, where the sequence of differences $a_i - a_{i+1}$ also contains every integer exactly once?

The "Just Do It" method turns some frightening-looking combinatorics problems, like this one, into jokes.

Homework: Recommended

Prerequisites: None.

Smol Results on the Möbius Function. (🌶🌶, Karen Ge, Thursday)
Our goal is to give an overview of classical analytic number theory techniques. In particular, we'll discuss the Mobius function in detail. The Mobius function appears in many unexpected places, such as in connection to the Riemann zeta function and the roots of unity. There are various open problems about its "randomness." It is also a useful tool for extracting values from certain types of summations and for analyzing other arithmetic functions. We will define many fundamental concepts about multiplicative functions and see how to use them to prove cute (smol) facts such as "the Mobius function can be expressed as the sum of the primitive roots of unity."

Homework: Optional

Prerequisites: None.

Topological groups. (🌶🌶🌶🌶, Aaron, Thursday–Friday)
Like groups? Like topology? Then you'll love topological groups! An topological group is just a topological space which is also a group.

We'll prove some neat facts about topological groups, such as showing that open subgroups are closed, finite index closed subgroups are open.

Homework: Recommended

Prerequisites: Group theory

1:10 Classes

Calculus Without Calculus. (🌶🌶, Tim!, Thursday–Friday)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Angela is 5 cubits tall and Misha is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Angela's head to the top of Misha's head that touches the ground in the middle. What is the shortest length of string you can use?
- Apurva rides a bike around an ellipical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters way along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Optional

Prerequisites: We won't use calculus (that's the point), but it would be good if you've seen it for context.

Fast Matrix Multiplication. (🌶🌶, Yuval, Wednesday)
Some of my favorite algorithmic questions are related to a task that might seem pretty uninteresting, namely that of multiplying together two big matrices. In this class we will see how you can multiply together matrices way faster than you might have expected. Additionally, we'll talk about a big open problem in the field, which basically says that multiplying together two matrices is no harder than just looking at them. Finally, we'll talk a bit about how people are trying to prove this conjecture, and about some recent theorems that say all such attempts simply can't work.

Homework: None

Prerequisites: Know how matrix multiplication works! Also, some small background with algorithms will be helpful, but not necessary.

Gödel's Incompleteness Theorem. (🌶🌶🌶, Steve, Tuesday–Friday)
In 1931, Kurt Gödel proved that there are true sentences of arithmetic which cannot be proven from the standard axioms for arithmetic; moreover, he proved that *no reasonable system* of axioms would

be free from this problem! This is totally wild: what exactly does it mean, and how on earth would one go about demonstrating it?

In this class we'll prove Gödel's Incompleteness Theorem, and talk about what it does—and does not—mean. Historically, this was almost the beginning of modern logic, so there's also some great stories from the time during which it was proved and we might talk about those too.

Homework: Recommended

Prerequisites: Comfort with formal proof, especially induction, and a little group, ring, or field theory.

Impossible Cohomology. (🌶🌶, Don, Wednesday)

Cohomology is a powerful tool, used to describe underlying properties of mathematical objects from topological spaces to groups and rings. It ultimately measures how a big thing is more complicated than the sum of its parts.

You may recognize this figure:

Each of the parts seems like a normal geometric object, but the way they fit together is wrong. With the power of cohomology, we'll figure out exactly how wrong!

Homework: Optional

Prerequisites: None

Problem Solving: Convexity. (🌶🌶🌶, Misha, Tuesday)

You see lots and lots of olympiad problems about inequalities. These are often solved by applying obscure theorems due to Scottish mathematicians.

In the real world, mathematicians study inequalities too, but their approach is different. Inequalities that arise from convex functions are much more important.

So in this class, we will take the best of both worlds and use convexity to solve olympiad math problems.

Homework: Optional

Prerequisites: None.

Quadratic Reciprocity. (🌶🌶🌶, Mark, Tuesday–Wednesday)

Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is q a square modulo p?"

(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Homework: Optional

Prerequisites: Some basic number theory (if you know Fermat's Little Theorem, you should be OK)

Quandles. (🌶🌶, Susan, Friday)
So… you know associativity? That thing that makes the operations in groups and rings at least reasonably nice? Well I say, phooey to associativity! Who needs it?

