CLASS DESCRIPTIONS—MATHCAMP 2020

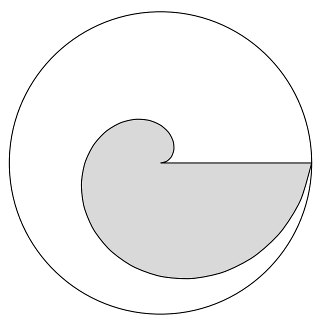
Classes

Ancient Greek calculus. (Yuval)

If you've ever seen a formal construction of the real numbers, you've probably heard of Dedekind cuts, named after the 19th-century German mathematician Richard Dedekind. However, he really doesn't deserve all the credit: 2000 years earlier, a Greek mathematician named Eudoxus of Cnidus came up with more or less the same definition. In my opinion, Eudoxus is the most important mathematician you've never heard of.

Even more, Eudoxus used his understanding of the real numbers to do what is essentially calculus; for instance, he was the first person to rigorously compute the volume of a cone. However, the mantle of ancient calculus was really picked up by Eudoxus's biggest fanboy, Archimedes. In my opinion, Archimedes is probably the most important mathematician you *have* heard of.

Building off of Eudoxus, Archimedes did some truly mind-blowing things. He computed the area of an arbitrary parabolic segment. He computed the volume and surface area of a sphere. He computed approximations of π. Perhaps most amazingly, he determined the area inside the following region, now called the *Archimedes spiral*.



If you've never seen this before, try it yourself—what fraction of the area of the circle is enclosed by the spiral? Even with modern integration techniques, the answer is not so easy to determine.

In this class, we'll get a sampling of ancient Greek proto-calculus. We'll start with Eudoxus's definition of the real numbers and we'll learn the "method of exhaustion", which was the proof technique he used to do calculus (it's more or less just evaluating an integral as a limit of Riemann sums). Then we'll move on to Archimedes and watch him do his magic, and we'll finish with his absolutely gorgeous argument for computing the area of a spiral.

Prerequisites: Having seen integrals and Dedekind cuts will be helpful, but not necessary.

An inquiry-based approach to group theory. (Katharine)

Groups are settings in which we can perform an operation, like addition or multiplication. Many groups are things you're already familiar with (we can add integers, or multiply real numbers), but we can also "add" symmetries of shapes, arrangements of objects, or states of a game. If you like Sudoku, logic grid puzzles, or arranging your bookshelf in creative ways, you might like groups!

We'll start by defining binary operations and groups, then look at tons of examples, prove some properties of groups, learn to present information about them in different ways, and explore relationships between groups. The "inquiry-based" part of the title means here that you can expect to use your time outside of class proving theorems and working out examples, and a lot of class time sharing your ideas and discussing ideas with the rest of the class (as a group, one might say).

Prerequisites: None.

A Rubik's cube-based approach to group theory. (Alan & Dennis)

The Rubik's cube is a very mathematical object! For example, if U, D, L, R, F, B denotes turning the up, down, left, right, front, back face 90 degrees clockwise (respectively), then we can write $R^4 = \text{Identity}$ to mean that doing R four times brings us back to where we started. We can also write more complicated equations, such as $(R^2U^2)^6 = \text{Identity}$.

In this class, we'll introduce some fundamental concepts in group theory through the Rubik's cube. By studying the structure of the Rubik's cube group, we will naturally be led to ideas such as group actions, permutations, symmetries, invariants, commutators, and semidirect products. With these and other tools, we will explain why certain states, such as having one flipped edge, are unreachable and derive the number of possible positions a Rubik's cube has.

In addition to using the Rubik's cube to motivate concepts in group theory, we'll also use group theory to help us understand the cube. We'll see how to use commutators to solve the cube. One benefit of this approach is that you can understand every move you are making—you don't have to memorize any magic "algorithms". We'll also see how permutations play an important role in both blindfolded cubing and fewest moves competitions.

Prerequisites: None officially. Some knowledge of basic group theory would help, but is not necessary. It would be helpful if you have a Rubik's cube and know how to solve it. If you would like to learn, here is a YouTube tutorial made by reputable speedcubers: `https://www.youtube.com/watch?v=1t1OL2zN0LQ`. Or (shameless self-promotion ahead) you could check out Alan's website: `http://learn2cube.com/beginners/intro`. Ask us if you have any questions!

A tour of Hensel's world. (Mark)

In one of Euler's less celebrated papers, he started with the formula for the sum of a geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and substituted 2 for x to arrive at the apparently nonsensical formula

$$1 + 2 + 4 + 8 + \cdots = -1.$$

More than a hundred years later, Hensel described a number system in which this formula is perfectly correct. That system and its relatives (for each of which 2 is replaced by a different prime number p), the p-adic numbers, are important in modern mathematics; we'll have a quick look around this strange "world".

Prerequisites: Some experience with the idea of convergent series.

Avoiding arithmetic triples. (Misha)

Three-term arithmetic progressions like $1, 2, 3$ or maybe $89, 97, 105$ are the worst. If you hate them as much as I do, you might be on board with my plan to "fix" the number line, get rid of some natural numbers, and avoid all these arithmetic triples.

You might be less on board with my plan if it turns out that my strategy is to keep only the numbers

$$1, 2, 4, 8, 16, 32, 64, \dots$$

and get rid of every number which isn't a power of 2.

Is there a less radical solution? Come to this class and find out!

Prerequisites: None.

Bairely complete. (Ben)

It's well-known that the real numbers are uncountable, due to the elegant diagonalization argument of Georg Cantor. It's also well-known that the Cantor set has the same cardinality as the real numbers. Can you write the reals as a union of countably many Cantor sets? You might be tempted to reach for measure theory here (which studies the lengths of sets), but measure theory will not help us here: there are Cantor sets with positive length, so infinitely many of them can still cover an infinitely long line.

There are some functions that are continuous, but nowhere differentiable. How abundant are these "Weierstrass functions"? Is the set of these "small" or "large"? What do we even mean by "small" and "large" in this context?

One generalization of Cantor's diagonalization argument is the Baire[1] Category[2] Theorem. This gives one possible answer to the question of what we mean by "small" and "large" here, and lets us figure out whether the set of nowhere differentiable functions is "small" or "large" in this sense.

It also lets us answer a lot of other questions. Can you find a function that's continuous at every irrational, and discontinuous at every rational? If you've done that, can you do the reverse, finding a function continuous at each rational number and discontinuous at each irrational? There are some functions whose derivatives are not everywhere continuous—but can we at least say that derivatives are "usually" continuous, that is, continuous on a "large" set? Or are there functions whose derivative is discontinuous on a "large" set?

Prerequisites: Some exposure to epsilon-delta and uniform convergence. Topology might help you orient yourself in the first days of this course, but is not necessary.

Block designs. (Emily)

Suppose you are running a scrabble tournament and 13 people show up to play. You wish to structure the tournament so that each game consists of four people, and each pair of people plays against each other in some game exactly once. Is this structure possible? If so, how many games must be played?

Now suppose the year is 1850 and you are Thomas Kirkman. You wish to solve the following problem: "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."

[1] Not that kind of bear.

[2] Not that kind of category.

Both of these problems can be understood using block designs: a set together with a collection of subsets that satisfy some certain conditions. We will explore some properties of block designs and how we can construct them; this will involve some combinatorics and in some cases projective planes.

Prerequisites: None. Some basic linear algebra may be helpful at a few points, but it is not required.

Brooks' theorem blues. (Misha)

(…and reds, and purples, and oranges, until we get to Δ colors…)

Brooks' theorem says that a graph with maximum degree Δ has chromatic number at most Δ as well, except for two cases: odd cycles, and complete graphs. In this class, we'll see several proofs of this theorem and its extensions, with detours to topics such as Kempe chains, list coloring, and complexity theory.

Prerequisites: Some graph theory: you should be able to understand and prove the statement that any graph with maximum degree Δ has chromatic number at most $\Delta + 1$. (One worse than Brooks' theorem.)

Cantor, Fourier, and the first uncountable ordinal. (Ben)

Where did set theory come from? The usual answer is that it was "motivated by Cantor's work in real analysis," to quote Wikipedia. Further investigation reveals that it was due to Cantor's investigations of trigonometric series, in particular, which motivated these discoveries. Such series were widely studied in the 19th century, due to the work of Fourier on the heat equation, but many basic questions about them remained open in the 1860s.

One of these questions is natural: Can a function be represented as a trigonometric series in more than one way? Some partial work had been done on this question before Cantor, but the general problem was wide open.

In this course, we will work through these partial results to catch up to the state of the art in 1870, when Cantor was working on precisely this question. We will then follow Cantor as he answers this question, and as he takes the first steps on the road to set theory.

Prerequisites: Knowledge of limits, series, and integrals. Having encountered uniform convergence would be helpful.

Cantor's leaky tent. (Ben)

One of the notorious counterexamples in point-set topology is called "Cantor's Leaky Tent" or the "Knaster–Kuratowski Fan." This space is connected! But there's one particular point that, when removed, makes the space *totally disconnected.* In this class, we'll go over all of these terms, put up our tent, and prove that it does exactly what it's supposed to.

Prerequisites: Some point-set topology. Knowing, or at least being willing to accept, the Baire Category Theorem.

Classifying complex semisimple Lie algebras. (Kayla)

In this class, we will be dipping our toes into the vast subject of Lie Theory! We will give some motivation why Lie theory is the intersection of all of mathematics and focus on Lie algebras. The goal of the week will be describing the structure of Lie algebras through showing that their eigenspaces have the beautiful combinatorial structure of a root system.

Prerequisites: You should be comfortable with linear algebra, specifically eigenspaces, linear transformations, actions of vector spaces. Also having an understanding of basic abstract algebra topics such as groups, conjugacy, definition of an algebra would be great. Lastly, knowing some point set topology would be great: definition of a topology, basic topological constructions, more to come.

Clopen for business: an inquiry-based approach to point-set topology. (Katharine)

Topology, like geometry, studies things we might describe as "shapes". However, topology regards shapes less rigidly than geometry does. We call these squishier shapes "spaces".

A topological space is made up of a set of points and a designation of which subsets of these points are "open". We can't necessarily measure distances or angles, nor can we necessarily draw our space and get information about it that way. So what can we do?? With only a definition of open sets, we can build things like sequences (and their limits), continuous functions, and decompositions of spaces into connected pieces. In this class, we will do all these things and more!

This is a presentation-based IBL class. Since the class is only four days long, I will ask you to look at some homework problems before the first class.

Prerequisites: Proof techniques, set notation

Combinatorial game theory. (Tim!)

Ania graffitied Mathcampus! She put 7 gnomes in the amphitheater, 10 aliens in the formal garden, and 12 bears on the President's lawn! You and a friend decide to clean it up, as a game. You alternate turns; on your turn you pick a location, and remove any number of pieces of graffiti (at least one) from (only) that location. You'd each like to be the one to remove the last of the graffiti, because that person will get all the credit and glory. Should you go first or second? What's your strategy?

The solution to this game is beautiful and surprising. And there are many other games with a similar flavor: these are called impartial combinatorial games. In fact, some of you played such a game against me in Week 1 relays! To solve these games, you need two tools. Part of the fun is figuring them out, so in this class, you will do that! Plus, we will see many examples of games, from the easy to the still-unsolved.

Prerequisites: None.

Combinatorics of tableaux. (Emily & Kayla)

Do you like combinatorics? Do you like changing hard complicated algebraic structures into pictures? We SCHUR do! Come learn about the combinatorics of tableaux with your favorite Minnesota dynamic duo. For the first day or two, we will introduce tableaux, various combinatorial identities associated to them, and RSK (Really Spicy Kombinatorics?).

t	a	b	l	e	a	u	x
e	m	i	l	y			
k	a	y	l	a			
f	u	n					

For the remainder of the class, we will be exploring the connection between tableaux and various group representations. Spicy! (3 chilis to be exact). Come get exposure to a little representation theory and algebraic combinatorics with us!

