WEEK 5 CLASS PROPOSALS, MATHCAMP 2017

Contents

Aaron's Classes

Fundamental Groups, Part 2. (🌶🌶🌶 → 🌶🌶🌶🌶, Aaron, 2–3 days)
This is a continuation of the week 1 fundamental groups course, with a view toward group theory.

We'll introduce van Kampen's theorem and use it to show that for every group G, there is a space X whose fundamental group is G. We'll then introduce the theory of covering spaces and use it to show that a finite index subgroup of a finitely generated group is again finitely generated.

Homework: Recommended

Prerequisites: The Fundamental Group, Group Theory

Infinitely Many Lines. (🌶🌶🌶, Aaron, 1 day)
In the colloquium from the first week, we saw that there are 27 lines on any smooth cubic surface. But why did we ask about cubic (degree 3) surfaces instead of quadric (degree 2) surfaces?

In this class, we'll show there are infinitely many lines on any quadric surface over the complex numbers. While the proof that any smooth cubic surface has 27 lines lies deep in algebraic geometry, the proof that any quadric surface has infinitely many lines is much more accessible. To prove it, we will take a nice detour into the theory of quadratic forms and projective geometry.

Homework: None

Prerequisites: Linear Algebra

Many Clubs Share People. (🌶🌶, Aaron, 1 day)
I'll explain and discuss the classic problems of oddtown and even town. In oddtown, there are n people. These people form clubs where each club has an odd number of members, and for any pair of clubs, the number of people they have in common is n. How many clubs can there be? In eventown, there are n people. These people form distinct clubs where each club has an even number of members, and any pair of clubs share an even number of members. How many clubs can there be? The answers to these two questions are surprisingly different, but both involve linear algebra over the finite field with two elements.
Homework: Recommended
Prerequisites: Linear Algebra

Solving Cubic and Quartic Equations Without the Mess. (🌶🌶 → 🌶🌶🌶, Aaron, 1 day)
Probably you are well acquainted with the quadratic formula. It is similarly possible to solve cubic and quartic equations in terms of radicals, but unfortunately, these formulas are incredibly messy.

How might you come up with these formulas yourself? In this class, we'll explain how to solve cubic and quartic equations using radicals via pure thought, without any messy equations.

The method I'll describe is heavily motivated by Galois theory, though you don't need to know any Galois theory to appreciate it.
Homework: None
Prerequisites: None.

The Spectres of Sequences are Haunting Mathcamp. (🌶🌶🌶🌶, Aaron, 1 day)
You may have heard that "A spectre is haunting Europe—the spectre of communism," but did you also know that the spectres of sequences are haunting mathcamp?

In this one-day course, we'll describe the idea of spectral sequences and how to use them to replace diagram chases as in the proofs of the five lemma and the snake lemma.
Homework: None
Prerequisites: Familiarity with exact sequences

Topological Groups. (🌶🌶🌶 → 🌶🌶🌶🌶, Aaron, 2 days)
Like groups? Like topology? Then you'll love topological groups! An topological group is just a topological space which is also a group.

We'll prove some neat facts about topological groups, such as showing that open subgroups are closed, finite index closed subgroups are open.
Homework: Recommended
Prerequisites: Group theory

APURVA'S CLASSES

Crash Course on Representation Theory. (🌶🌶🌶🌶, Apurva, 2 days)
One way to study groups is by making them do things to vector spaces and see what happens. Like every other useful thing in algebra this is inappropriately named as Representation theory.

For finite groups, study it we shall.
Homework: None
Prerequisites: Group theory, Linear algebra.

How to Quantize? (, Apurva, 1 day)
When it works (rarely), there is a very simple, mechanical way of taking a classical system and converting it into a quantum system. However it is still not clear why the quantization process works the way it does and how to generalize this to complicated systems.

This class can be thought of as an intro class to quantum mechanics.

Homework: None

Prerequisites: Linear algebra, some calculus

Projective and Injective Resolutions. (, Apurva, 2 days)
This course is dedicated to Frank's Halloween costume.

For a mysterious reason that I do not yet fully understand, modules (eg. abelian groups and vector spaces) arise and are studied in a collection as what are called chain complexes. In the world of chain complexes projective and injective modules become the main characters and are used to define algebraic invariants.

This is going to be a very boring course. We'll get confused together. It will be a lot of fun.

Homework: None

Prerequisites: At the very least solid group theory and linear algebra knowledge, it will be better if you knew what rings and modules are. (Hint: See the chilis)

The Maths Behind JPEG: Wavelets. (, Apurva, 1 day)
The new JPEG compression algorithm uses a very clever linear algebra technique called the Wavelet transform to achieve good compression ratio.

Images are especially bad to analyze mathematically as they tend to have sharp edges and hence are not differentiable, Wavelet transform provides a means to use the discontinuities themselves to compress the image, like a boss.

Homework: None

Prerequisites: Linear algebra

Waves $= \lim_{n\to\infty} \frac{n \textbf{ coupled harmonic oscillators}}{n}$**.** (, Apurva, 1 day)
It is possible to derive the wave equation starting from harmonic oscillators (mass-spring systems) by taking n harmonic oscillators close to each other and letting n tend to ∞.

We'll do just that.

Homework: None

Prerequisites: Basic calculus, Nobel prize in Physics

What is Homology? (, Apurva, 2 days)
Enough said.

Homework: None

Prerequisites: Linear algebra, and when I inadvertently utter the phrase 'topological space' in class you should be happy and not sad.

Beatriz's Classes

And That Is Why Birds Shouldn't Drink Alcohol. (, Beatriz, 2 days)
In the words of Kakutani, "A drunk man will eventually find his way home, but a drunk bird may get lost forever." In this class we will study random walks on a d-dimensional lattice $\mathbb{Z}^d$. We will prove

that for $d \leq 2$, the random walk on $\mathbb{Z}^d$ is recurrent (meaning the walker returns to its starting point with probability one), yet for $d \geq 3$, it is transient (meaning there is a positive probability that the walker may not return to its starting point).

Homework: None

Prerequisites: Series

Ergodic Theory. (🌶🌶, Beatriz, 2 days)
Imagine you are stirring your tea in a very regular way. Suppose X is a measure space that represents your cup, and $T : X \to X$ is a function that represents where the particles of the tea go after one stirring motion. So, after 5 motions, a point $x \in X$ will go to the point $T(T(T(T(T(X)))))$. Now, if you put some a bit of honey into your tea, and stir, will the honey distribute evenly across your tea? Well, this won't happen if we secretly have two cups, and only put the honey in one (in other words, if there is some subspace of positive measure honey $Y \subset X$ such that T never takes points in Y outside Y). However, it turns out that in all other cases, the answer is yes! After proving this, we might say something about how quickly, and how evenly the honey gets distributed, and what other nice properties T might have.

Homework: None

Prerequisites: A little measure theory, know what an integral is

Every Natural is "Fibonacci": Games, Miles and Kilometres. (🌶, Beatriz, 1 day)
In this class we will prove Zeckendorf's theorem which states that every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This theorem has many very interesting application. In this class we will see how Fibonacci numbers can help us win the following game:

There is one pile of n stones. The first player may remove as many as they like on their first turn as long as they remove at least one and leave at least one. From then on, the next player may remove no more than their opponent did on their previous turn. The player who takes the last coin wins.

Also, because Mathcamp is so international, and I want to help you understand each other, we'll learn how to use Fibonacci numbers to easily convert from miles to kilometres and vice versa.