In this class we'll learn about a weird-looking nonassociative algebraic structure called a Quandle. We'll talk about how quandles behave, how they misbehave, and how they're related to knot theory. If you want just a little bit more brain-bending algebra before the end of camp, then this is the class for you.

Homework: None

Prerequisites: None.

The Stable Marriage Problem. (🌶, Beatriz, Thursday)
Imagine we run a dating agency, and we must match n men with n women. We ask each man to rank the women in order of preference; similarly, each woman is asked to rank the men. Is there an algorithm to guarantee the best possible matches? In fact, to attract more clients, our agency offers one million dollars to those whose matched partner leaves them for another client; can we build an algorithm that ensures we will never have to pay up? We'll also learn why it's not a good idea for women to wait for a man to propose. Can our algorithm be adapted to solve other similar problems, such as the roommate problem?

Homework: None

Prerequisites: None

The Szemerédi Regularity Lemma. (🌶🌶🌶🌶, Yuval, Thursday–Friday)
The Szemerédi Regularity Lemma is one of the deepest and most powerful tools in combinatorics. Roughly speaking, it tells us that all "big" graphs look "the same", and that we can actually more or less forget about where the edges of our graphs are. This may sound stupid, but its power cannot be overstated.

In this class, we'd learn the Regularity Lemma and see some of its applications in number theory, for which it was originally developed. By the end of the class, you'll have some ideas that go into the famous (and famously difficult) Green-Tao Theorem on arithmetic progressions in the primes.

Homework: Optional

Prerequisites: Intro to Graph Theory

Welzl's Theorem on Graph Homomorphisms. (🌶🌶, Jalex, Tuesday)
A *graph homomorphism* $f : G \to K$ is a function from the vertex set of G to the vertex set of K such that if (x, y) is an edge of G, then $(f(x), f(y))$ is an edge of K. Say that $G < K$ if there exists a homomorphism from G to K but there does not exist a homomorphism from K to G. In this class, we'll prove a theorem of Welzl: The partially order set of finite graphs with $<$ is dense. In other words, if $G < K$, then we can always find H such that $G < H < K$.

Homework: None

Prerequisites: Know what a graph is. Know what a function is.

2:10 Classes

How to Build a Calculator. (🌶, Lotta, Tuesday)
I have a little box with buttons on it that's probably made of magic. When I push buttons like '2,' '3,' '4,' '+,' '3,' '4,' it returns '268.' How does it know? Probably there is a little person inside doing arithmetic by hand.

In this class, we will talk about circuits and gates and how you can put gates together to build an adder or anything you want.

Homework: Optional

Prerequisites: None

Many Clubs Share People. (🌶🌶, Aaron, Wednesday)
I'll explain and discuss the classic problems of oddtown and even town. In oddtown, there are n people. These people form clubs where each club has an odd number of members, and for any pair of clubs, the number of people they have in common is n. How many clubs can there be? In eventown, there are n people. These people form distinct clubs where each club has an even number of members, and any pair of clubs share an even number of members. How many clubs can there be? The answers to these two questions are surprisingly different, but both involve linear algebra over the finite field with two elements.

Homework: Recommended

Prerequisites: Linear Algebra

I click "Random Page" in OpenProblemGarden.org until we get one we understand: then we spend 50 minutes trying to solve it. (🌶 → 🌶🌶🌶🌶, Linus, Thursday)
See title.

Homework: None

Prerequisites: Probably combinatorics.

How Not to Solve Tricky Integrals. (🌶🌶, Steve, Friday)
The function e^{-x^2} doesn't have an elementary antiderivative, but the definite ("Gaussian") integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

can be solved in a variety of ways. One quite nice approach (due to Poisson) is via change of coordinates. It's a very easy trick, and it's natural to ask what other problems we could solve this way.

It turns out the answer is—*absolutely none*! In a precise sense, Poisson's trick only works once. Useful knowledge is overrated! Come see how to solve the Gaussian integral, and how not to solve tricky integrals in general!

If we have time, we'll see a broad generalization of Poisson's trick. This still doesn't work anywhere else, but it's really fancy.

Homework: Recommended

Prerequisites: Multivariable calculus (specifically, change of variable in multiple integrals)

Crash Course on Representation Theory. (🌶🌶🌶🌶, Apurva, Tuesday–Wednesday)
One way to study groups is by making them do things to vector spaces and see what happens. Like every other useful thing in algebra this is inappropriately named as Representation theory.