Prerequisites: Linear algebra, group theory

Complex analysis. (Alan)

We'll define what a contour integral in the complex plane is, and prove a nonempty subset of the following fundamental theorems from complex analysis: Cauchy's integral theorem, Cauchy's integral formula, analyticity of holomorphic functions, residue theorem.

Prerequisites: Multivariable calculus, specifically contour integrals (a.k.a. line integrals) and Green's theorem. Uniform convergence.

Complex dynamics: Julia sets and the Mandelbrot set. (Neeraja)

If $p(z)$ is a polynomial, the sequence of *iterates* is the sequence

$$p(z), p(p(z)), p(p(p(z))), p(p(p(p(z)))), \ldots$$

for a fixed complex number z. For what values of z does this sequence converge? Diverge? For what values of z is it periodic? These questions led mathematicians Julia and Fatou to define certain sets, one of which is the filled Julia set, the set of all z for which the sequence of iterates is bounded. In this class, we'll draw some pictures and study some properties of filled Julia sets. In the process, we'll also come across the Mandelbrot set, which has been called "the most fascinating and complicated subset of the complex plane."

Prerequisites: Complex numbers (taking the modulus, writing a complex number in polar form). If you don't know this but would like to attend the class, please talk to me!

Complexity theory. (Linus)

Finding elegant proofs of theorems is hard. Playing chess is hard. Solving Sudoku puzzles is hard. But which is hardest?

The answer is chess.

It turns out that if you can solve an arbitrary $n \times n$ Sudoku puzzle quickly, then you can also find a short proof of an arbitrary theorem quickly (as long as one exists). So, Sudoku is at least as hard as doing math research.

However, for quite involved reasons, mathematicians believe that being able to solve Sudoku puzzles does *not* enable you to solve a chess problem quickly. If you manage to prove them correct, then you will become famous overnight. On the contrary, chess is known to be exactly as hard as e.g. Portal 2.

If an all-powerful alien descended to earth claiming that chess is a win for the first player, then how would it convince us? Just beating us over and over again wouldn't be convincing—that'd only prove the alien is good at chess, not that its strategy is optimal. But it turns out there is a way the alien could do it, using polynomials over finite fields.

Topics: P vs. NP vs. PSPACE vs. EXP, and a host of others; oracles; zero-knowledge proofs; how not to prove $\mathsf{P} \neq \mathsf{NP}$; and more! (Though not the alien chess thing. It's too hard.)

Note: despite the frivolous nature of this blurb, this class will be about the theory of computational hardness, not about fun games like Portal 2.

Prerequisites: Know what an algorithm is. Grasp intuitively the cliff between reading a 400-page book and reading all possible 400-page books. It'd be cool if you were reasonably familiar with big-O notation.

Computing trig functions by hand. (Misha)

When you learn about trig functions, you typically memorize a few of their values (for 30° or 45°, say) and if you want to know any of the other values, you get pointed to a calculator.

Has that ever seemed unsatisfying to you? If so, take this class, in which we'll see that finding some of these values is as easy as solving polynomials, and approximating all of them is as easy as multiplication. If time allows, we'll learn how to compute inverse trig functions, and also how to quickly find lots of digits of π.

Prerequisites: Be familiar with the formula $e^{ix} = \cos x + i \sin x$.

Conflict-free graph coloring. (Pesto)

You put some cell towers on a graph, each broadcasting at some frequency. Every vertex in the graph is a cell phone that needs to be able to listen at at least one frequency, but can't listen at a frequency if two towers adjacent to it are both trying to broadcast at that frequency. How many cell towers and how many distinct frequencies do you need?

This problem defines a version of graph coloring called "conflict-free" graph coloring. For this new version of graph coloring, we'll prove an analogue of the most important unsolved problem in graph theory, generalize the four-color map theorem, and prove that we (probably) can't solve it efficiently in general.

Prerequisites: None, but having seen a proof of NP-hardness would be nice.

Congruences of Bernoulli numbers and zeta values. (Eric)

The Riemann zeta function is a wonderful thing that packages many of the mysteries surrounding prime numbers into the form of an analytic function. Among the many interesting things about this function are its values at integers: there is a precise formula for $\zeta(2n)$ in terms of powers of 2, π, some factorials, and a Bernoulli number. You may be familiar with the equality $\zeta(2) = \pi^2/6$. In the 1850s Kummer found a deep connection between these special values of ζ and the arithmetic of the integers, allowing him to prove some cases of Fermat's last theorem by understanding congruences of these special values.

We'll start the class by learning what Bernoulli numbers are and relating them to special values of ζ (we will be pretty cavalier about issues of convergence in this part). With that in hand we'll develop the theory of integration on $\mathbb{Z}/p^k\mathbb{Z}$ in order to give a clean proof of the Kummer congruences: if $n \equiv m \mod (p-1)$ then $B_n/n \equiv B_m/m \mod p$. (In fact we'll extend this statement to similar congruences mod p^k.) Secretly what we'll be doing is showing that there is a p-adic Riemann zeta function built by interpolating special values of the usual ζ function.

A note on format: I'd like to try and have this class run using in-class worksheets and mini lectures instead of regular lecturing, so expect that for the first few days at least. If it seems like that is not working well we may switch to a more traditional lecture format.

Prerequisites: Modular arithmetic at the level of knowing which elements of $\mathbb{Z}/n\mathbb{Z}$ are invertible. You should be happy with Fermat's little theorem ($x^{p-1} \equiv 1 \mod p$ if $x \not\equiv 0 \mod p$). Having seen a formal power series before would be nice, maybe at the level of knowing the power series expansion of e^x. Knowing the definition of integrals through Riemann sums is not necessary but many things will make much more sense if you do.

Connections to category theory. (Katharine)

Imagine yourself in a 9am group theory class. Your teacher defines the direct product of groups, $G \times H$, as the set of all pairs of an element of G and an element of H, with componentwise multiplication. "Huh," you think to yourself, "this sure seems like a Cartesian product of sets, but with groups". At 10am you go to a graph theory class where your teacher tells you about the tensor products of graphs. This, too, feels strangely familiar. After lunch, you're in a topology class where your teacher defines a product space. "What??" you think, "Am I trapped in some sort of weird time loop? Are the mentors so tired they could only write one set of course notes between them?"

There are many wild and wonderful parts of math that are peculiar to their particular fields. (Fortunately, we don't actually write only one set of course notes, so you can learn all about them!) There are also many constructions that are wonderful in part because they pop up repeatedly across so many fields. Category theory gives us language to precisely define these kinds of constructions in a way that can implemented in a multitude of settings. We'll focus on the basic language of categories and universal properties. Universal properties are a way of defining mathematical objects by what they **do** rather than how they're built.

Because we're focusing on drawing connections between different areas of math, it will definitely help to have seen a few different areas. I'll talk about groups sets, topological spaces, graphs, vector spaces, and more. However, it's not crucial that you are familiar with every single example. As long as you are comfortable hearing some things you don't understand, you should feel free to skip a prereq or two.

Prerequisites: Groups and group homomorphisms, sets and set maps, at least 1 of: topological spaces, graphs, vector spaces, rings

Continued fraction expansions and e. (Susan)

The continued fraction expansion of e is

$$1 + \cfrac{1}{0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{8 + \cfrac{1}{\ddots}}}}}}}}}}}}}}$$

Okay, but seriously, though, why?!?! Turns out we can find a simple, beautiful answer if we're willing to do a little integration. Or maybe a bit more than a little? No previous experience with continued fractions necessary. Come ready to get your hands dirty—it's gonna be a good time!

Prerequisites: None.

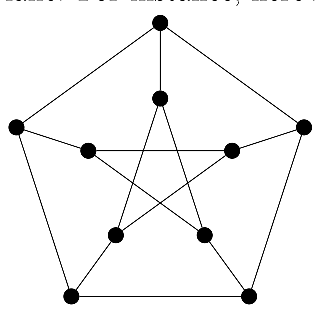
Counting, involutions, and a theorem of Fermat. (Mark)

Involutions are mathematical objects, especially functions, that are their own inverses. Involutions show up with some regularity in combinatorial proofs; in this class we'll see how to use counting and an involution, but no "number theory" in the usual sense, to prove a famous theorem of Fermat about primes as sums of squares. (Actually, although Fermat stated the theorem, it's uncertain whether he had a proof.) If you haven't seen why every prime $p \equiv 1 \pmod 4$ is the sum of two squares, or if you would like to see a relatively recent (Heath-Brown 1984, Zagier 1990), highly non-standard proof of this fact, do come!

Prerequisites: None.

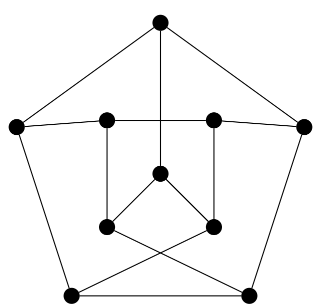
Crossing numbers. (Yuval)

We really like drawing graphs in the plane. For instance, here's a drawing of the Petersen graph.



Sometimes, when we draw graphs in the plane, some of the edges cross, which is a real bummer. Even worse, this is often unavoidable—if a graph is non-planar, then we will *always* have a crossing, no matter how we do our drawing.

Nevertheless, we can still try to do better. For instance, here's a different drawing of the Petersen graph:



As you can see, this drawing has only two crossings, which is better than the five crossings we had earlier. As it turns out, two crossings is the best we can do for the Petersen graph: its *crossing number* is 2.

Somewhat surprisingly, studying crossing numbers is an enormously fruitful activity. In this class, we'll prove a fundamental result about crossing numbers, and then use this result to say many interesting things about apparently unrelated areas of math. For instance, we'll use this to attack one of my favorite open problems in all of math: what is the largest number of unit distances that can exist among n points in the plane?

Prerequisites: Basic graph theory: you should know what it means for a graph to be planar. A bit of probability will be helpful but not required.

Cubic curves. (Mark)

A curve in the x, y-plane is called a cubic curve if it is given by a polynomial equation $f(x, y) = 0$ of degree 3. Compared to conic sections (which have degree 2), at first sight cubic curves are unpleasantly diverse and complicated; Newton distinguished more than 70 different types of them, and later Plücker made a more refined classification into over 200 types. However, as we'll see, by using complex numbers and points at infinity we can bring a fair amount of order into the chaos, and cubic curves have many elegant and excellent properties. One of those properties in particular, which is about intersections, will allow us to prove a beautiful theorem of Pascal about hexagons and conic sections, and it will also let us define a group structure on any cubic curve—well, almost. We may have to leave out a singular ("bad") point first, but a cubic curve has at most one such point (which may be well hidden; for example, $y = x^3$ has one!), and most of them don't have any. Cubic curves without singular points are known as elliptic curves, and they are important in number theory, for example in the proof of the Fermat–Wiles–Taylor theorem (a.k.a. "Fermat's Last Theorem"). However, in this week's class we probably won't look at that aspect at all, and no knowledge of number theory (or even groups) is required. With any luck, along the way you'll pick up some ideas that extend beyond cubic curves, such as how to deal with points at infinity (using "homogeneous coordinates"), what to expect from intersections, and where to look for singular points and for inflection points.

Prerequisites: A bit of differential calculus, probably including partial derivatives; complex numbers; a bit of experience with determinants.