Homework: None

Prerequisites: None

Jumping Frogs and Powers of Two. (🌶, Beatriz, 1 day)
In this class we we'll study how close a jumping frog gets to a crack in the sidewalk, and we'll also take a look at the distribution of the initial digits of powers of two. Are you wondering how are frogs and powers of two connected? One of the awesome things about math is how that very often seemingly unrelated ideas turn out to be two different aspects of the same thing. In this class we will learn about some number-theoretic (powers of 2) applications of ergodic theory (jumping frogs).

Homework: None

Prerequisites: None

On the Irreducibility of Certain Trinomials: (🌶🌶🌶, Beatriz, 1–2 days)
We will study the irreducibility of polynomials of the form $x^n \pm x \pm 1$ for $n \geq 2$ in $\mathbb{Q}[x]$. For general $n \geq 2$, the standard irreducibility criteria fail. For example, the Eisenstein criterion is inapplicable, and reduction mod n only works when n is prime. We will prove a theorem by Selmer (1956) that shows that the polynomial $x^n - x - 1$ is irreducible in $\mathbb{Q}[x]$ for every $n \geq 2$, and that the polynomial $x^n + x + 1$ is irreducible in $\mathbb{Q}[x]$ if and only $n \not\equiv 2 \bmod 3$.

Homework: None

Prerequisites: None

The Stable Marriage Problem. (🌶, Beatriz, 1–2 days)
Imagine we run a dating agency, and we must match n men with n women. We ask each man to rank the women in order of preference; similarly, each woman is asked to rank the men. Is there an algorithm to guarantee the best possible matches? In fact, to attract more clients, our agency offers one million dollars to those whose matched partner leaves them for another client; can we build an algorithm that ensures we will never have to pay up? We'll also learn why it's not a good idea for women to wait for a man to propose. Can our algorithm be adapted to solve other similar problems, such as the roommate problem?

Homework: None

Prerequisites: None

Don's Classes

"Finite" Sets. (🌶🌶, Don, 1–3 days)
Hyperreal numbers have beautiful applications to proofs in calculus; they let us actually divide by infinitesimals, which eliminates all of the futzing around with ϵs and δs.

One of my favorite proofs from class, though, really relied on the fact that the set $\{1, 2, 3, \ldots, n\}$ has the same properties whether n is finite or infinite. Such sets are called Hyperfinite, or "Finite" (with big air quotes), and in this class, we'll explore the many ways that they are like finite sets. For being uncountable, there are surprisingly many ways!

Homework: Optional

Prerequisites: Hyperreal Numbers

Finite "Sets". (🌶🌶🌶 → 🌶🌶🌶🌶, Don, 2 days)
What makes something finite? Is it that it has finitely many elements? What if it doesn't, techncially speaking, have any elements?

Come learn how Category Theory can tell us what properties really make an object finite - and then see some objects that, despite having infinitely many elements, have put in the hard work to truly earn the title of Finite.

Homework: Optional

Prerequisites: None

"Group Theory". (🌶🌶🌶, Don, 2 days)
"Groups" are sets with an associative operation, an "identity" for that operation, and "have inverses."

More seriously, when studying semigroups (which are like groups, but less so), you can't invert elements, but much of the same structure still applies, and you get a whole slew of lovely new examples; come see why bicycles are simple, and if you're in a band, you have lots of inverses.

Homework: Optional

Prerequisites: Group Theory

How to "Divide". (🌶🌶🌶, Don, 1–2 days)
Adding two sets together is pretty easy - we just take their union, with an extra copy of anything they share.

If $A+B$ is the same size as $A+C$, and A is finite, it's possible, but not obvious, to show that B is the same size as C — that is, to subtract!

Multiplying two sets is also pretty easy — we take their cartesian product.

But, how do we divide? It turns out, entirely without a need for the axiom of choice, as long as we're clever enough. Instead, we get by with one secret ingredient — love[1].

Homework: None

Prerequisites: None

Impossible Cohomology. (🌶🌶, Don, 1 day)
Cohomology is a powerful tool, used to describe underlying properties of mathematical objects from topological spaces to groups and rings. It ultimately measures how a big thing is more complicated than the sum of its parts.

You may recognize this figure:

Each of the parts seems like a normal geometric object, but the way they fit together is wrong. With the power of cohomology, we'll figure out exactly how wrong!

Homework: Optional

Prerequisites: None

My Favorite Magic Trick. (🌶🌶, Don, 1 day)
I've always liked magic, especially mathematical magic — a good mathematical magic trick is essentially a live-action puzzle, with particularly good flavortext. There are a number of tricks that I hold dear to my heart, but this class is about my favorite one: the one where you do all the work. Come perform this trick, and then figure out how it works!

Homework: Optional

Prerequisites: None

Schemes. (🌶🌶🌶🌶, Don, 4 days)
A scheme is

> *"A large-scale systematic plan or arrangement for attaining some particular object or putting a particular idea into effect."*

That's not what a mathematical scheme is, but it is a pretty good description of this class, whose entire goal is to define one particular object, and promote an idea.

That object is a scheme, a "geometric" device that algebraists often like to pretend is something like a manifold.

[1]Where by love, I may or may not mean topological cobordism theory

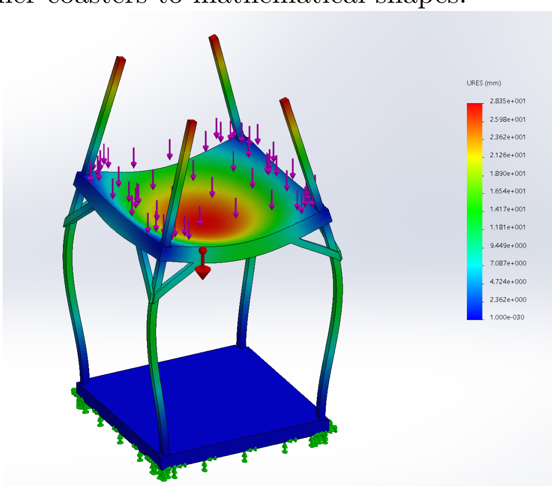
The idea I'll be propagating is that commutative rings, far from being the totally arbitrary constructs you might have considered them, are all really just collections of functions on a space — that space being a scheme! At least, that's what *they* want me to want you to believe - I'm still not sure geometry exists.

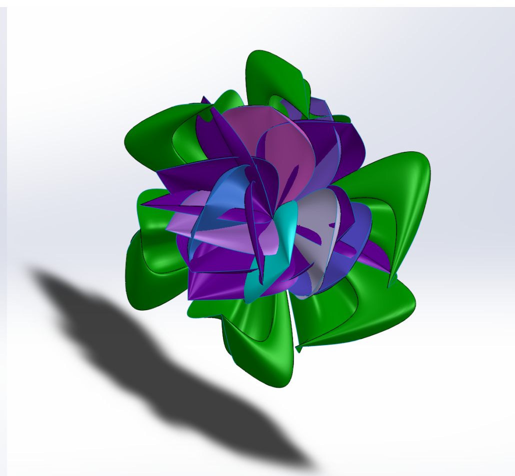
Homework: Optional

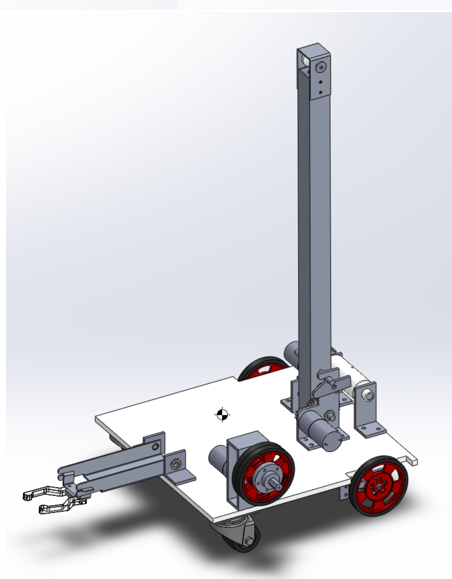
Prerequisites: Ring Theory

Elizabeth's Classes

Computer Aided Design. (🌶, Elizabeth, 2–4 days)
Computers are awesome! They can do so many cool things! In particular, if you can imagine some shape or machine, you can make a computer draw it in 3D. Once the computer knows what it is, then it can show you what it would like from any angle, and you can tweak it without having to redraw the whole thing. You can also turn it colors and zoom in on small details. Basically, anything you can do in your head, you can show to other people, with the computer. In addition, once you have told the computer about it, the computer can print out pictures or files that let machinists or machines make the part in real life. Computer aided design is useful for all kinds of things, from making robots to roller coasters to mathematical shapes.