For finite groups, study it we shall.

Homework: None

Prerequisites: Group theory, Linear algebra.

Finite "Sets". (🌶🌶🌶, Don, Thursday–Friday)
What makes something finite? Is it that it has finitely many elements? What if it doesn't, techncially speaking, have any elements?

Come learn how Category Theory can tell us what properties really make an object finite - and then see some objects that, despite having infinitely many elements, have put in the hard work to truly earn the title of Finite.

Homework: Optional

Prerequisites: None

Permutation Statistics. (🌶🌶, Kevin, Tuesday–Friday)
Here's a classic conundrum. The first passenger to board a full flight can't remember what seat to sit in and picks a random seat. Every passenger thereafter sits in the correct seat if available, or a random seat otherwise. You're the last passenger to board. What's the probability you end up in the correct seat?

Here's a more difficult one. Suppose you and 99 of your best "friends" are incarcerated by Don for being too janky, and Don offers to release you if you meet the following challenge. Don secretly places each of your names in 100 different boxes then lays the boxes out in a row in his office. One by one, you and your 99 friends can enter Don's office and peek inside 50 boxes, but you can't tell your friends what you see. If you and all 99 of your friends each manage to find your own name, you'll be released!

Everyone individually has a 1/2 chance of succeeding; if you pick randomly and independently, then there's a vanishingly small chance that you all win the game. But you can decide a strategy ahead of time so that you actually have a decent shot of being released!

In this class, we'll use generating functions to study permutation statistics, things like average number of cycles in a permutation, chances of two numbers being in the same cycle, and maybe some more exotic properties of permutations as well, and we'll see how they answer these questions.

Homework: Optional

Prerequisites: Generating functions are useful to have seen, but I can help catch you up if necessary.

Related to (but not required for): Generating Functions and Regular Expressions (W1)

What's The Deal With e? (🌶🌶🌶, Susan, Tuesday–Wednesday)
The continued fraction expansion of e is

$$1+\cfrac{1}{0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{\ddots}}}}}}}}}}}}}}$$

Okay, but seriously, though, why?!?! Turns out we can find a simple, beautiful answer if we're willing to do a little integration. Or maybe a bit more than a little? Come ready to get your hands dirty—it's gonna be a good time!

Homework: Recommended

Prerequisites: Familiarity with integration by parts, and basic partial fractions. Campers who are already familiar with continued fractions can skip the first day.

The Combinatorial Nullstellensatz. (🌶🌶🌶🌶, Yuval, Thursday–Friday)
You might well ask, "What is the combinatorial Nullstellensatz?" That would certainly be a question. It may well even be a question that we will address in this class. When I say "we", I refer to you, the students who will attend this class, and Yuval, who will teach it. However, "we" does not include me, Ari, the person writing this blurb at Yuval's request, because I do not know what the combinatorial Nullstellensatz is. I suspect that it similar to the plain old Nullstellensatz, but somewhat more combinatorial. Note that the word "Nullstellensatz" comes from the Tagalog word "Null", meaning "zero", and "stellensatz", meaning "stellensatz". Anyway, you should come to this class, unless you're not interested in it, but honestly I think I'll be okay either way[2].
Homework: Recommended
Prerequisites: None.

Colloquia

Math, Games, and Strategy. (*Po-Shen Loh*, Tuesday)

What is the secret to finding the best path through real life? Hint: it might involve math. In this talk, we'll talk about some games, and use them to inspire discussions about strategy, which ultimately connect math and real life. You might even learn how to win at Monopoly.

Visitor Bios

Po-Shen Loh. Po-Shen Loh is a math enthusiast and evangelist. He is the national coach of the USA International Mathematical Olympiad team, a math professor at Carnegie Mellon University, and the founder of `expii.com`, an educational technology platform which delivers free personalized learning on every smartphone. His math research considers a variety of questions that lie at the intersection of combinatorics (the study of discrete systems), probability theory, and computer science.

[2]Here is a worse, but potentially more informative, blurb:
The Combinatorial Nullstellensatz is one of the most versatile tools I know, and it can be used to solve a huge range of problems and prove a huge range of theorems. This is particularly surprising because it's nothing more than a very simple statement about how the roots of polynomials can be arranged.
In this class, we will be focusing on the applications of this theorem, primarily ones from algebra, graph theory, and number theory.