Cut that out! (*Zach Abel*)

Let's cut shapes into other shapes and assemble them into even more shapes! We'll have a plethora of pretty pictures and a panoply of perplexing puzzles, including:

- Can you divide a square into (any number of) polygons that are similar to each other but all have different sizes? Can you do it with just 2 pieces? 3? Try it!
- Can you cut a square into (any number of) polygons and rearrange them exactly into an equilateral triangle? What about funkier polygons? If we allow curved cuts, can you rearrange the square into a circle? What about a cube into a tetrahedron using 3D pieces? What if we allow infinitely many cuts? Or fractal cuts? or ...

Shaaaapes!

Prerequisites: None.

Determinantal formulas. (Kayla)

How can we use linear algebra to help us answer counting problems? For example, how many ways can we stack boxes into the corner of an $n \times n \times n$ room? Why do we mathematicians care about such problems? In this course, we will be exploring how to express solutions to counting problems as determinants of matrices!

Prerequisites: None

Dirac delta function. (Alan)

The Dirac delta function, a.k.a. the unit impulse function, is the "function" which satisfies

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

This may seem like nonsense, but this function shows up naturally in many physical problems.

In this class, we'll talk about the theory of distributions (note that "distribution" has many different meanings in mathematics), which will allow us to describe the delta function rigorously and make sense of statements such as $\frac{d^2}{dx^2}|x| = 2\delta(x)$. In fact, we'll learn how to differentiate *any* function. Then we'll see some applications of all this.

Prerequisites: Single variable calculus, integration by parts.

Dominant eigenvalues and directed graphs. (*Yuyuan*)

Suppose we have the following vector equation:

$$Ax + b = x$$

for some positive invertible matrix A and nonnegative vector b, does there exist a non-negative solution for x? This question can be answered with the help of the Perron–Frobenius theorem, which states that for an irreducible matrices, the dominant eigenvalue (i.e. the eigenvalue with the greatest magnitude) is real and positive, and its corresponding eigenvector has all positive entries. This theorem has many practical applications, such as in the fields of population modeling, statistical mechanics, and economic modeling. However, most proofs of this theorem require a lot of linear algebraic machinery. In this class, we will see a proof of this theorem that does not involve lots of algebraic manipulations; instead, we will assemble a proof through constructing directed graphs from matrices and converting the process of matrix multiplication into a process on graphs.

Prerequisites: Familiarity with directed graphs, big-O notation, and eigenvalues.

Don't worry, these cats don't bite! (Basic category theory). (Dennis)

Categories have always been seen as an extremely high level concept, requiring a basic knowledge of several fields of mathematics already to understand. In this class, we'll try to explain some basic categorical concepts without anything more than a very rudimentary knowledge of what "sets" are. After all, the categorical language is incredibly intuitive once you get the hang of it; it's much more aligned to our usual intuition than how classical set theory constructs things. We'll be going over many examples, at a leisurely pace, of classical set theoretic constructions and how they are more naturally expressed in terms of categorical means.

Prerequisites: Intuition about sets as collections of things, of functions/maps as a rule assigning things of one set to another. Hopefully this makes it as accessible as possible.

Exploring the Catalan numbers. (Mark)

What's the next number in the sequence $1, 2, 5, 14, \ldots$? If this were an "intelligence test" for middle or high schoolers, the answer might be 41; that's the number that continues the pattern in which every number is one less than three times the previous number. If the sequence gives the answer to some combinatorial question, though, the answer is more likely to be 42. We'll look at a few questions that do give rise to this sequence (with 42), and we'll see that the sequence is given by an elegant formula, for which we'll see a lovely combinatorial proof. If time permits, we may also look at an alternate proof using generating functions.

Prerequisites: None, but at the very end generating functions and some calculus might make an appearance.

Extremal graph theory. (Mia)

Do you like to live on the *edge*? Are you an *edge-maximal*[3] sort of mathematician? Well, I've got a class for you. In extremal graph theory, we'll consider what happens when we take graphs to their max.

Let H be your favorite graph. We'll consider the following question: Given a graph G with n vertices, what is the minimum number of edges you need to guarantee that H is a subgraph? This question will lead us to the proof of Mantel's theorem, Turán's theorem, and finally, the statement of the Erdős–Stone–Simonovits theorem, which gives beautiful bound on the number of edges required. However, the E–S–S theorem fails to give useful information for one major class of graphs. Whomp. Not to be deterred, we'll spend the last day and a half looking at partial fixes for that failure!

If you came to my class during Sneak Peek, there will be some overlap with the first day, but all subsequent days will be new material.

Prerequisites: Graph theory. Familiarity with AM–GM and Jensen's inequality will be helpful but not required.

Extremal set theory: intersecting families. (Neeraja)

In an n-element set, what is the largest number of subsets of which no one subset contains any other? What is the largest number of k-element subsets of which every pairwise intersection is nonempty? What if every pairwise intersection must have size exactly t? This course will answer some of these questions! We'll prove some classical results about families of subsets of $\{1, 2, 3, \ldots, n\}$ which intersect or fail to intersect each other in specific ways. Possible results include Sperner's theorem, Dilworth's theorem, Erdős-Ko-Rado theorem and Bollobás' two families theorem.

Prerequisites: Set notation (union, intersection, complement) and proof by induction.

Extreme extremal graph theory. (Mia)

A typical question in extremal graph theory asks, given a graph G with n vertices, how many edges does G need to guarantee that H is a subgraph? But what if I want not one graph H, but MANY? What if I want ALL of the cycles C_k, up to some fixed k? This class will look at a delightful proof of Bondy's theorem, which gives conditions that guarantee not one cycle, but all of them.

Prerequisites: Graph theory.

Fair squares (mod p). (Maya)

A number a is a square mod p if there is some x with $x^2 \equiv a \bmod p$. For example, 2 is a square mod 7, because $2 \equiv 16 = 4^2 \bmod 7$.

The distribution of squares mod p seems arbitrary, and in this class, we will show that it is in fact very similar to a random distribution. For example, knowing that a is a square mod p tells us almost nothing about whether $a + 1$ is. Along the way, we will build up machinery to count Pythagorean triples, solutions to the equation $x^2 + y^2 \equiv z^2 \bmod p$.

Prerequisites: Comfortable working with multiplicative inverses mod p

Finding the center. (Pesto)

Given n points in the plane, how can we find the center of the smallest circle containing them if:

(1) Programmer time is the main constraint;
(2) Worst-case runtime is the main constraint;
(3) Average-case runtime is the main constraint?

These have different answers!

Prerequisites: Understand the statement "An algorithm runs in time $O(n^2)$".

Fourier analysis. (Alan)

Around 1800, the French mathematician Jean-Baptiste Joseph Fourier accompanied Napoleon through Egypt. Egypt was very hot, and Fourier became interested in heat, so he developed Fourier series to solve the differential equation known as the "heat equation." (This is a story I heard from Elias Stein, the mathematician who taught me Fourier analysis.)

The central idea of Fourier series is to decompose a periodic function into pure oscillations (i.e. sine waves):

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

This is what our ears do when we listen to music; it explains why the C-sharp of a piano sounds different from same C-sharp of a violin. (In class, we'll see this with some demonstrations using the software Audacity.)

Fourier analysis has wide applications to other areas, including signal processing (e.g., wireless communication), number theory (e.g., Dirichlet's theorem on primes in arithmetic progressions), quantum mechanics (e.g., the Heisenberg uncertainty principle, which Neeraja will cover in Week 4), and Boolean functions (as in Tim!'s Week 1 class).

In this class, we will learn how to find the Fourier series of any periodic function, prove some basic properties, and see how this can be used to solve differential equations. We will also look at the Fourier transform, which is an analogue of Fourier series for functions which are not periodic. With the remaining time, we'll discuss some of the many applications.

Prerequisites: Single variable calculus: know integration by parts and what a partial derivative is

Fourier something something boolean functions. (Tim!)

At its most basic, a computer program takes in a string of ones and zeros, and outputs either *accept* or *reject*. Such a program evaluates a *boolean function*; that is, a function $f\colon \{0,1\}^n \to \{0,1\}$. Perhaps because of the computer connection, and also because the definition is so fundamental, people have been asking questions about boolean functions for a long time.

But how do you study boolean functions? How do you even represent them? Consider the majority function on five variables; this is the function $\text{Maj}_5\colon \{0,1\}^5 \to \{0,1\}$ given by

$$f(x) = \begin{cases} 0 & \text{if the input } x \text{ has three or more 0s} \\ 1 & \text{if the input } x \text{ has three or more 1s} \end{cases}$$

Instead of this succinct description, you could instead write the majority function as the polynomial $\text{Maj}_5(x_1,x_2,x_3,x_4,x_5) = 3x_1x_2x_3x_4x_5 - 3x_1x_2x_3x_4 - 3x_1x_2x_3x_5 - 3x_1x_2x_4x_5 - 3x_1x_3x_4x_5 - 3x_2x_3x_4x_5 + x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_3x_4x_5$. Some people would call this a waste of time. But others would call it *Fourier Analysis*.

Somehow, this actually has a bunch of applications. You can study election systems: Which voting systems (majority, electoral college, etc.) give voters the most power? Which voting systems are more vulnerable to tampering? You can prove Arrow's Impossibility Theorem that there is no "fair" voting system for an election with three candidates. You can study learning theory and cryptography.

A few notes:

- We will be doing Fourier analysis on $\mathbb{F}_2^n$. A lot of folks like to do Fourier analysis on $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ instead (go to Alan's class in Week 3!). They are connected but different. It's nice to see both, because there are beautiful underlying ideas that aren't necessarily apparent if you just see the real version.
- Fair warning that this class will have quite a few definitions to learn. But it's worth it! You'll want to do the homework so that the definitions don't become a jumble.
- I would have called this class "Fourier Analysis of Boolean Functions" but the Academic Ordinators have made it clear that I should not call this "analysis" to avoid confusion with actual analysis. They've used such strong language as "maybe leave out the word analysis?" and "to be clear, it's really not a big deal to put the word analysis in".

Prerequisites: None.

Functions you can't integrate. (Ben)

In AP calculus, it always seems as if differentiation is a lot easier than integration. In particular, for all of our old friends like sine, cosine, e^x and so forth, we can take their derivatives and write them down in terms of other old friends by following some simple rules. Integration, on the other hand, has a lot more "tricks" and weird techniques.[4]

Here, we'll explore these difficulties in integration, and prove that some easy-to-write-down functions, such as e^{x^2}, don't have an easy-to-write-down integral. To show this, we won't be doing any sort of analysis: there will be no ϵs or δs in this course. Instead, we'll be using the tools of ring theory to study this question. Along the way, we'll see a very nice way to describe the "functions we can write down" or the so-called "elementary functions" in terms of field extensions.

Prerequisites: Introductory ring theory, some linear algebra, and knowing the product rule for derivatives.

FUNdamental groups and friends: an introduction to topological invariants. (Katharine)

In topology, we often consider shapes ("spaces") up to some idea of equivalence (that is, we consider some spaces to be "the same"). This can make it difficult to tell spaces apart! A topological invariant is a machine that takes in topological spaces and spits out something more understandable—for example, a number or a group. If two spaces are "the same" then they will result in the same output. (However, if two spaces are not "the same" then we might still get the same output.) We'll look at a bunch of examples: the Euler characteristic, scissors congruence, fundamental groups, and homology groups (very roughly).

Prerequisites: Proof techniques, some group theory (definition of a group, some examples). A little point-set topology (definition of a space, continuous maps of spaces) is helpful, but not required.

Geometric programming. (Misha)

Geometric programming is a moderately obscure kind of optimization problem. Maybe you've heard of linear programming; it's a tiny bit like that but completely different.

It is about solving problems using the AM–GM inequality. A classic easy example: "If you have 40 feet of fence, what's the largest area you can fence off?"

When these problems have more variables and more constraints, there are multiple ways to apply AM–GM, and this leads to a beautiful duality theory that's a distant cousin of the LP dual and the Lagrangian dual. (If these words made no sense, good; one of my goals in this class is to show you how dual problems arise in optimization.) Geometric programming is a fun way to dip your toes into operations research without much background required.