If you took this class last year, days $n > 1$ will still be interesting, because they will be free time to work on CAD, but day 1 will probably not be, because it will be roughly the same tutorial as last year.
Homework: Optional
Prerequisites: None.

Jalex's Classes

An Embedding Theorem for Finitely Presented Groups. (🌶🌶🌶🌶, Jalex, 4 days)
Think of your favorite finitely-presented group. Now for each pair of variables that appear in the same relation, add a relation saying that they commute. I'll bet that your new group is much less complicated than your old group, even if you try a bunch of examples. So here's a surprising fact: Every finitely presented group embeds into a group with a presentation of the above form.

William Slofstra proved this in 2016[2] by "using 50 pages dense with commutative diagrams, quotient maps, and other Serious Math Stuff."[3]

In this class, we'll dig straight into the heart of this proof, generalizing the theory of topological graph minors to hypergraphs and exploring some nice connections with van Kampen diagrams.
Homework: Recommended
Prerequisites: Know enough about groups that the first paragraph of this blurb surprises you.

Pseudotelepathy Workshop. (🌶🌶🌶, Jalex, 2–4 days)
At the end of the week 3 Pseudotelepathy class, we saw a glimpse of van Kampen diagrams. I claimed that they could be used to generalize a result of Alex Arkhipov[4]. In this class, we'll start by playing with van Kampen diagrams and getting some intuition. Then I'll state the main theorem and let you prove it. With any extra time, we'll ask questions about the correspondence between the combinatorics of a system of linear equations and the structure of the associated solution group.
Homework: Optional
Prerequisites: The week 3 Pseudotelepathy class.

Surreal Numbers. (🌶, Jalex, 1 day)
Typical constructions of the reals go like this:

- Start with 0.
- Put in 1.
- Use addition to get $\mathbb{N}$
- Use subtraction to get $\mathbb{Z}$.
- Put in fractions to get $\mathbb{Q}$.
- Complete the Cauchy sequences to get $\mathbb{R}$.

That's a big number of steps! In this class, we'll start from the following definition:

- A Number is an ordered pair (L — R) where L and R are sets of numbers, and no number in L is bigger than any number in R,

and get not only the reals, but a much richer extension that has a proper subfield of every cardinality.

Homework: Recommended
Prerequisites: None.

[2] https://arxiv.org/abs/1606.03140
[3] http://www.scottaaronson.com/blog/?p=2820
[4] https://arxiv.org/abs/1209.3819

Welzl's Theorem on Graph Homomorphisms. (🌶🌶, Jalex, 1 day)
A *graph homomorphism* $f : G \to K$ is a function from the vertex set of G to the vertex set of K such that if (x, y) is an edge of G, then $(f(x), f(y))$ is an edge of K. Say that $G < K$ if there exists a homomorphism from G to K but there does not exist a homomorphism from K to G. In this class, we'll prove a theorem of Welzl: The partially order set of finite graphs with $<$ is dense. In other words, if $G < K$, then we can always find H such that $G < H < K$.

Homework: None

Prerequisites: Know what a graph is. Know what a function is.

Kevin's Classes

Partially Ordered Sets. (🌶🌶 → 🌶🌶🌶, Kevin, 1–4 days)
A poset is simply a set along with a partial order, a relation that lets us compare some (but not necessarily all) pairs of elements. A whole universe of exciting math develops from this seemingly simple definition! We'll see how posets can be used to study everything from generating functions to polytopes to graphs and more. If you took Hyperplane Arrangements, you've already seen one instance of the power of posets; come see more!

Homework: Optional

Prerequisites: None.

Related to (but not required for): Young's Lattice (W5)

Permutation Statistics. (🌶🌶 → 🌶🌶🌶, Kevin, 1–4 days)
Here's a classic conundrum. The first passenger to board a full flight can't remember what seat to sit in and picks a random seat. Every passenger thereafter sits in the correct seat if available, or a random seat otherwise. You're the last passenger to board. What's the probability you end up in the correct seat?

Here's a more difficult one. Suppose you and 99 of your best "friends" are incarcerated by Don for being too janky, and Don offers to release you if you meet the following challenge. Don secretly places each of your names in 100 different boxes then lays the boxes out in a row in his office. One by one, you and your 99 friends can enter Don's office and peek inside 50 boxes, but you can't tell your friends what you see. If you and all 99 of your friends each manage to find your own name, you'll be released!

Everyone individually has a 1/2 chance of succeeding; if you pick randomly and independently, then there's a vanishingly small chance that you all win the game. But you can decide a strategy ahead of time so that you actually have a decent shot of being released!

In this class, we'll use generating functions to study permutation statistics, things like average number of cycles in a permutation, chances of two numbers being in the same cycle, and maybe some more exotic properties of permutations as well, and we'll see how they answer these questions.

Homework: Optional

Prerequisites: Generating functions are useful to have seen, but I can help catch you up if necessary.

Related to (but not required for): Generating Functions and Regular Expressions (W1)

Series with Complex Analysis. (🌶🌶🌶 → 🌶🌶🌶🌶, Kevin, 1–3 days)
Let $x_1, x_2, x_3, \ldots$ be the nonzero values of x where $\tan x = x$. There are infinitely many such points, and they tend to be very irrational. Yet the reciprocal of the sum of the squares of their reciprocals is an integer.

The central binomial coefficients $\binom{2n}{n}$ have a beautiful generating function, and one of the nicest ways to find it is with a contour integral.

In this class, we'll talk about surprising instances like these where complex analytic methods help us with seemingly unrelated series.
Homework: None
Prerequisites: Functions of a Complex Variable (both weeks; we need the residue theorem)

Young's Lattice. (🌶🌶 → 🌶🌶🌶, Kevin, 1–4 days)
Young's lattice is the partially ordered set of Young diagrams ordered by inclusion. We'll see how thinking about this poset allows us to write questionable but useful equations like

$$f^{D\lambda} = f^{\lambda},$$

and how these equations actually help us prove real results like the hook length formula (which we proved in Young Tableaux, but it can be done other ways as well!).
Homework: Optional
Prerequisites: Young Tableaux is helpful but not required.
Related to (but not required for): Young Tableaux (W3); Partially Ordered Sets (W5); Young "Tableaux" (W5)

Young "Tableaux". (🌶🌶 → 🌶🌶🌶, Kevin, 1–4 days)

In Week 3, I taught about standard Young tableaux, which are fillings of nice shapes with numbers like this:

1	3	4	6	9
2	7	8		
5				

.