Prerequisites: Mostly none; I will spend a bit of time on one day of the class talking about partial derivatives. If you're not comfortable with those, this shouldn't affect your enjoyment of the rest of the class.

Geometry of lattices. (*J-Lo*)

You are standing at an intersection in the town of Skewvillle. Like in many towns, Skewvillle has two sets of streets, each set consisting of evenly spaced parallel lines. Unlike in most towns however, the two sets of streets are nowhere near perpendicular, and the distance between adjacent intersections is somewhat absurd (see Figure 1).

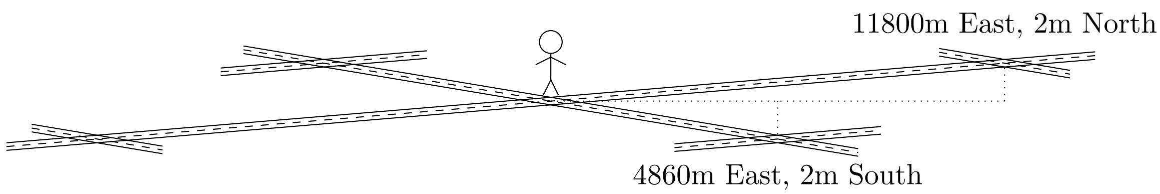


Figure 1. You and adjacent intersections (not to scale).

Your friend claims to be at an intersection that's only 52 meters away from you. If you have to stay on the roads,[5] how far will you need to travel in order to meet your friend?

Lattices are what you get when you take linear algebra and try to make it discrete. In some lattices, like the intersections in Skewvillle, "actual distance" (*as the crow flies*) and "step distance" (*follow the roads*) can be very different. We will prove as much as we can about the relationship between these two notions of distance, including two important theorems by Minkowski, a complete classification of all 2-d lattices, and an introduction to lattice basis reduction. We'll end with a bit of cryptography: finding your friend in a 200-dimensional version of Skewvillle may be so hard that the problem could save internet security as we know it today from quantum destruction.

Prerequisites: Linear algebra: you should know how to interpret a matrix as a linear map, be comfortable with multiplying matrices, and understand what the determinant of a linear map means geometrically. (This class did not make it onto the Prerequisites chart, so if you wanted to take this course but didn't take linear algebra, I or any of the mentors would be happy to get you up to speed.)

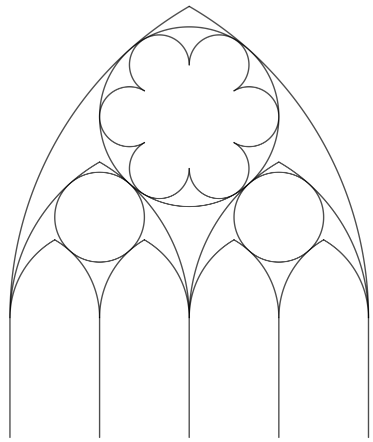
Gothic windows. (Kinga)

You probably have already heard about the cathedral Notre Dame in Paris. Maybe you've even visited it. When I was in Paris a few years ago, one of the things I remembered most were its huge, beautiful rose windows. But that's not the only building that has memorable windows! There are many more.

In this class, we'll take a closer look at gothic traceries. What were some frequently used patterns and shapes? You'll learn how to construct a few of the most common ones using only a compass and

[5]The law in Skewvillle requires you to stay exactly in the middle of the road at all times; no cutting corners.

a ruler, but there are also some that can't be constructed this way. We'll calculate lengths and talk about geometry. There will be lots of pictures!



Prerequisites: None

Grammatical group generation. (Eric)

Do you like silly word games[6]? Normal subgroups and presentations of groups[7] got you down? Come to this *extremely* light-hearted romp through the world of grammatically generated groups!

In this class, *based on a real actual published math paper*, we will use group theory to understand how many homophones and anagrams the English language has. If you think this sounds silly, it's because it is silly. But we'll do it anyways. Be prepared for terrible jokes and words you will never see used in any other context.

Prerequisites: It would be nice if you've seen normal subgroups and quotient groups before. If you're not super comfortable with them that is great! This class is a very gentle way to get better thinking about them.

Graphs on surfaces. (Marisa)

Suppose you want to draw a graph on the plane with no edge crossings, but your graph is not planar. Well, you could give up. Or you could change the rules of the game: start drawing, and before two edges cross, add an overpass to your plane, and then send one of those edges up the overpass. Problem solved, under your new rules (a.k.a. on the torus)!

This class will explore the question: given a graph, on what surfaces can we draw it without crossings? Our toolkit will be highly combinatorial, even including our definitions of surfaces. Our goal will be to fully answer this question for several families of graphs by the end of the week. We'll also prove an analogue to the Four-Color Theorem for every closed surface *except* the plane.

Style notes. This class will run in a hybrid format: short lecture, some group work, and a recap together.

Prerequisites: Misha's Introduction to graph theory, or familiarity with planar graphs and the proof of Euler's formula. (All the concepts from topology are self-contained and/or black boxed.)

Hilbert's space-filling curve. (Ben)

It is well-known that the unit interval in the real line (that is, the interval $[0,1] = \{x : 0 \leq x \leq 1\}$) has the same cardinality as the unit square in the plane (the set $[0,1] \times [0,1]$, that is). But if you look at the usual ways of showing that these are the same size, you'll find that the function you construct is not continuous everywhere. (For example, if you construct this bijection by "interleaving decimals", you'll need to make some arbitrary choices for numbers that have multiple decimal expansions. This doesn't matter much for whether it's a bijection, but it matters a LOT for whether it's continuous!)

In some ways this makes sense. Such a continuous bijection would sort of say that our spaces not only had the same *size* (which we knew) but also the same *shape*, which seems sillier. After all, a line looks different than a plane; in particular it seems like a curve should have a length, but not an area.

But some curves are too long to have a length, and instead fill up two dimensions of space. This course will show one construction of such a curve, which was one of the early examples of a fractal in mathematics. Nowadays, Hilbert's Space-Filling Curve can be found helping companies such as Google store multi-dimensional data (e.g. a map) in memory (which usually "looks like" something single-dimensional, more or less).

Prerequisites: Familiarity with uniform convergence (Introduction to analysis will be enough to follow this)

Homotopy colimits. (Dennis)

Let's take the circle S^1, and crush it down to a point. Now let's do it twice. Nothing changed, right? We just get the boring point as a result.

Well, let's look into this further. Crushing the circle down to a point, homotopically it's the same as gluing in a disk, so the circle is now filled in. After all, the disk is trivial homotopically; it's contractible! (This is exactly like with the fundamental group). If we do this process twice, we now have *two* disks, glued to the circle. This gives us an upper hemisphere and a lower hemisphere, glued at the equator, in other words S^2, the sphere! The sphere is definitely *not* a point. What happened?

The first case is a "strict" pushout, while the second is what's called a "homotopy" pushout. The second case is actually much better: for one, it remembers that we tried "crushing the circle to the point" *twice*, while the first one doesn't! Secondly, the second construction is a homotopy invariant! That means if you replace S^1 with a different, but homotopy equivalent, space, and perform the construction again, you'll get something homotopic to S^2 back, while the first one fails.

In this class, we'll go over some important constructions—like gluing, quotienting, taking special subspaces—of topological spaces. We'll also observe how they aren't great! They are almost *never* invariant under homotopy! We'll then go over the general process of correcting this, making everything "homotopical", or squishier, than it was before.

Prerequisites: Knowledge of topological spaces, homotopy, some gluing of topological spaces.

How not to prove the Continuum Hypothesis (week 1 of 2). (Susan)

When Cantor did his pioneering work in set theory he discovered that there are infinitely many different sizes of infinity—in particular, the real numbers are larger than the natural numbers. However, the obvious follow-up question—whether there are sizes of infinity in between—went unresolved for more than fifty years. This question become known as the Continuum Hypothesis.

In this class, we will explore what it means for a subset of the real numbers to be "small" or "large." We'll explore the mysteries of Cantor's middle-thirds set and discuss why logicians like to think of the real numbers as a tree rather than a line.

Finally, we'll discuss an alternative to the Axiom of Choice called the Axiom of Determinacy. This axiom allows us to express many questions about subsets of the real numbers in terms of two-player games, and to prove a result that looks an awful lot like the Continuum Hypothesis.

Prerequisites: None

How not to prove the Continuum Hypothesis (week 2 of 2). (Susan)

This is a continuation of last week's class!

Prerequisites: How not to prove the Continuum Hypothesis (week 1 of 2)

How Riemann *finally* understood the logarithms. (Apurva)

Logarithms are hard to define for complex numbers. (If you came to Jon's talk you know this all tpo well.) Euler settled the question by saying that the logarithm is a multi-valued function. But functions aren't allowed to be multi-valued! What's going on?

Riemann realized that the way to fix this is by not thinking about functions but instead studying graphs of functions. This led to the definition of a Riemann surface and resulted in the creation of half a dozen new branches of mathematics.

In this class, we will see how Riemann fixed the multi-valued logarithm problem and prove that an elliptic curve is a torus.

Prerequisites: You should know the polar decomposition of complex numbers and Euler's identity.

How to ask questions. (Eric)

In this class you will learn about asking questions and also ask questions, though possibly not in that order. You will have the opportunity to learn practical wisdom on how to ask questions in a mathematical context and how to be intentional about your question asking.

Your homework will be to ask questions, in this class and others.

Prerequisites: None.

How to glue donuts. (Apurva)

Mathematicians routinely encounter higher dimensional geometric objects. But our brains are incapable of imagining anything in higher dimensions. One way we circumvent this obstacle is by breaking up complex higher dimensional objects into simpler lower dimensional ones, like donuts.

In this class, we'll see how a donut can be a powerful tool in the hands of a mathematician and learn how to visualize four dimensional objects.

Prerequisites: None.

Hyperplane arrangements. (Emily)

They sound fancy, but hyperplane arrangements are pretty simple to define. In $\mathbb{R}^2$, they are collections of lines; in $\mathbb{R}^3$, they are collections of planes (and we can keep going into higher dimensions!). For example, cutting a pizza into slices produces a hyperplane arrangement, where the cuts are the hyperplanes. We will discuss how to classify the different pieces of hyperplane arrangements, and how to do operations on them.

Another thing that we will explore is how to count the number of slices that hyperplanes cut $\mathbb{R}^n$ into. This is obviously very easy in the case of a pizza, but in general it is not always so nice (especially when we are constructing arrangements that we cannot easily visualize). Some tools that we will use are posets, the Möbius function, and characteristic polynomials.

Prerequisites: None.

Infinitesimal calculus. (Tim!)

If you've learned the definition of *continuous function*, you may have learned that a function f is continuous if an infintely small change in x results in an infinitely small change in $f(x)$. This is a pretty good definition: it's short, and you can picture it on a graph, and you can see the connection to more geometric descriptions ("a function is continuous if you can draw its graph without lifting your pen"). It's also how many of the pioneers of calculus thought about the subject.

But if you take a proof-based calculus class, you might see this definition instead: *A function f is continuous at c if for all $\epsilon > 0$, there is a $\delta > 0$ such that for all x with $|x - c| < \delta$, we have that $|f(x) - f(c)| < \epsilon$.* What an ugly definition! To be sure, it's correct, and is often useful, but nevertheless it's clunky and counterintuitive. Why would any class use it instead of the "infinitely small" definition? The problem is that there is *no such thing* as an infinitely small (or *infinitesimal*) real number.

Most proof-based calculus classes usually throw in the towel on infinitesimals at this point and haul out ϵ and δ instead. But not us. We'll just add some infinitesimal numbers to the real numbers to get the *hyperreal numbers*. And we'll get to have nice definitions like the one at the start of this blurb. We'll go through the process of defining the hyperreals. Then, we'll visit some of the the highlights of a calculus class, with proofs that are correct and often much simpler than the standard ones, but which nevertheless are alien and bizarre.

You see, when you start playing with with the fundamental building blocks of reality, things can start going totally bananapants. And perhaps we'll come to understand why most calculus classes shy away from infinitesimals.

Prerequisites: Some calculus.

Information theory. (Mira)

In 1948, Claude Shannon published a paper called "A Mathematical Theory of Communication." By the time the paper came out as a book in 1949, its name had changed to "The Mathematical Theory of Communication." It took only a year for people to realize that what Shannon had invented was *the* theory—now usually called information theory.

All sorts of communication channels existed in Shannon's day: telegraph, telephone, radio, and TV, not to mention plain old human writing and speech. Shannon's insight was that all these different media could be analyzed within a single mathematical framework: the transmission of *information*, a concept that could be defined mathematically. Shannon showed that any channel—even a very noisy one, with lots of errors and distortion—has a certain rate at which it can transmit information virtually error-free. Anything up to that rate is possible, at least in theory; anything beyond it is hopeless.

Shannon's paper was the mathematical foundation of the digital revolution: every digital device that you've ever used runs on information theory just as surely as it runs on electricity. But the basic framework of information theory is actually quite elementary. In this course, I hope to let you discover a lot of it on your own—while solving some really fun problems along the way. To begin with, we'll have to define exactly what we mean by "information"; for this we'll need some probability theory, which we'll pick up as we go. We'll prove Shannon's Noiseless Coding Theorem, and while we may not get to the full proof of the Noisy Coding Theorem (aka the Channel Capacity Theorem), we'll definitely get far enough that you'll understand the statement and the intuition behind the proof.

Prerequisites: Basics of probability theory: discrete random variables, expected value, joint and conditional probabilities, Bayes' Rule. *You can still take the class if you are not solid on these concepts: we won't review them in class, but you can learn them through the homework and ask me about them at TAU. However, in this case, you should consider the class to be homework-required!*

Integration on manifolds. (Neeraja)

In single-variable calculus, integrals look something like this: $\int_a^b f(x)\,dx$. In this class, we will define and learn to evaluate integrals of the form $\int_M f$, where M is a manifold (e.g. a sphere or a torus). In order to define this integral, we will introduce differential forms, which turn out to be the "correct" objects to integrate on a manifold. After this we will prove the generalized Stokes' theorem, from which many classical integration results, such as Green's theorem, the classical Stokes' theorem and the divergence theorem, can be easily derived. We will derive at least one of these classical integration results and evaluate some integrals on manifolds.

Prerequisites: Familiarity with vectors (addition, scalar multiplication, dot product) and single variable calculus up to the fundamental theorem of calculus.

Introduction to analysis. (Alan)

This class is a rigorous introduction to limits and related concepts in calculus. Consider the following questions:

(1) Every calculus student knows that $\frac{d}{dx}(f+g) = f' + g'$. Is it also true that $\frac{d}{dx}\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n'$?
(2) Every calculus student knows that $a+b=b+a$. Is it also true that you can rearrange terms in an infinite series without changing its sum?

Sometimes, things are not as they seem. For example, the answer to the second question is a resounding "no." The Riemann rearrangement theorem, which we will prove, states that we can rearrange the terms in infinite series such as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ so that the sum converges to π, e, or whatever we want!

To help us study the questions above and many other ones, the key tool we'll use is the "epsilon-delta definition" of a limit. This concept can be hard to work with at first, so we will study many examples and look at related notions, such as uniform convergence. Being comfortable reasoning with limits is central to the field of mathematical analysis, and will open the door to some very exciting mathematics.

Prerequisites: Single-variable calculus (you should know what derivatives and integrals are)

Introduction to combinatorial topology. (Kayla)

Do you like combinatorics? Do you like topology? Ever wondered if there is any intersection between the two areas? It turns out that we can make topological spaces out of poset structures! Learn about what these topological spaces look like when we discuss the property of being shellable.

Prerequisites: Having some familiarity with topology is nice! Also having seen posets and Hasse diagrams would be good.

Introduction to Coxeter groups. (Kayla)

I want to begin by defining Coxeter systems and state the classification of Coxeter systems. Then I want the campers to play with a bunch of examples (I have about 10 in mind as of right now that would be good for them to work through). We will start with looking at what the Coxeter groups of simple Coxeter graphs are and work our way up to more technical examples like reflection groups and potentially discuss Weyl groups of roots systems. Time permitting and if there is interest, I would like to talk about some of the combinatorics of Coxeter groups introducing Bruhat order and weak order.

Prerequisites: Group theory.

Introduction to graph theory. (Misha)

A graph is an object with a bunch of things (called "vertices"), some of which have connections between them (called "edges"). You could argue that just about anything is a graph. So graph theory is the most important subject in all of mathematics.

There are some problems which it's more useful to study with graphs than others. For example, `https://en.wikipedia.org/wiki/Switzerland#Cantons` has a map of Switzerland's 26 cantons, colored with 6 colors. Can we use fewer, without coloring adjacent cantons the same color? (We'd like to at least get it down to 5, since it's confusing that both Wallis and the adjacent Lake Geneva are similar shades of light blue.)

This class is an overview of some of these problems. We won't spend too long on any of them, but we'll try to catch a glimpse of many different topics that are really graph theory under the hood.

Prerequisites: None.

Introduction to linear algebra. (Linus)

How indeed? It turns out one can rotate an image 45 degrees by using Paint's Resize tool to (1) skew 45 degrees horizontally, (2) skew -26 degrees vertically, and (3) stretch 50% horizontally. (Try it!) (But what if you want to rotate 30 degrees. . . ?)

These skews and stretches are 2-dimensional examples of *linear transformations*. And it is impossible to escape these beasts. So far this quarantine, I, personally, have used linear transformations for: algebraic number theory; machine learning; Markov chains; problem 5c on the qualifying quiz; and helping the EHT decide where to build a $2 million telescope. (Brag.)

This class will consist of $\sim 33\%$ lecture and $\sim 66\%$ learning through problem-solving.

Topics: vector spaces, rank, eigenvalues and eigenvectors, determinant, diagonalization, special matrices (e.g. symmetric, positive semidefinite), and more.

Prerequisites: Complex numbers.

Introduction to number theory. (Mark)

How do you *know* that 12345678913579147159161718192468258262728293693738394849 can be written as a product of primes in only one way (except for the order of the primes)? (There are number systems in which the analog of this is not true; for example, notice that $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, but it's not too hard to show that in the set of numbers of the form $a + b\sqrt{-5}$ with a, b integers, none of the numbers $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ can be factored further except in trivial ways.) Which integers are the sum of two (or the sum of three, or the sum of four) perfect squares? What postages can you get (and not get) if you have only 8 cent and 17 cent stamps available? How does the RSA algorithm (used for such things as sending confidential information, such as your credit card number, over the Internet) work? What happens if you take the Fibonacci sequence $1, 1, 2, 3, 5, 8, \dots$ and study how it repeats modulo n, for different values of n? We may not get to all these questions (and one of the answers is not even completely known), but we will touch on several of them as we explore some basic, beautiful, and subtle properties of our old friends, the integers. For thousands of years professional and amateur mathematicians have been fascinated by number theory (by the way, some of the amateurs, such as the 17th century lawyer Fermat and the modern-day theoretical physicist Dyson who passed away very recently, are not to be underestimated!) and chances are that you, too, will enjoy it quite a bit. Why 3 chilis? Because we'll move relatively fast!

Prerequisites: None (beyond modular arithmetic). Anyone who took the Mathcamp crash course in week 1 should certainly be fine.

Introduction to ring theory. (Eric)

A ring is a set of objects that you can "add" and "multiply." Many examples of rings come from two sources, arithmetic (rings whose elements are like numbers) and geometry (rings whose elements are like functions). We'll explore many familiar concepts (like modular arithmetic, prime factorization, division with remainder), what they mean in these two worlds, and the varied and interesting ways in which they can break down.

We'll start by recontextualizing how we think of the 4 arithmetic operations $(+, -, \times, \div)$, and figuring out how to make "modular arithmetic" work in any setting where we can "add" and "multiply." Then we'll work through a hierarchy of rings with extra structure, seeing how various nice properties die out as we relax how much extra structure we have, but also keeping some things alive by expanding our idea of what prime factorizations should be.

Homework is listed as required, but only 1 or 2 problems a day will be required. The required problems will be discussed at the start of each class to set up the day's lecture. Plenty of non-required problems will be available to explore examples and gain familiarity with concepts.

Prerequisites: None; in particular we'll run without group theory. Some examples will assume familiarity with other areas of math, hopefully with enough examples that everyone gets to interact with things they know.

King chicken theorems. (Marisa)

Chickens are incredibly cruel creatures. Whenever you put a bunch of them together, they will form a pecking order. Perhaps "order" is an exaggeration: the chickens will go around pecking whichever chickens they deem to be weaker than themselves, and whenever chickens encounter one another, it's a peck-or-be-pecked situation. Imagine you're a farmer, and you're observing the behavior of your chickens. You would like to assign blame to the meanest chicken. Is it always possible to identify the meanest chicken? Can there be two equally mean chickens? Are there pecking orders in which all the chickens are equally mean?

Prerequisites: None.

Let's reverse-engineer photoshop. (*Olivia Walch*)

In this two-day class, we're going to try to reverse engineer the math behind image editing software as best we can. We'll talk about color channels, pixel operations, the difference between raster and vector formats, how to make nice looking strokes with Bézier curves, and image compression. At the end of the class, everyone will be required[8] to draw me a beautiful picture with what they've learned.

Prerequisites: None.

Majorizing-Comparisons Solving of Problems. (Pesto)

High-school olympiads usually try to choose problems relying on as little prior knowledge as possible. In inequalities problems, they usually fail completely; training is necessary to solve most and sufficient to solve many of them. We'll go over the common olympiad-style inequalities, and solve problems like the following:

(1) Prove that if a, b, and c are positive and $ab + bc + cd + da = 1$, then $\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}$.

(2) [USAMO 2004] Prove that if a, b, and c are positive, then $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to olympiad-style problems you'll've solved as homework the previous day.

Prerequisites: None

Many Counterexamples, Some Pathology. (Some staff)

Do you want to see your instructors talk about awful stuff? In this class, various teachers will make reasonable-sounding statements and then tell you why they were wrong.

Some of the MYRIAD, CONFUSING, STRANGE POSSIBILITIES are listed below!

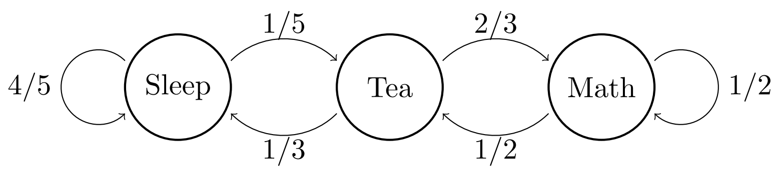
- Katharine is happy to do pathological spaces.
- Linus will show you a combinatorial geometry conjecture 'the answer to this problem is exactly $n+1$,' and then how the answer is really, really not $n+1$.
- Mira will state statistics that will stupefy you.
- Susan will show you a ring that does a truly terrible thing.
- Come hear from Ben, about bad things in analysis!

Prerequisites: None.

Markov chains and random walks. (Misha)

When (for whatever reason) I stay at home, can't leave my apartment, and don't have a regular schedule, I lose all track of time. I spend all my hours sleeping, drinking tea, and doing math.