In this class, we'll discuss janky shapes like

1	2	4	7
	3	6	
		5	

,

and maybe some "tableaux" like

1	3	'1'	'3'
2	'2'		

.

Homework: Optional
Prerequisites: None; you will miss some connections if you don't know about Young tableaux from my class, but you'll be fine if you're willing to believe things I say about them.
Related to (but not required for): Young Tableaux (W3); Young's Lattice (W5)

Lara's Classes

Eisenstein's Criterion. (🌶🌶, Lara, 1–2 days)
In this class, we'll state and prove Eisenstein's criterion and Gauss' lemma: conditions for determining the irreducibility of polynomials over rings and their fraction fields. We'll also give a couple of interesting applications of this criterion: 1. Showing that $\mathbb{C}[x_1, \dots, x_d]/(x_1^n + \cdots + x_d^n)$ is a domain for $d \geq 3$ 2. Understanding the minimal polynomials of primitive roots of unity.
Homework: None
Prerequisites: You should know what rings and ideals are.

Laws of Large Numbers. (🌶🌶 → 🌶🌶🌶, Lara, 2–3 days)
In this class, for $X_1, \ldots, X_n$ independent and identically distributed random variables, we'll prove the convergence of expressions like $\frac{X_1+\cdots+X_n}{n}$ in different senses. We'll use these results to answer a few real-world problems such as:

(1) How long do we expect it to take us to collect an entire set of stamps if we acquire one stamp at random per time interval?
(2) How much should we pay to play a game where we win 2^j dollars if we get $j-1$ consecutive tails and it takes 2^j coin tosses to get heads for the first time?

Homework: None

Prerequisites: You should know what a random variable is, what it means to integrate a random variable and what it means for random variables to be independent.

Modules and Integrality. (🌶🌶 → 🌶🌶🌶, Lara, 2–3 days)
This class will give an introduction to modules, where we'll get to understand what they are and give lots of examples. We'll explore the basic properties of modules and of maps between them.

If R is a subring of R', then an element $b \in R'$ is integral over R if there is a monic polynomial $f(X) \in R[X]$ such that $f(b) = 0$. We'll use the theory of modules to understand integrality: e.g. to show that sums and products of integral elements in R' are also integral over R.

Homework: None

Prerequisites: You should know what rings, ideals and vector spaces are.

The Baire Category Theorem. (🌶🌶 → 🌶🌶🌶, Lara, 2 days)
In this class we'll prove the Baire category theorem, which tells us that a countable intersection of dense open sets is dense and explore some of its amazing consequences. Examples of these are:

(1) Not only are the irrationals uncountable, but they can't even be written as the countable union of closed sets.
(2) There exists functions continuous at all the rationals but discontinuous at all the irrationals, but not vice versa.
(3) Any infinitely differentiable function on $[0, 1]$ that has some derivative vanish at each point of the interval must be a polynomial.

Homework: None

Prerequisites: You should know what a metric space is and be comfortable with the idea of open sets and closed sets and what the interior and closure of sets are.

Linus's Classes

I click "Random Page" in OpenProblemGarden.org until we get one we understand: then we spend 50 minutes trying to solve it. (🌶 → 🌶🌶🌶🌶, Linus, 1 day)
See title.

Homework: None

Prerequisites: Probably combinatorics.

Machine Learning (NO Neural Networks). (🌶🌶🌶, Linus, 1–2 days)
Emphasis on theory over practice. Possible topics:

- Learning a linear classifier (with noise)
- Models of learning: PAC, Mistake Bound, Bayesian

- VC dimension (a combinatorial property: "simple", "easy to learn" concepts tend to have low VC dimension)

Homework: Recommended

Prerequisites: Thinking in n-dimensional space

Related to (but not required for): Not Your Grandparents' Algorithms Class (W1)

Lotta's Classes

How to Build a Calculator. (🌶, Lotta, 1 days)
I have a little box with buttons on it that's probably made of magic. When I push buttons like '2,' '3,' '4,' '+,' '3,' '4,' it returns '268.' How does it know? Probably there is a little person inside doing arithmetic by hand.

In this class, we will talk about circuits and gates and how you can put gates together to build an adder or anything you want.

Homework: Optional

Prerequisites: None

Mark's Classes

Existence and Uniqueness. (🌶🌶🌶, Mark, 1 day)
Suppose a particle can only travel along the x-axis and the force on it is given by $F(x) = 6m \cdot x^{1/3}$, where the constant m is the mass of the particle. If the particle starts by being at rest at the origin, we get the initial value problem

$$x'' = 6 \cdot x1/3, x(0) = 0, x'(0) = 0$$

telling us what the particle is going to do, so this has a unique solution, right? Wrong! You can easily check that both $x(t) = t^3$ and $x(t) = -t^3$ are solutions, as is $x(t) = 0$ (for all t). So what is going on?

It turns out that this kind of unexpected and unpleasant "unpredictability" of solutions can also occur for some first-order equations $x' = f(t, x)$, but that imposing some fairly mild conditions on the function f takes care of things: it can no longer happen. We'll see what the famous Existence and Uniqueness Theorem (which shows this) says, and get at least the general idea of the proof.

Homework: None

Prerequisites: Comfort with derivatives and integrals.

Galois Theory Crash Course. (🌶🌶🌶🌶, Mark, 4 days)
In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully.

Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later). If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we just *might* be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class but which may be shown in a separate week 5 class.) Even if we

don't get this far, the so-called Galois correspondence (which we should be able to get to, and prove) is well worth seeing!
Homework: Optional
Prerequisites: Group theory; linear algebra; some familiarity with fields and with polynomial rings.

Multiplicative functions. (🌶🌶 → 🌶🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which makes that set (except for one worthless function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.
Homework: Optional
Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* required.)

Perfect Numbers. (🌶, Mark, 1 day)
Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called *perfect*, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, the so-called *Mersenne primes* — a search that has largely been carried out, with considerable success, by a far-flung network of individual "volunteer" computers.
Homework: None
Prerequisites: None

Primitive Roots. (🌶🌶, Mark, 1 day)
Suppose you are working modulo n and you start with some integer a and multiply it by itself repeatedly. For instance, if $n = 17$ and $a = 2$ you get $2, 4, 8, 16, 15, 13, 9, 1$ and then you're back where you started. Note that on the way we haven't seen all the nonzero integers mod 17; however, if we had used $a = 3$ instead we would have gotten $3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$ and cycled through all the nonzero integers mod 17. In general we can ask when (that is, for what values of n) you can find an a such that every integer mod n that's relatively prime to n shows up as a power of a (such an a is called a *primitive root* mod n). We may not get much beyond the case that n is prime, but even in that case the analysis is interesting. In particular, we'll be able to show that a exists in that case without having any idea of how to find a, other than the flat-footed method of trying $2, 3, \ldots$ in turn until you find a primitive root.
Homework: None
Prerequisites: Modular arithmetic; a bit more number theory might be helpful, but you should be able to manage without. If you have taken the class on finite fields, you should probably *not* take this class: you'll know too much.

Quadratic Reciprocity. (🌶🌶🌶, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is q a square modulo p?"

(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Homework: Optional

Prerequisites: Some basic number theory (if you know Fermat's Little Theorem, you should be OK)

Simplicity Itself: A_n and the "other" A_n. (🌶🌶 → 🌶🌶🌶, Mark, 2 days)

The monster group (of order roughly $8 \cdot 10^{53}$) gets a lot of "press", but it's not the largest finite simple group; it's the largest *exceptional* finite simple group. (Reminder: A simple group is one which has no normal subgroups other than the two "trivial" ones; by using homomorphisms, all finite groups can be "built up" from finite simple groups. The complete classification of finite simple groups was a monumental effort that was completed successfully not far into our new millennium.)