Every hour, I make a random decision to change the activity I'm doing, according to the following diagram (where the numbers on the arrows denote probabilities):



This is an example of a Markov chain, and questions we might ask about it include the following:

- What fraction of the time am I drinking tea?
- When I wake up, how long do I stay awake before I go back to sleep?
- What's the probability that I'll go for a whole day without doing any math?

All of these questions can be answered in a boring way: by solving systems of linear equations. In this class, we'll learn to solve them in more exciting ways. These include making up a betting game about what I'm doing—or even transforming the Markov chain into an electric network!

Prerequisites: None; I'll make some offhanded references to linear algebra and graph theory, but neither one is needed to follow the class.

Math and literature. (Yuval)

Many Mathcampers (including me!) love reading, but we often think that reading and doing math are fundamentally different things. Though this is sometimes the case, there are many instances in which math and literature are inextricably related. In this class, we'll explore some incredible pieces of literature and discuss some of the math that went into their creation.

Among the amazing feats we'll see are the following.

- How one author wrote a book of sonnets that contains more poetry than the rest of humanity has ever produced, combined.
- How a nearly 200-year old conjecture due to Euler was eventually disproven, and how this disproof led to one of the most remarkable novels of the 20th century.
- How the plot of a book can correspond to the steps of a proof by contradiction, including how the big plot twist at the end yields the contradiction.

Note: Though I will provide some of the literature for you to read, doing so is totally optional. In particular, feel free to come even if you aren't comfortable reading in English!

Prerequisites: None

Mathcamp crash course. (Susan)

There are two fundamental parts to doing mathematics: the toolbox of notation and techniques that go into proofs, and the ability to communicate your ideas through writing and presentation. Most math books, papers, and classes (including at Mathcamp!) take these things for granted; this is the class designed to introduce and reinforce these fundamentals. We'll cover basic logic, basic set theory, notation, and some proof techniques, and we'll focus on writing and presenting your proofs. If you are new to advanced mathematics, or just want to make sure that you have a firm foundation for the rest of your Mathcamp courses, then this class is *highly* recommended. If you want to build up confidence in working with others mathematically, from simply asking questions in class to writing proofs for others to read to presenting at a blackboard, this class may also be right for you.

Here are some problems to test your knowledge of this fundamental toolbox:

(1) Negate the following sentence without using any negative words ("no", "not", etc.): *"If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."*
(2) Given two sets of real numbers A and B, we say that A *dominates* B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two disjoint, nonempty sets A and B such that A dominates B and B dominates A.
(3) Prove that there are infinitely many prime numbers.
(4) Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
(5) Define rigorously what it means for a function to be increasing.
(6) Prove that addition modulo 2013 is well-defined.
(7) What is wrong with the following argument (aside from the fact that the claim is false)?

    Claim: On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

    Proof: We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. QED.
(8) Explain what it means to say that the real numbers are uncountable. Then prove it.

If you would not be comfortable writing down proofs or presenting your solutions to these problems, then you can probably benefit from this crash course. If you found this list of questions intimidating or didn't know how to begin thinking about some of them, then you should *definitely* take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Prerequisites: None.

Matrix completion. (Linus)

Can you find the pattern and fill in the question marks in the following matrix??

$$\begin{pmatrix} 10 & 10 & ? & 4 \\ 6 & 10 & 9 & ? \\ ? & 8 & 6 & 5 \\ 3 & 7 & 7 & 9 \end{pmatrix}$$

Did you get it? The numbers represent how much (columns) me, my boyfriend, and my roommates enjoy (rows) Nichijou, Dark Souls, The Lobster, and League of Legends. I haven't seen The Lobster, so if you figure out that ?, please let me know.

Well... okay. We can't hope to solve this exactly. But with enough $\mathfrak{B}$ig $\mathfrak{D}$ata, and a dose of linear algebra, we can find a good approximation. It's machine learning!

Prerequisites: Linear algebra: you should be able to define rank, and prove that if $A : \mathbb{R}^m \to \mathbb{R}^k$ and $B : \mathbb{R}^k \to \mathbb{R}^n$ are linear maps then BA has rank at most k.

Modeling computation. (Mia)

How does one mathematically model a computer? Well, computers are really large, complicated, and gnarly, so instead, mathematicians work with idealized computers, called *computational models.* This class examines three important computational models, exploring the capacities and limitations of each. We'll start by examining deterministic finite automata, studying the computations they can execute and building clever DFA's to perform the computations of our choosing. From there, we will study nondeterministic finite automata and then pushdown automata. With these three models in hand, some natural questions arise: Are there computations that one model can perform but not the others? Are there computations that none can perform?

To make this more concrete, we turn to ... slackbot. As you may have noticed, slackbot performs the following *very important* operation, given a string of character, it determines if the string contains the word "yucca" in it (and then responds appropriately). This is a computation! One question you might ask is, can we build automata, of our given type, that determine if a string contains the word "yucca"? What about determining if the string contains an even number of "yucca"'s? Or what about determining if the string contains the same number of "yucca"'s as "cactus"'s? Come to this class and find out!

Prerequisites: None

Oh the sequences you'll know. (*Zach Abel*)

> I do not like Fib-nacho man,
> I'd rather talk of Catalan.
> Or Look and Say, even Farey,
> Oh 1,1,2,3 let me be!

We'll survey a variety of sequences that aren't as well known as the Fib****** sequence, but should be! Each day will explore a different sequence in detail, finding surprising links to number theory, geometry, combinatorics, and more!

Some sequences that may be covered: Thue–Morse, Lucas, Beatty sequences, Bell numbers, Up/Down numbers, derangements, Prüfer,...

Prerequisites: None

Perceptron. (Linus)

For the third year in a row, I refuse to teach neural networks at Mathcamp. (There'd be almost nothing I can prove.)

But I'll skirt the edges, by teaching the simplest unit inside a neural network: a perceptron. These babies solve the following problem: given points in some n-dimensional space labeled $+$ and $-$, how can we efficiently find a hyperplane separating all the $+$ from all the $-$ (assuming one exists)?

(What if only *most* of the $+$ and $-$ obey the rule, but there are some outliers? What if, instead of a hyperplane, a more complicated boundary (e.g. a polynomial) separates the $+$ from the $-$?)

Prerequisites: Vectors, dot products.

Perfect numbers. (Mark)

Do you love 6 (a big number!) and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called perfect, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes—a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Prerequisites: None.

Posets and the Möbius function. (Kayla)

Give an example-heavy introduction to posets and lattices in algebraic combinatorics following `https://arxiv.org/pdf/1409.2562.pdf` section 4. We will discuss types of posets and where they arise in math!

Prerequisites: None.

Quantum mechanics. (*Andrew Guo*)

Quantum mechanics is weird and nobody understands it[9]. But with the right mathematical tools, almost anybody can understand it well enough to use it! In this course, we will learn about the weird phenomenon of electrons that behave as if they can exist in two diametrically opposed states at the same time—i.e. in a quantum "superposition". We'll then see how superposition allows computers that run on quantum mechanics to perform computations in parallel, which gives them the power to solve certain problems more efficiently than any computer built on ordinary, classical mechanics (such as the one you're currently reading this on). Along the way, we will attempt to axiomatize quantum mechanics in the language of linear algebra and construct an example of a quantum algorithm with an exponential quantum speed-up.

Prerequisites: Linear Algebra

Ramanujan graphs, quaternions, and number theory. (*Dan Gulotta*)

Suppose you are designing a computer network. You would like to design the network so that information can be sent efficiently between any pair of computers. In theory, you could directly connect every pair of computers, but this could get very expensive. So in practice, each computer will only have a direct connection to a few others.

There are various ways of measuring the efficiency of a network. One of these is the *spectral gap*. Define a matrix A so that $A_{ij} = 1$ if computers i and j are connected, and $A_{ij} = 0$ otherwise. If each computer is connected to k others, then one of the eigenvalues of A will be k. The network is called a *Ramanujan graph* if all of the other eigenvalues have absolute value at most $2\sqrt{k-1}$. (This is, in a certain precise sense, almost the smallest that the eigenvalues can be.)

I will show how to use ideas from number theory to construct Ramanujan graphs. I will describe some surprising connections between these graphs and some seemingly very different kinds of objects. In particular, the fact that these graphs are Ramanujan is a consequence of the Ramanujan–Petersson conjecture, an important theorem in the theory of modular forms.

See more at the course website: `https://people.maths.ox.ac.uk/gulotta/mc20.html`

Prerequisites: Linear algebra, basic number theory (modular arithmetic), basic graph theory (just the definition of a graph). Knowing some group theory might be helpful.

Random walks and electric networks. (Misha)

In this class, I will tell you a few basic rules about how voltage and current in an electric network behave.

Then, we'll see that some properties of a random walk on such a network follow the same rules—and prove that any two objects that follow these rules must be the same object.

You don't need to have taken my Markov Chains class in week 2. If you did take that class, you will see completely new things about random walks, so you don't need to worry about being bored.

Prerequisites: None.

Regular expressions and generating functions. (Linus)

To cheat at Mathcamp's famed week 4 puzzle hunt, I use regular expressions. For example, if I know a puzzle answer uses the letters d, u, c, and k in that order, I can type the regular expression "*d*u*c*k*" into `onelook.com` to get a list of all English words it could be.

To count anything, e.g. the number of domino tilings of a $4 \times n$ rectangle, I use generating functions, a magical tool in combinatorics.

Learn how regular expressions and generating functions are the same thing, and use them together to instantly solve a bunch of problems like:

- "What's the most chicken nuggets I can't order if they come in 5-piece and 8-piece boxes?"
- "Why do rational numbers have repeating decimals?"
- Problem 5c on this year's Qualifying Quiz

NOTE: The first two days of this class will cover regular expressions and finite automata, which overlaps with Mia's class last week.

Prerequisites: none

(Relatively) prime complex numbers. (Emily)

What does it even mean for a complex number to be prime? Well first, we must restrict ourselves to some sort of "complex integers," specifically the Gaussian integers $\mathbb{Z}[i]$. There are multiple different categories of primes in this ring, and once we understand them all we can do lots of interesting things, such as computing prime factorizations!

Now what does it mean for two Gaussian integers θ and η to be relatively prime? It means exactly what you would expect based on the regular old integers: $\gcd(\theta, \eta) = 1$. The more interesting question to ask is "given $\theta \in \mathbb{Z}[i]$, what is the number of Gaussian integers relatively prime to and less than θ?" But wait, how do we know when a complex number is less than another?? Can I even ask if $-4 + 3i < 3 - 5i$??? It turns out we can answer this more interesting question without even talking about "less than"; instead, it comes down to understanding quotient rings in $\mathbb{Z}[i]$, and defining something called the Euler phi function.

We won't stop at just the Gaussian integers; we can explore an entirely different type of complex integers, the Eisenstein integers $\mathbb{Z}[\rho]$ where $\rho = e^{2\pi i/3}$. All the ideas we used to understand primes and the Euler phi function for the Gaussian integers extend *fairly* nicely to the Eisenstein integers!

Prerequisites: Ring theory, number theory, and familiarity with arithmetic in the complex numbers. Talk to me if you are unsure if you have the required background.

Representation theory of finite groups (week 1 of 2). (Mark)

It turns out that you can learn a lot about a group by studying homomorphisms from it to groups of linear transformations (if you prefer, groups of matrices). Such a homomorphism is called a representation of the group; representations of groups have been used widely in areas ranging from quantum chemistry and particle physics to the famous classification of all finite simple groups. For example, Burnside, who was one of the pioneers in this area along with Frobenius and Schur, used representation theory to show that the order of any finite simple group that is not cyclic must have at least three distinct prime factors. (The smallest example of such a group, the alternating group A_5 of order $60 = 2^2 \cdot 3 \cdot 5$, is important in understanding the unsolvability of quintic equations by radicals.) We may not get that far, but you'll definitely see some unexpected, beautiful, and important facts about finite groups in this class, along with proofs of most or all of them. With any luck, the first week of the class will get you to the point of understanding character tables, which are relatively small, square tables of numbers that encode all the information about the representations of particular finite groups; these results are quite elegant and very worthwhile, even if you go no further. In the second week, the chili level may ramp up a bit (from about $\pi + 0.4$ to a true 4) as we start introducing techniques from elsewhere in algebra (such as algebraic integers, tensor products, and possibly modules) to get more sophisticated information.

Prerequisites: Linear algebra, group theory, and general comfort with abstraction.

Representation theory of finite groups (week 2 of 2). (Mark)

This is a continuation of last week's class with the same name. If you would like to join, please check with me first; we can talk through what you may need to catch up on, and I can also give you access to the notes and the problem sets from last week.

Prerequisites: Linear algebra, group theory, general comfort with abstraction, and week 1 of the class.

Skolem's paradox. (Susan)

Holy Axiomatizations, Batman! A ninja has snuck into the Museum of Real Numbers and stolen all but countably many of them. You, the curator, have a huge exhibition tomorrow. What are you going to do? Why, it's simple! You'll use the Lowenheim–Skolem theorem to build a countable model of set theory, complete with the real numbers. From inside the museum, no one will be able to tell that it's countable. To keep real number ninjas from interfering in your life, come to our class.

Prerequisites: None.

Solving equations with origami. (Eric)

Put a piece of paper in front of you. Mark two points on it. Pretend that your paper is a piece of the complex plane, and that your two marked points are 0 and 1. Which other points in the plane can you construct by folding your paper and marking where your existing points fold to? If you allow arbitrary folds you can hit anything, but what if we restrict ourselves to folds we can "line up" using our existing points and lines?

We'll be able to answer this question precisely by developing a system of axioms for one-fold origami and analyzing their algebraic potential. Along our journey to understanding the limits of the algebraic power of origami we'll travel the world (Japan to India to Austria to Italy and back), gently encounter some flavours of math from abstract algebra and algebraic geometry, and employ a truly wonderful piece of 19th century mathematics to solve equations by shining lasers on turtles. If that's not enough for you, one of the historical characters we'll encounter has possibly the greatest name in mathematics: Margherita Piazzola Beloch (arguably the first person to really understand the algebraic power of origami).

Classes will be a mix of lectures and paper folding activities, so come prepared with a stack of paper you can fold!

Prerequisites: You should be comfortable with the concept of the dimension of a vector space. You do not need any prior knowledge of origami to follow this course.

So you like them triangles? (Dennis)

Are (topological) spaces weird? Haunting? Scary? These great untamed beasts of manifold shapes and forms have terrorized us for too long. Luckily for us, the humble triangle (simplex) has vowed to help us understand these horrific monsters. But how?

Here's the idea: we "triangulate" a space by cutting it up into simplices. Once we've done so, we can perform calculations, such as find a space's Euler characteristic, or more generally, its cohomology groups, which are most importantly *not spaces*! Instead they are objects of algebra, so we can use our favorite equation solving techniques to help us.

In this course, we'll go over what simplicial complexes (or delta complexes, or simplicial sets; there are many variants) are and how to use them to compute cohomology of a space. We'll focus on example calculations and using them to deduce things about some basic spaces. Using our tools, we can formally prove some cool results like Brouwer's fixed point theorem and the Jordan curve theorem, two theorems that are intuitively pretty obvious, but mathematically difficult. We'll also go over some of the main themes of algebraic topology, connecting our construction to the fundamental (or homotopy) groups.

Prerequisites: Some linear algebra, and perhaps some group theory. Calculating kernels and quotients will be important.

Spectral graph theory. (Ania)

When you think about graphs, you probably imagine some connected dots. However, graphs can be also represented as matrices! Spectral graph theory is a way of turning problems about graphs into linear algebra by associating a matrix to a graph (called the adjacency matrix) and studying its eigenvalues. In this class, we'll see applications of this method to prove some cool facts about graphs!

Prerequisites: Intro linear algebra, basics of graph theory

Stirling's formula. (Neeraja)

Stirling's formula gives an asymptotic estimate for $n!$, i.e. an approximation for $n!$ as n gets large. The formula first arose from correspondence between Stirling and de Moivre in the 1720s, when de Moivre used a version of the formula to discover what was essentially the central limit theorem in probability! (These days the central limit theorem is usually proved without using Stirling's formula.) In this class, we'll prove Stirling's formula and use it to solve some fun problems, such as showing the recurrence of the random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$.

Prerequisites: Single-variable calculus (integration by parts).

Teaching math to computers. (Apurva)

Wouldn't it be great if computers could automatically generate proofs of complex mathematical theorems? Or even come up with new math? But right now computers can't even *understand* basic undergraduate math curriculum. There has been a lot of progress in the recent years to improve this appalling situation and there is a lot more that needs to be done.

In this class, we will teach math to computers using the Lean proof assistant. Lean is a programming language that is almost human readable and looks very close to math. Example Lean code:

```
theorem fermats_last_theorem
(n : N)
(n_gt_2 : n > 2)
:
not(exists x y z : N,
    (x^n + y^n = z^n) and
    (x > 0) and (y > 0) and (z > 0))
:=
begin
    sorry,
```

```
end
```

We will teach computers some basic number theory results like the infinitude of primes and that $\sqrt{2}$ is irrational. The goal of this course is to simply introduce you to this fascinating world of computer assisted theorem proving in Lean so that you might do a project on it afterward. If you get *really* interested, then you can eventually even contribute to the official Lean math library. (This isn't far fetched as many contributions to the Lean math library are being made by undergrads.)

This will a DIY course. You'll learn the language by coding yourself. I'll simply provide the assignments and give short 10-minute introductions each day.

See the class website `https://apurvanakade.github.io/courses/lean_at_MC2020/introduction.html` to explore more.

Prerequisites: You do not require any special software to run Lean. You can run it in a web browser. You also do not require any serious coding experience but you should be interested in coding and find it enjoyable.

You *do* need to be very comfortable with the following terms: logical operators (and, or, if ... then ..., not, iff), quantifiers (for all, there exists), proof by contradiction, contrapositive, principle of mathematical induction.

The bell curve. (Mira)

> I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason.
>
> Sir Francis Galton, 1889

Human heights; SAT scores; errors in scientific measurements; the number of heads you get when you toss a million coins; the number of people per year who forget to write the address on a letter they mail.... What do all of these (and numerous other phenomena) have in common?

Empirically, all of them turn out to be distributed according to "the bell curve":

The bell curve, known in the 19th century as the "Law of Error", is now usually called the *normal* or *Gaussian* distribution. It is the graph of the function $e^{-x^2/2}/\sqrt{2\pi}$ (scaled and translated appropriately). We will see how Gauss derived this function from a completely backward argument—a brilliant leap of intuition, but pretty sketchy math. We'll see how the great probabilist Laplace explained its ubiquity through the Central Limit Theorem. (Maybe you've learned about CLT in your statistics class ... but do you know the proof?) We'll talk about how the normal distribution challenged the nineteenth century concept of free will. Finally, we'll look at some other mathematical contexts in which the normal distribution arises—it really is everywhere!

Prerequisites: Integral calculus. (There will be a *lot* of integrals!)

The Hilbert cube. (*Harini*)

You're probably familiar with a one dimensional cube—the closed unit interval. And two dimensional cubes are also easy—unit squares. Three dimensional cubes are normal, and maybe you've seen hypercubes somewhere or the other. That's fine and good, because these are all finite dimensional. What happens when we try to take an infinite dimensional cube? How do things work there? This is called the Hilbert Cube, and it turns out that generalizing properties of finite dimensional cubes to it can be done, but with a little bit of cleverness. For example, we CAN define distance in this cube! Not only that, but when we do so, we can find a copy of ANY sufficiently small metric space inside it. And even more surprisingly, we can actually write down how to find this copy! Come to my class to find out how to do this!

Prerequisites: None.

The John Conway hour (week 1 of 2). (Pesto & Tim!)

John Conway was a great mathematician who proved deep and important theorems, while also managing to work in a variety of fields. Incredibly, on top of this, he had a penchant for making math entertaining and accessible. Even more incredibly, every year, for many years, he spent a whole week at Mathcamp, teaching classes on his favorite topics and playing games with campers. Conway died this year from COVID-19. Mira, Misha, Pesto, and Tim! are teaching two weeks of classes in his memory. We all have memories from Conway's visits and classes that we would like to share with you.

The topics of Conway's classes were always "NTBA"—*not* to be announced. When you showed up to his class, you wouldn't know what he was about to talk about, and sometimes he wouldn't either (but he always made it exciting). We will bend this traditional structure a little bit; this week, Pesto and Tim! will be talking about:

- Monday: Rational Tangles
- Tuesday: Wallpaper Groups
- Wednesday: Conway's Soldiers
- Thursday: Doomsday Algorithm (for the day of the week)
- Friday: Look-and-Say Sequence

The days are independent; you can show up to one without having been to the others.

Prerequisites: None except for day 5, for which you should know what an eigenvalue is.

The John Conway hour (week 2 of 2). (Mira & Misha)

The John Conway extravaganza continues! This week, we'll be talking about one of Conway's favorite topics: games. As before, all the days are independent (although the last three days can also be taken as a unit).

- Day 1 (Misha, ): PRIMEGAME. Find out why the finite sequence
$$\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, 55$$
is actually a machine that generates prime numbers!
- Day 2 (Misha, ): Conway's Game of Life. A game in which cells in a grid evolve according to simple rules that can create complex patterns from simple starting conditions. This cellular life might reproduce to fill the plane, die off, or even perform arbitrary calculations.
- Day 3 (Mira, ): Numbers and Games. All numbers are secretly games, though not all games are numbers.
- Day 4 (Mira, ): Games and Codes. All impartial games are secretly error-correcting codes (another area in which Conway was famous), though not all codes are games.
- Day 5 (Mira, ): Games Conway played. John Conway invented several games you can actually play, including Phutball (Philosopher's Football) and Sprouts. Also, every time he came to visit Mathcamp, he would challenge campers to play Dots and Boxes—a game he did not invent, but almost always won. I'll tell you a little bit of the math behind these games, and also some Conway stories.

Prerequisites: None. Days 3–5 may have some overlap with Tim!'s Combinatorial Game Theory class, but neither class depends on the other.

The Kakeya needle problem, projective geometry, and fractal dimensions. (Alan)

Let's go through the three topics in the course title.

(1) The Kakeya needle problem asks the following question: Suppose you have a unit line segment (a "needle") in the plane and you'd like to rotate it 180 degrees, so that it points in the opposite direction. What is the area of the smallest region you can do this in? This problem can be solved with elementary geometric techniques, and the answer may not be what you expect!

(2) The real projective plane is like the Euclidean plane, except parallel lines intersect at "points at infinity." In this space, there is a magic trick. If you wave your wand and say the magic words ("point-line duality"), then all the points will transform into lines and vice versa. Furthermore, any theorem about points and lines that was true will still remain true!

(3) We can generalize the notion of "dimension" to talk about s-dimensional sets for any nonnegative real number s. This allows us to better understand sets such as fractals. For example, the Koch snowflake (look it up!) has infinite length and zero area, and turns out to be neither 1-dimensional nor 2-dimensional. It is actually $\log_3 4$-dimensional!

For the first few days, we will discuss these three topics independently of each other. Then we will see the surprising connections they have to each other, as well as to the Kakeya conjecture, a famous unsolved problem in analysis.

Prerequisites: None

The lemma at the heart of my thesis. (Eric)

In the words of my thesis advisor "mathematics is not about proving theorems, it's about proving lemmas." I'll tell you the story (and prove the lemma!) of the lemma at the heart of my thesis. (Spoilers: it's a lemma about the structure of quotient rings of $\mathbb{Z}[\zeta]$ where ζ is a root of unity.)