What about the unexceptional finite simple groups? They come in infinite families, and in this class we'll look in some detail at two of those families: the alternating groups A_n and one class of groups of "Lie type", related to matrices over finite fields. (If you haven't seen finite fields, think "integers mod p" for a prime p.) By the way, the simplicity of the alternating groups plays a crucial role in the proof that in general, polynomial equations of degree 5 and up cannot be solved by radicals (there is no "quintic formula").

We'll prove that A_n is indeed simple for $n \geq 5$, and we should be able to prove simplicity for the other class of groups also, at least for 2×2 matrices.

Homework: None

Prerequisites: Basic group theory and linear algebra; familiarity with finite fields would be helpful, but not really necessary.

So What *Is* a Determinant, Really? (🌶🌶🌶, Mark, 2 days)

You've probably seen determinants (of a matrix, or of a linear transformation), but you may not have seen a good definition, or much motivation. The goal of this class is to give a definition of determinant that's both motivated and rigorous, to prove its main properties (such as Laplace expansion), and as time permits, to give a few applications, such as general formulas for the inverse of a matrix and for the solution of n linear equations in n unknowns ("Cramer's Rule").

Homework: Optional

Prerequisites: A bit of linear algebra (in particular, the concepts of basis and linear transformation). Also, a little experience with permutations would help.

The Cayley-Hamilton Theorem. (🌶🌶🌶, Mark, 1 day)

Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Homework: None

Prerequisites: Linear algebra, including a solid grasp of determinants (the "So What *Is* a Determinant?" class would take care of that).

The Delta "Function". (🌶🌶🌶, Mark, 1 day)
In introductory books on quantum mechanics, you can find "definitions" of a "function", called the *delta function*, which has two apparently contradictory properties: Its value is zero for all nonzero x, and yet its integral over any interval containing 0 is 1. This "function" was introduced by the theoretical physicist Dirac and it turns out to be quite useful in physics, but how can we make any mathematical sense of such a creature?

Homework: None

Prerequisites: Integration by parts. The concept of a linear transformation will also help.

Misha's Classes

Choosing Random Numbers. (🌶, Misha, 1 day)
If you want to pick a random number, but you don't care how, you might as well just roll a die. If you *do* care how, you had better do some math first. We'll do a few random things in this class, but the main goal is to see a very cute algorithm for doing what seems like magic.

You may know that factoring a 1000-digit number would take a computer thousands of years. But in this class, I will teach you how to factor a randomly chosen 1000-digit number in just a few minutes on a computer.

Homework: None

Prerequisites: Willingness to take on faith a few of my arbitrary claims about the values of infinite sums and products.

Problem Solving: Convexity. (🌶🌶🌶, Misha, 1 day)
You see lots and lots of olympiad problems about inequalities. These are often solved by applying obscure theorems due to Scottish mathematicians.

In the real world, mathematicians study inequalities too, but their approach is different. Inequalities that arise from convex functions are much more important.

So in this class, we will take the best of both worlds and use convexity to solve olympiad math problems.

Homework: Optional

Prerequisites: None.

Problem Solving: Euler Characteristic. (🌶🌶, Misha, 1 day)
In Week 2, Apurva taught some of you how useful Euler characteristic is for doing actual serious math.

In this class, we're going to be using Euler's formula to solve olympiad math problems.

Homework: Optional

Prerequisites: The equation $V - E + F = 2$ should not be a mystery to you.

Problem Solving: Linear Algebra. (🌶🌶🌶, Misha, 2–3 days)
Most high school math contests (the IMO included) do not use any topic considered to be too advanced for high school, such as linear algebra. This is a shame, because there have been many beautiful problems about linear algebra in undergraduate contests such as the Putnam Math Competition.

In this class, we will look at linear algebra from a new perspective and use it to solve olympiad problems.

Homework: Recommended

Prerequisites: Linear algebra. In particular, you should know what eigenvalues are.

Queueing Theory. (🌶🌶 → 🌶🌶🌶, Misha, 2–3 days)
We will look at continuous versions of Markov chains and their applications to queueing theory: the study of how long you will wait in line.

Homework: Recommended

Prerequisites: None—in particular, my class on Markov chains is *not* a prerequisite.

Solving 3SAT Faster. (🌶🌶🌶, Misha, 2–4 days)
The Boolean satisfiability problem is the computational task of taking a formula in n logical variables and checking if there is some way of assigning the values "true" and "false" to each of them to make the formula true.

We can do this by brute force, by choosing all 2^n possible assignments and checking each one. This is slow. We currently don't think that there's a fast (polynomial-time) algorithm. But in this class, I'll talk about several ways we can improve on brute force.

Homework: Optional

Prerequisites: None

The Fractional Part of a Normal Distribution. (🌶🌶🌶, Misha, 1 day)
Take a random number from the standard normal distribution. Let $\boldsymbol{X}$ be its fractional part.

Mathematica refuses to tell me how $\boldsymbol{X}$ is distributed without shenanigans. After doing some sketchy manipulations, I can get it to say that

$$\Pr[0 < \boldsymbol{X} < \frac{1}{3}] \approx 0.333\,333\,334\,071.$$

That's supposed to be $\frac{1}{3}$, right?

In this class, we will learn a bit about the Fourier transform and how we can use it to answer this question.

Homework: None

Prerequisites: You should understand what integrals do and what the normal distribution is.

The Game Chromatic Number. (🌶🌶, Misha, 1 day)
To figure out the chromatic number of a graph, you color its vertices, and try to avoid adjacent vertices getting the same color. So far, so good.

Now suppose Harry Potter and Voldemort want to color a graph together. They take turns coloring a vertex. But Voldemort is a jerk; his goal (while obeying the rule that adjacent vertices must get different colors) is to make sure that they get stuck and run out of colors.

How many colors does Harry Potter need to have for the graph to get colored? In this class, we'll see what the answer is for some classes of graphs, and find out that some aspects of this problem are still embarrassingly unknown.

Homework: None

Prerequisites: Enough graph theory to know about chromatic numbers.

Sam's Classes

Algorithms to Generate Randomish Numbers. (🌶 → 🌶🌶, Sam, 1 days)
Suppose you wanted to generate a truly random sequence of numbers. You sit down to start flipping a coin for forever, but shortly realize that this is boring. Instead, suppose you wanted to generate a sequence of numbers that looked random (so that you could fool your friend Tim!, say, into thinking that you were playing rock-paper-scissors randomly). How might you do this?

In this class, we'll introduce some of the methods for generating pseudorandom numbers. We'll also look at why some of these were terrible!

Homework: None

Prerequisites: Basics of modular arithmetic

Mathematics in Crisis: The St. Petersburg Paradox. (🌶🌶, Sam, 1 day)
By the early 18th century, probability was starting to develop as a field of its own right. Core to this development was the concept of *expectation* (expected value). The hope was that expectation would become the tool for making rational decisions: the decisions a mathematician would make. In 1713, however, Nicolas Bernoulli came along and proposed what is now known as the St. Petersburg paradox: a game that no mathematician would pay even $100 to play, but for which mathematics dictated that anyone should jump at the chance to pay any amount of money to play. Many of the initial resolutions to this paradox (posed by serious mathematicians) might seem crazy today. For example, an event having different **mathematical** and **moral** probabilities.

In this class, we'll briefly trace out the early history of probability, leading up to the crisis that was the St. Petersburg Paradox. We'll then spend the rest of the class talking about the historical resolutions proposed.

Homework: None

Prerequisites: You should know what expected value is.

Pattern Avoidance in Latin Squares. (🌶🌶, Sam, 1 day)
Pattern avoidance is a classic question of combinatorics. Classically, it's studied by looking at permutations and leads to a beautiful way of counting the Catalan numbers. But you know what's cooler than a permutation? A Latin Square! Why? It's easy to count the number of permutations of n elements ($n!$, yay!), but much harder to count the number of Latin squares using n elements. Moreover, each row and column of a Latin Square is a permutation!

In this class, we'll drop the classical theory and study pattern avoidance in Latin Squares. We'll see a cute result, and then discuss a whole bunch of open questions that I'd love a camper to solve!

Homework: Optional

Prerequisites: None.

Saving Calculus: Cauchy, Rigor, and Epsilon to the Rescue. (🌶🌶, Sam, 1 day)
In the early 18th century, Bishop Berkeley was fed up with mathematicians calling religion irrational. His response was to write the wonderfully titled *The Analyst, or a Discourse Addressed to an Infidel Mathematician.* Here he ripped into the philosophical foundations of Newtons and Leibniz's calculus. Several mathematicians then tried to respond by putting calculus on a rigorous foundation, culminating in Cauchy's epsilon-delta formalization. In addition, Cauchy was a wonderful and eclectic mathematician. He taught engineers at the Ecole Polytechnique where he was notorious for forcing them to learn modern analysis instead of computational methods; he basically wrote the book on analysis; and he stuck to his principles (rightly or wrongly) at all costs.

TL;DR: in my book, Cauchy is the quirky hero who saved calculus. Come to this class to find out why!

Homework: None

Prerequisites: Basic familiarity with calculus. Having seen the epsilon-delta definition of a limit will make you appreciate this class more.

Steve's Classes

Aperiodic Tilings. (🌶🌶, Steve, 2–3 days)
It's easy to tile the plane with squares; it's also pretty boring. More complicated tiles lead to more interesting tilings—tiling with hexagons is pretty neat—but of particular interest are those tilings where *every part looks unique*: although they tile the plane with few shapes, the resulting tiling has few or no symmetries at all. It is not obvious that (interesting) tilings of this type exist, but they do. One particularly famous one is the *Penrose tiling*, which is a tiling (or rather, class of tilings) with very little actual symmetry but lots of tantalizing structure; if you haven't seen this before, it's worth googling (it's really quite pretty). We'll look at a couple simple examples of "symmetry-less" tilings, and then dive into a mathematical explanation of the Penrose tilings and their various interpretations. Come for the tilings, stay for the projections of five-dimensional cubes!

Homework: Recommended

Prerequisites: None

From Carrying to Cohomology. (🌶🌶, Steve, 1–2 days)
A lot of abstract algebra comes from taking some mathematical operation we already know about (e.g. addition, multiplication, division) and extracting some abstract idea from it (e.g. groups, rings, fields). In this class, we'll look at a particular mathematical operation which is very basic, but algebraically weird: *carrying* in addition. When we add 19 and 32, the units digit is 1 because that's how arithmetic modulo 10 works, but why is the tens digit 5 instead of 4? Of course this is obvious arithmetically, but what sort of strange abstract algebra is "going on behind the curtain?"

In this class we'll find the key identity that carrying satisfies. From this humble beginning emerges *group cohomology*: this is a way to study a group by studying how it can be "extended" in the same way that arithmetic-mod-ten extends to arithmetic-mod-a-hundred via the carrying operation. While we won't be able to prove anything about cohomology in general, it's still pretty cool that we can motivate and define such a powerful and complicated tool with elementary arithmetic!

Most of this class will be two chilis; the last ten minutes (or 20, if it runs for 2 days) will talk about group theory, and will be three chilis.

Homework: None

Prerequisites: Group theory

Godel's Incompleteness Theorem. (🌶🌶🌶, Steve, 4 days)
In 1931, Kurt Godel proved that there are true sentences of arithmetic which cannot be proven from the standard axioms for arithmetic; moreover, he proved that *no reasonable system* of axioms would be free from this problem! This is totally wild: what exactly does it mean, and how on earth would one go about demonstrating it?

In this class we'll prove Godel's Incompleteness Theorem, and talk about what it does—and does not—mean. Historically, this was almost the beginning of modern logic, so there's also some great stories from the time during which it was proved and we might talk about those too.

Homework: Recommended

Prerequisites: Comfort with formal proof, especially induction, and a little group, ring, or field theory.

How Not to Solve Tricky Integrals. (🌶🌶, Steve, 1 day)
The function e^{-x^2} doesn't have an elementary antiderivative, but the definite ("Gaussian") integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

can be solved in a variety of ways. One quite nice approach (due to Poisson) is via change of coordinates. It's a very easy trick, and it's natural to ask what other problems we could solve this way.

It turns out the answer is—*absolutely none*! In a precise sense, Poisson's trick only works once. Useful knowledge is overrated! Come see how to solve the Gaussian integral, and how not to solve tricky integrals in general!

If we have time, we'll see a broad generalization of Poisson's trick. This still doesn't work anywhere else, but it's really fancy.

Homework: Recommended

Prerequisites: Multivariable calculus (specifically, change of variable in multiple integrals)

Spaces and Algebras. (🌶🌶🌶, Steve, 3 days)
Universal algebra is the study of kinds of algebraic structures which can be defined by equations. For instance, the axioms for groups are each equations (assuming we have symbols for the identity and the inverse operation, as well as for the group operation), but the axioms for fields are not (because multiplicative inverses must be treated differently for 0).

Recently, attention has turned to universal algebra *over topological spaces*: given a space X and a set of equations $\mathcal{E}$, we ask when there are continuous functions on X which satisfy those equations. For instance, the group axioms are satisfiable on the real line in this sense, since addition of real numbers is continuous and is a group. The group axioms are also satisfiable on the circle (think e.g. about multiplication of complex numbers of the form $e^{i\theta}$).

In this class we'll talk about topological universal algebra. Although it's an outgrowth of universal algebra, no prior knowledge of universal algebra is needed: we'll just look at shapes and algebraic structures that "fit onto" those shapes. In particular, we'll be interested in telling when a set of axioms is *not* satisfiable on a given space, and what kinds of spaces have "few" algebraic structures.

Homework: None

Prerequisites: Group theory, and knowing what a "continuous map" is in a general metric (ideally, topological) space

Susan's Classes

Quandles. (🌶🌶, Susan, 1 day)
So... you know associativity? That thing that makes the operations in groups and rings at least reasonably nice? Well I say, phooey to associativity! Who needs it?

In this class we'll learn about a weird-looking nonassociative algebraic structure called a Quandle. We'll talk about how quandles behave, how they misbehave, and how they're related to knot theory. If you want just a little bit more brain-bending algebra before the end of camp, then this is the class for you.

Homework: None

Prerequisites: None.

Symmetry Odds and Ends and Odds Symmetry. (🌶🌶, Susan, 1–2 days)
In our Wallpaper Patterns class, we took a fast track to the classification of the wallpaper groups. But there are a couple of pretty things we missed along the way. There are an infinite number of distinct symmetry patterns on the sphere, but it turns out that we can still neatly classify all of them. We can also use them to classify the seven different types of frieze pattern. And wouldn't it be nice to see some pretty pictures of actual orbifolds that we can't make out of paper? If you enjoyed the Wallpaper Patterns class, come on out and we can tie up some loose ends.