Prerequisites: You should know what a quotient ring is.

The matrix exponential and Jordan normal form. (Dennis)

You've heard of the exponential function. You might have heard that the exponential function helps us solve differential equations, especially of the form $y' = ay$. What if you have not just one equation, but a whole system of them? What do you do then?

Well, first of all, if they are all linear, you'd probably think to use linear algebra; namely to write the system as one matrix equation $x' = Ax$, where now x is a vector and A is a matrix. How do we solve this? If you guessed, as in the above case, that we should use "e^{At}", whatever that means, you'd be right!!

In this class we'll explore exactly what this matrix exponential is and how it helps us solve differential equations. Along the way we'll need the Jordan normal form, which is a generalization of diagonalization, that at least puts matrices in upper triangular form (not necessarily diagonal form). But it's enough for us to actually *compute* the matrix exponential!

If we have time, we'll also go over how the matrix exponential ties together the Lie group GL_n and its Lie algebra, as well as all the other matrix groups. (Don't worry if you have no idea what this means!)

Prerequisites: Basic linear algebra (familiarity with diagonalization), some calculus 2. For the proof of Jordan normal form only, we'll need some knowledge of polynomial rings.

The Plünnecke–Ruzsa inequality. (Milan)

You probably know how to add integers, but what about sets of integers? A natural definition for the sum of two sets of integers A and B is

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

What can we do with this? Given any set of integers A, it is not too hard to prove that $|A+A| \geq 2|A|-1$. Furthermore, we have equality exactly when A is an arithmetic progression. What about when we are close to equality? Intuitively, A should still have some arithmetic structure. The Plünnecke–Ruzsa inequality tells us that in some sense our structure is still preserved when we iterate adding A to itself many times. In this class, we will develop the necessary tools to understand the Plünnecke–Ruzsa inequality precisely, and then we will see a clever recent elementary proof.

Prerequisites: None

The puzzle of the superstitious basketball player. (Tim!)

Here's one of my favorite math puzzles. It's from Mike Donner, and it was published on FiveThirtyEight.

> A basketball player is in the gym practicing free throws. He makes his first shot, then misses his second. This player tends to get inside his own head a little bit, so this isn't

> good news. Specifically, the probability he hits any subsequent shot is equal to the overall percentage of shots that he's made thus far. (His neuroses are very exacting.) His coach, who knows his psychological tendency and saw the first two shots, leaves the gym and doesn't see the next 96 shots. The coach returns, and sees the player make shot No. 99. What is the probability, from the coach's point of view, that he makes shot No. 100?

I remember solving it. I had to do a bit of tedious calculation to arrive at the final answer. And when I saw the answer, I was astounded. It was so simple. I thought I was done with the puzzle, but really I was just beginning. Such a simple answer had to have a simple explanation, right? There are in fact a few simple explanations, each more satisfying than the previous.

In the end, I will make the following claim: even if we accept the scenario described by the puzzle, the basketball player's view of the world is totally wrong, and he is probably just superstitious. Perhaps there is a lesson here that we can take back with us to our real lives.

Prerequisites: None, but we'll spoil the answer to the puzzle pretty early in the class, so if you'd like to think about the puzzle yourself (which I wholly recommend), do it beforehand!

The redundancy of English. (Mira)

NWSFLSH: NGLSH S RDNDNT! (BT DN'T TLL YR NGLSH TCHR SD THT...)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting, or to have a conversation in a noisy (Zoom) room. The redundancy is a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

But can we quantify exactly *how* redundant English is? In other words, how much information is conveyed by a single letter of English text, relative to how much could theoretically be conveyed? We will answer this question in the way that Claude Shannon, the father of information theory, originally answered it: by (1) generating a bunch of amusing gibberish; (2) playing a word game that I call Shannon's Hangman, and using it as a way of communicating with our imaginary identical twins.

Prerequisites: The definition of information, which was covered on Day 1 of my Week 3 Information Theory class. If you were not in that class, you can take a look at the slides for Day 1, and DM me if you have any questions. The definition is pretty straightforward, and you don't need to know anything else from that class to enjoy this one.

The Riemann zeta function. (Mark)

Many highly qualified people believe that the most important open question in pure mathematics is the Riemann hypothesis, a conjecture about the zeros of the Riemann zeta function. Having been stated in 1859, the conjecture has outlived not only Riemann and his contemporaries, but a few generations of mathematicians beyond, and not for lack of effort! So what's the zeta function, and what's the conjecture? By the end of this class you should have a pretty good idea. You'll also have seen a variety of related cool things, such as the probability that a "random" positive integer is not divisible by a perfect square (beyond 1) and the reason that $-691/2730$ is a useful and interesting number.

Prerequisites: Some single-variable calculus (including integration by parts) and some familiarity with complex numbers and infinite series; in particular, geometric series.

The Sylow theorems. (Mia)

Suppose I give you a mystery group and all I tell you about it is its order. What can you tell me? A surprising amount, actually! For example, if I tell you that a group has order 77, you can tell me that it has exactly one subgroup of order 11 and exactly one of order 7. In fact, you can even tell me that it is an Abelian group, isomorphic to $\mathbb{Z}_7 \oplus \mathbb{Z}_{11}$. All you need are the Sylow theorems.

In this class, we'll learn not only how to perform this group sleuthing, but also why it works. We'll start by developing two extremely useful tools for partitioning groups, cosets and conjugacy classes, which give deep insights into the structure of a group. Then, we'll move on to prove Lagrange's theorem, which states that the order of a subgroup divides the order of a group, and its partial converse, the Sylow theorems. Throughout the class, we'll look at what fascinating facts we can deduce about our mystery groups.

Prerequisites: Group theory.

Tridiagonal symmetric matrices, the golden ratio, and Pascal's triangle. (Emily)

Tridiagonal symmetric matrices are a type of Toeplitz matrix, which is a matrix in which every diagonal descending from left to right is constant. We will study a specific family of these matrices, namely $n \times n$ matrices with ones on the superdiagonal and subdiagonal and zeroes elsewhere:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Now where do the golden ratio and Pascal's triangle come in? It turns out that for certain values of n, the golden ratio (and its friends) appear as eigenvalues, and Pascal's triangle can tell us what the characteristic polynomials will look like! We will explore and prove these phenomena using a combination of linear algebra, trigonometry, and combinatorics.

Prerequisites: Linear algebra (should know characteristic polynomials and eigenvalues), comfortable with summation notation and $\binom{n}{k}$ related to Pascal's triangle.

Uncertainty principle. (Neeraja)

A physicist is stopped for speeding. "No, officer, I don't know how fast I was going. But I do know exactly where I am." That's a reference to Heisenberg's Uncertainty Principle, which you might have seen stated in something like the following form in a physics class:

$$\Delta x \cdot \Delta p \geq \frac{h}{2\pi}.$$

Here Δx is the error in the measurement of the position of a particle, Δp is the error in the measurement of its momentum and h is Planck's constant. One way to actually witness this principle in action is to listen for "interference beats" while tuning a musical instrument. If two notes are almost in tune, i.e. the difference in frequency is very small, then we hear fewer interference beats, which are pulsations of loudness, per second. So the smaller the frequency difference, the more time we need to observe it.

The mathematical version of the uncertainty principle involves the Fourier transform, which is a map that sends certain well-behaved functions $f : \mathbb{R} \to \mathbb{C}$ to other functions of the same form. Heuristically, the uncertainty principle says that the "spread" of a function and its Fourier transform are inversely proportional. If most of the mass of the original function is clustered tightly in one area, the mass of the Fourier transform of the function must be spread out more widely. In this class, we'll define the Fourier transform and use some of its properties to prove the uncertainty principle. We'll also discuss the interpretation of the uncertainty principle in quantum mechanics and in the musical context mentioned above.

Prerequisites: Single-variable calculus (change of variables and integration by parts), complex numbers (Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$). Although not strictly necessary, prior exposure to Fourier series or the Fourier transform (for example in Alan's class in Week 3!) will be very helpful.

Voting theory 101. (Pesto)

"The only fair voting system is a dictatorship". What properties would make a voting system "fair'? What sorts of (non-dictatorship) voting systems are pretty good, even if they're not "fair"?

Prerequisites: None.

Wallis and his product. (*Jon Tannenhauser*)

John Wallis (1616–1703) published what is essentially the infinite product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$$

in his 1656 treatise *Arithmetica infinitorum*. His work mixed suggestive analogy, intuitive leaps, and sheer moxie—and sparked reactions ranging from awe to skepticism to spluttering rage. We'll trace Wallis's argument (day 1; 🌶🌶) and discuss how and why it touched off a battle in a long-running 17^{th}-century war over infinitesimals, which was actually—although no one knew it at the time—a struggle over the foundations of calculus (day 2; 🌶). Then we'll look at a recent (October 2015) derivation of the Wallis product via the quantum mechanics of the hydrogen atom (day 3; 🌶🌶🌶🌶)!

Prerequisites: None.

Weierstrass approximation. (Neeraja)

If you're familiar with Taylor series, you already know that an infinitely differentiable function can be approximated uniformly by polynomials. But infinitely differentiable functions are a relatively small subclass of functions, and it's natural to ask if this very useful property could be extended to a larger class. It turns out, as Weierstrass first proved in 1885, that all continuous functions defined on a closed interval can be approximated uniformly by polynomials. To place this result in context, note that the class of continuous functions contains many "monsters", such as the Weierstrass function, which is everywhere continuous and nowhere differentiable! In this class we'll give a constructive proof, due to Bernstein, of the Weierstrass approximation theorem.

Prerequisites: Epsilon-delta definition of continuity, uniform continuity (if you'd like to take this class but don't know what uniform continuity is, please talk to me!)

What the continuum *cannot* be. (*Steve Schweber*)

One of the oldest questions in set theory is exactly how big the set of real numbers is. We know that $2^{\aleph_0}$ is uncountable, but can we narrow that down at all? In particular, is there any set whose cardinality is strictly between $\aleph_0$ and $2^{\aleph_0}$?

It is now known that very little can be proved about $2^{\aleph_0}$ from ZFC (the usual axioms of set theory) alone. However, there is one important thing which can be proved outright: there are certain types of uncountable infinity which the cardinality of $\mathbb{R}$ *cannot* be. In this class we'll examine exactly what ZFC can prove about $2^{\aleph_0}$.

Prerequisites: Understand the notation "$2^{\aleph_0}$" and have a vague notion of what an ordinal is—basically, be happy with the "big number line"

$$0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega \cdot 2, \ldots, \omega \cdot 3, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots\ldots\ldots\ldots$$

Which things are the rationals? (Ben)

Do you know what the rationals look like, as a topological space? Can you recognize them in different guises? For example, which of the following spaces are homeomorphic to the rationals?

- $\mathbb{R}$, the real numbers?
- $\mathbb{Z}$, the integers?
- $\mathbb{Q}^2$, the space of rational points in the plane?
- The algebraic numbers (that is, the real numbers which are solutions to polynomial equations)?
- The Cantor set?

Oh, wait, some of those questions are really hard! Some of them we can deal with easily: the reals and the Cantor set, for example, are uncountable. The integers, on the other hand, are "discrete." Both of those let us tell that these things are not the rationals. But those don't let us do anything about the other two. These don't even let us figure out whether the sets $(0, 1) \cap \mathbb{Q}$ and $[0, 1] \cap \mathbb{Q}$ are homeomorphic!

In this course, we'll learn which things are the rationals, and which things are Cantor sets[10]. These questions are answered by a theorem of Sierpinski and a theorem of Brouwer.

Prerequisites: Some kind of point-set topology, and some introductory group theory.