Homework: Recommended
Prerequisites: Wallpaper Patterns

What's The Deal With e? (🌶🌶🌶, Susan, 2 days)
The continued fraction expansion of e is

$$1+\cfrac{1}{0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{\ddots}}}}}}}}}}}}}}$$

Okay, but seriously, though, why?!?! Turns out we can find a simple, beautiful answer if we're willing to do a little integration. Or maybe a bit more than a little? Come ready to get your hands dirty—it's gonna be a good time!
Homework: Recommended
Prerequisites: Familiarity with integration by parts, and basic partial fractions. Campers who are already familiar with continued fractions can skip the first day.

Tim!'s Classes

Calculus without Calculus. (🌶🌶, Tim!, 2–4 days)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Angela is 5 cubits tall and Misha is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Angela's head to the top of Misha's head that touches the ground in the middle. What is the shortest length of string you can use?
- Apurva rides a bike around an ellipical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters way along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!
Homework: Optional
Prerequisites: We won't use calculus (that's the point), but it would be good if you've seen it for context.

Guess Who? (🌶 → 🌶🌶, Tim!, 1 day)
Guess Who? is a simple children's game: There's a deck of 24 cards, each of which has a cartoon face on it. You and I are each dealt a card, then we take turns asking yes/no questions to guess the other person's card. Also, there's these fun plastic flippy cards. It's so easy, you probably already know the winning strategy: use each question to cut the possibilities in half.

Except you're wrong. For no interpretation of the rules is this the best strategy. Under one interpretation of the rules, someone playing the optimal strategy finish in t turns with probability p according to the following table:

t	4	5	6	7	8	9	10
p	$\frac{65}{129}$	$\frac{31}{129}$	$\frac{17}{129}$	$\frac{7}{129}$	$\frac{5}{129}$	$\frac{1}{129}$	$\frac{2}{129}$

Also, Problem 7 on this year's qualifying quiz was basically a simplified version of *Guess Who?*, but the solution is not straightforward.

Homework: None

Prerequisites: None.

Intersecting Polynomials. (🌶🌶, Tim!, 1–2 days)
You might think that everything there is to know about one-variable real polynomials has been known for hundreds of years. Except, in 2009, while bored at a faculty meeting, Kontsevich scribbled down a brand new fact about polynomials. You'll discover it.

Homework: None

Prerequisites: None.

Viv's Classes

How Not To Prove That a Group Isn't Sofic. (🌶🌶🌶🌶, Viv, 2 days)
Cayley's Theorem tells us that finite groups are all subgroups of finite permutation groups. A sofic group is a possibly-infinite group that we sort of maybe want to have the same property roughly speaking. The group can't always be a subgroup of a finite permutation group, so instead we just require that all finite subsets of our sofic group act kind of like finite subsets of permutation groups. This ends up being a super-useful definition, but we're left with a burning question: do non-sofic groups exist? We don't actually know the answer to this one. We'll spend the class talking about a hopeful candidate for a non-sofic group and one great way not to prove that it isn't sofic.

Homework: Recommended

Prerequisites: Group Theory

Yuval's Classes

Crossing Numbers. (🌶🌶, Yuval, 1–3 days)
Planar graphs are great—we can draw them without any edges intersecting, and this is a very nice property. Non-planar graphs are less great—no matter how we draw them, some edges will intersect. But how many?

It turns out that this question, which basically tries to understand *how* non-planar a non-planar graph is, leads to some extremely deep and important results in graph theory. In this class, we'll see the Crossing Number Inequality, which gives us a rough answer to this question, and then see some of its very many applications to fields like geometry and number theory.

Homework: Recommended

Prerequisites: Intro to Graph Theory

Dirty Tricks with Ultrafilters. (🌶🌶🌶, Yuval, 1–2 days)
Here are some sad facts:

- Some bounded sequences of real numbers don't converge.
- We want to say that half of all integers are even, but we can't consistently assign such a "density" to every set of integers.
- When learning calculus, we pretend that infinitesimally small numbers exist, but they really don't.
- Democracy is really hard when you have infinitely many voters.

Luckily, ultrafilters can solve *all* these problems! (In fact, if you were in Hyperreal Numbers in Week 3, then you've already seen them solve the infinitesimals problem). In this class, we'll see how we can use ultrafilters as a very big hammer with which we can smash all our problems into oblivion.
Homework: Optional
Prerequisites: Know what a non-principal ultrafilter is. For instance, Set Theory (W1) or Hyperreal Numbers (W3) suffice.

Fast Matrix Multiplication. (🌶🌶, Yuval, 1 days)
Some of my favorite algorithmic questions are related to a task that might seem pretty uninteresting, namely that of multiplying together two big matrices. In this class we will see how you can multiply together matrices way faster than you might have expected. Additionally, we'll talk about a big open problem in the field, which basically says that multiplying together two matrices is no harder than just looking at them. Finally, we'll talk a bit about how people are trying to prove this conjecture, and about some recent theorems that say all such attempts simply can't work.
Homework: None
Prerequisites: Know how matrix multiplication works! Also, some small background with algorithms will be helpful, but not necessary.

The Combinatorial Nullstellensatz. (🌶🌶🌶🌶, Yuval, 1–2 days)
You might well ask, "What is the combinatorial Nullstellensatz?" That would certainly be a question. It may well even be a question that we will address in this class. When I say "we", I refer to you, the students who will attend this class, and Yuval, who will teach it. However, "we" does not include me, Ari, the person writing this blurb at Yuval's request, because I do not know what the combinatorial Nullstellensatz is. I suspect that it similar to the plain old Nullstellensatz, but somewhat more combinatorial. Note that the word "Nullstellensatz" comes from the Tagalog word "Null", meaning "zero", and "stellensatz", meaning "stellensatz". Anyway, you should come to this class, unless you're not interested in it, but honestly I think I'll be okay either way[5].
Homework: Recommended
Prerequisites: None.

The Hadwiger Conjecture. (🌶, Yuval, 1 day)
The Hadwiger Conjecture is one of the biggest open problems in Graph Theory today. Despite being quite simple to state and seeming very plausible, it turns out to imply the famous Four-Color Theorem, and is in fact *way* stronger than that result. In this class, we'll state the Hadwiger Conjecture, see

[5] Here is a worse, but potentially more informative, blurb:
The Combinatorial Nullstellensatz is one of the most versatile tools I know, and it can be used to solve a huge range of problems and prove a huge range of theorems. This is particularly surprising because it's nothing more than a very simple statement about how the roots of polynomials can be arranged.
In this class, we will be focusing on the applications of this theorem, primarily ones from algebra, graph theory, and number theory.

how it's related to the Four-Color Theorem, and talk about what small amounts of partial progress are known.
Homework: None
Prerequisites: Intro to Graph Theory

The Kakeya Maximal Conjecture. (🌶🌶🌶🌶, Yuval, 1 day)
Do you like the Kakeya Conjecture, but feel that it's too plain? If so, you might be interested in taking it… to the *max*.

In this class, we'll learn about the Kakeya Maximal Conjecture, which is an important strengthening of the Kakeya Conjecture. Much of the recent progress on the Kakeya Conjecture (including the proof of the conjecture for the case $n = 2$ that we saw on the last day of class in Week 1) has actually been progress on the Kakeya Maximal Conjecture.

In this class, we'll learn what the Maximal Conjecture is, and why it's such a useful framework for proving Kakeya-type results.
Homework: Optional
Prerequisites: The Kakeya Conjecture

The Szemerédi Regularity Lemma. (🌶🌶🌶🌶, Yuval, 2 days)
The Szemerédi Regularity Lemma is one of the deepest and most powerful tools in combinatorics. Roughly speaking, it tells us that all "big" graphs look "the same", and that we can actually more or less forget about where the edges of our graphs are. This may sound stupid, but its power cannot be overstated.

In this class, we'd learn the Regularity Lemma and see some of its applications in number theory, for which it was originally developed. By the end of the class, you'll have some ideas that go into the famous (and famously difficult) Green-Tao Theorem on arithmetic progressions in the primes.
Homework: Optional
Prerequisites: Intro to Graph Theory

Camper-taught Classes

Non-Classical Logic. (🌶🌶, Joshua Schluter, 1 days)
Mathematics has this long-standing tradition of creating new systems. When the reals weren't enough, we invented the complex numbers. When geometry was too strict, we invented topology. But what happens if we want to change the very laws of logic? In this class, we'll do just that.

For the most part, we'll stick to a type of logics called modal logics. These logics can help us make sense of statements like "Misha must be the crowman," among other statements. We might also touch on other interesting types of logics near the end of class.
Homework: None
Prerequisites: No Prior knowledge necessary.

Smol Results on the Mobius Function. (🌶🌶, Karen Ge, 1 days)
Our goal is to give an overview of classical analytic number theory techniques. In particular, we'll discuss the Mobius function in detail. The Mobius function appears in many unexpected places, such as in connection to the Riemann zeta function and the roots of unity. There are various open problems about its "randomness." It is also a useful tool for extracting values from certain types of summations and for analyzing other arithmetic functions. We will define many fundamental concepts about multiplicative functions and see how to use them to prove cute (smol) facts such as "the Mobius function can be expressed as the sum of the primitive roots of unity."

Homework: Optional

Prerequisites: None.

Cap-Set Problem: Recent Progress. (🌶🌶🌶, Paweł Burzyński, 2 days)
Have you ever played the card game SET? If not, run to the game lounge and play it. The picture on every card has four different features (number, shape, color, filling), each of which can take three values. We call three cards a *SET* when they are all different or all the same in each feature. The goal of the game is to find a SET among 12 cards on the table. As it turns out, it may not always be possible. One may wonder: "How many cards without a SET can you have on the table?"

Figure 1. All 3 cards have different shapes, colors, and numbers and the same shading. Hence they form a SET.

A mathematician can imagine a SET card as a point in $\mathbb{F}_3^n$, n-dimensional space over $\{0, 1, 2\}$, for $n = 4$. Three points form a SET if they are collinear. The Cap-Set Problem asks, "What is the maximum number of points in $\mathbb{F}_3^n$ without a collinear triple?". For more than 30 years in the history of analytic combinatorics, nobody could even tell if the answer is of much smaller magnitude than 3^n. However, in 2016, Ellenberg and Gijswijt in their groundbreaking paper showed that indeed it is. What is more, in contrast to the previous Fourier Analysis approaches, they use only elementary methods.

In this class, I will show you the whole argument that every subset of $\mathbb{F}_3^n$ without a collinear triple is exponentially smaller than 3^n. The basic idea of this approach is a smart use of polynomials and ranks of matrices.

Homework: Optional

Prerequisites: Linear algebra (familiarity with vector space and its dimension and matrix and its rank). Understanding the meaning of $P(x)$, where P is a polynomial in n variables and x is a point in F_q^n.

Co-taught Classes

Lie Groups and Lie Algebras. (🌶🌶🌶, Aaron & Apurva, 2 days)
We'll look at Lie groups $SU(n), SO(n), Sp(n)$ and understand the relations to their Lie algebras and learn how to do compute their properties like dimensions. Time permitting we'll look at the classification theorem of compact Lie groups.

Homework: None

Prerequisites: Group theory, Linear algebra

Maths Behind the Quantum Oscillator. (🌶🌶🌶, Aaron & Apurva, 2 days)
We'll start with understanding the mathematical description of the quantum harmonic oscillator (mass-spring system). We'll see how harmonic oscillators connect to the matrix group $SU(2)$ and learn about the representations of $SU(2)$ and their relation to physics.

Homework: None

Prerequisites: Group theory, Linear algebra

What Makes Quantum Weird? When Probabilities Go Negative. (🌶🌶🌶, Linus & Jalex, 2 days)
How do quantum computers perform their sorcery?

- The magazine catchphrase: "They try all possibilities in parallel (universes)."
- The true but simplified answer: "Via clever math, give the quantum program a 1/2 chance of outputting the wrong answer. And separately, a -1/2 chance of outputting the wrong answer. In total, it's never wrong."
- The true but nonsimplified answer: take the class. BONUS: We'll explain how, if you have one button that does nothing and one that explodes the Earth, you can decide which is which with probability 75%.

Homework: Recommended

Prerequisites: None.

Related to (but not required for): Pseudo-Telepathy via Representation Theory of Finite Groups (W3)

Problem Solving: the "Just Do It" Method. (🌶🌶, Linus or Misha, 1–2 days)
Example Problem: Is there a sequence of integers $a_1, a_2, a_3, ...$ which contains every integer exactly once, where the sequence of differences $a_i - a_{i+1}$ also contains every integer exactly once?

The "Just Do It" method turns some frightening-looking combinatorics problems, like this one, into jokes.

[If this is a two-day class, the second day will be 3 chilis and focus on harder variants, such as transfinite Just Do It.]

Homework: Recommended

Prerequisites: None.

Your Grandparents' Algorithms Class. (🌶, Sam and Tim!, 1 day)
Listen here, sonny. Sit down, have some warm milk. Back in my day, we didn't have these fancy newfangled iComputers and robot phones. When we debugged a computer, we were literally removing insects. And we couldn't just swipe right on our touch screens. If we wanted to ruin someone's day, we'd reorder their punch cards. And if we accessed different parts of our hard drive too fast, the whole thing would start shaking. I'm telling you kiddo, you don't know how good you have it.

Homework: None

Prerequisites: None.

Martin's Axiom. (🌶🌶🌶🌶, Steve & Susan, 2 days)
There are lots of senses in which countable sets are "small." For example, for any countable set of functions $\mathbb{N} \to \mathbb{N}$, there is another function which is eventually greater than all of them. But what if we try to go beyond the countable? For example, how many functions do I need before no single function grows faster than all of them?

It turns out that these sorts of questions are often not decidable using the usual axioms of set theory. We won't address this issue; instead, we'll see how a new hypothesis—*Martin's axiom* —solves the problem. Martin's axiom says roughly that under certain circumstances we can "do uncountably many things at once" when trying to build some object (like a function from $\mathbb{N}$ to $\mathbb{N}$). In this class we'll introduce Martin's Axiom and show how it can be used.

Homework: None

Prerequisites: Familiarity with ordinal numbers (= know how to count up to ω_1)

Randomized Algorithms. (🌶🌶🌶, Sam & Yuval, 1–2 days)
Are your algorithms too boring? Too slow? Do they always output the same answer? Do they always get the answer *right?*

If any of these troubles are plaguing you, you might be interested in randomized algorithms. These are algorithms that do not need to make every decision deterministically; they are allowed to toss coins, roll dice, and so on, and use the outcomes to affect how they run. This means that they are liable to make errors, but also comes with enormous advantages: many of these algorithms are truly beautiful feats of imagination, and some of them easily solve problems that we just don't know how to solve deterministically.

In this class, we'll take a quick tour through the world of randomized algorithms, seeing both some of the general theory and some of the most beautiful and powerful examples of such tools.

Homework: Optional

Prerequisites: Some experience with algorithms will be helpful, but not strictly necessary.