CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2020

Contents

9 am Classes

Combinatorics of tableaux. (Emily & Kayla, Monday–Friday)
Do you like combinatorics? Do you like changing hard complicated algebraic structures into pictures? We SCHUR do! Come learn about the combinatorics of tableaux with your favorite Minnesota dynamic duo. For the first day or two, we will introduce tableaux, various combinatorial identities associated to them, and RSK (Really Spicy Kombinatorics?).

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e	m	i	l	y			
k	a	y	l	a			
f	u	n					

For the remainder of the class, we will be exploring the connection between tableaux and various group representations. Spicy! (3 chilis to be exact). Come get exposure to a little representation theory and algebraic combinatorics with us!

Chilis:

Homework: Recommended.

Prerequisites: Linear algebra, group theory

Cluster: Counting things.

Graphs on surfaces. (Marisa, Tuesday–Friday)
Suppose you want to draw a graph on the plane with no edge crossings, but your graph is not planar. Well, you could give up. Or you could change the rules of the game: start drawing, and before two edges cross, add an overpass to your plane, and then send one of those edges up the overpass. Problem solved, under your new rules (a.k.a. on the torus)!

This class will explore the question: given a graph, on what surfaces can we draw it without crossings? Our toolkit will be highly combinatorial, even including our definitions of surfaces. Our goal will be to fully answer this question for several families of graphs by the end of the week. We'll also prove an analogue to the Four-Color Theorem for every closed surface *except* the plane.

Style notes. This class will run in a hybrid format: short lecture, some group work, and a recap together.

Chilis:

Homework: Optional.

Prerequisites: Misha's Introduction to graph theory, or familiarity with planar graphs and the proof of Euler's formula. (All the concepts from topology are self-contained and/or black boxed.)

Cluster: Graph theory.

Introduction to number theory. (Mark, Monday–Friday)
How do you *know* that 12345678913579147159161718192468258262728293693738394849 can be written as a product of primes in only one way (except for the order of the primes)? (There are number systems in which the analog of this is not true; for example, notice that $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$, but it's not too hard to show that in the set of numbers of the form $a + b\sqrt{-5}$ with a, b integers, none of the numbers $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ can be factored further except in trivial ways.) Which integers are the sum of two (or the sum of three, or the sum of four) perfect squares? What postages can you get (and not get) if you have only 8 cent and 17 cent stamps available? How does the RSA algorithm (used for such things as sending confidential information, such as your credit card number, over the Internet) work? What happens if you take the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ and study how it repeats modulo n, for different values of n? We may not get to all these questions (and one of the answers is not even completely known), but we will touch on several of them as we explore some basic, beautiful, and subtle properties of our old friends, the integers. For thousands of years professional and amateur mathematicians have been fascinated by number theory (by the way, some of the amateurs, such as the 17th century lawyer Fermat and the modern-day theoretical physicist Dyson who passed away very recently, are not to be underestimated!) and chances are that you, too, will enjoy it quite a bit. Why 3 chilis? Because we'll move relatively fast!

Chilis: 🌶🌶🌶

Homework: Recommended.

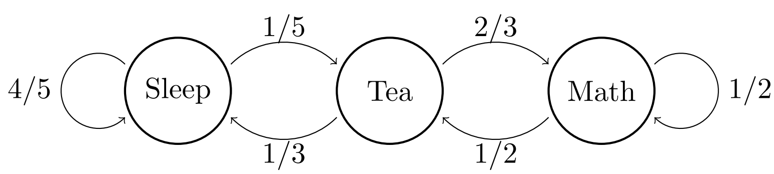
Prerequisites: None (beyond modular arithmetic). Anyone who took the Mathcamp crash course in week 1 should certainly be fine.

Cluster: Number theory.

Required for: Congruences of Bernoulli numbers and zeta values (W3); (Relatively) prime complex numbers (W4); Fair squares (mod p) (W4)

Markov chains and random walks. (Misha, Monday–Friday)
When (for whatever reason) I stay at home, can't leave my apartment, and don't have a regular schedule, I lose all track of time. I spend all my hours sleeping, drinking tea, and doing math.

Every hour, I make a random decision to change the activity I'm doing, according to the following diagram (where the numbers on the arrows denote probabilities):



This is an example of a Markov chain, and questions we might ask about it include the following:

- What fraction of the time am I drinking tea?
- When I wake up, how long do I stay awake before I go back to sleep?
- What's the probability that I'll go for a whole day without doing any math?

All of these questions can be answered in a boring way: by solving systems of linear equations. In this class, we'll learn to solve them in more exciting ways. These include making up a betting game about what I'm doing—or even transforming the Markov chain into an electric network!

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: None; I'll make some offhanded references to linear algebra and graph theory, but neither one is needed to follow the class.

Cluster: Probability and statistics.

Oh the sequences you'll know. (*Zach Abel*, Monday–Friday)

> I do not like Fib-nacho man,
> I'd rather talk of Catalan.
> Or Look and Say, even Farey,
> Oh 1,1,2,3 let me be!

We'll survey a variety of sequences that aren't as well known as the Fib****** sequence, but should be! Each day will explore a different sequence in detail, finding surprising links to number theory, geometry, combinatorics, and more!

Some sequences that may be covered: Thue–Morse, Lucas, Beatty sequences, Bell numbers, Up/Down numbers, derangements, Prüfer,. . .

Chilis: 🌶

Homework: Optional.

Prerequisites: None

10 am Classes

Clopen for business: an inquiry-based approach to point-set topology. (Katharine, Tuesday–Friday)

Topology, like geometry, studies things we might describe as "shapes". However, topology regards shapes less rigidly than geometry does. We call these squishier shapes "spaces".

A topological space is made up of a set of points and a designation of which subsets of these points are "open". We can't necessarily measure distances or angles, nor can we necessarily draw our space and get information about it that way. So what can we do?? With only a definition of open sets, we can build things like sequences (and their limits), continuous functions, and decompositions of spaces into connected pieces. In this class, we will do all these things and more!

This is a presentation-based IBL class. Since the class is only four days long, I will ask you to look at some homework problems before the first class.

Chilis: 🌶🌶🌶

Homework: Required.

Prerequisites: Proof techniques, set notation

Cluster: Topology.

Required for: Classifying complex semisimple Lie algebras (W3); So you like them triangles? (W4)

Conflict-free graph coloring. (Pesto, Tuesday–Friday)

You put some cell towers on a graph, each broadcasting at some frequency. Every vertex in the graph is a cell phone that needs to be able to listen at at least one frequency, but can't listen at a frequency if two towers adjacent to it are both trying to broadcast at that frequency. How many cell towers and how many distinct frequencies do you need?

This problem defines a version of graph coloring called "conflict-free" graph coloring. For this new version of graph coloring, we'll prove an analogue of the most important unsolved problem in graph theory, generalize the four-color map theorem, and prove that we (probably) can't solve it efficiently in general.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: None, but having seen a proof of NP-hardness would be nice.

Cluster: Graph theory.

Hilbert's space-filling curve. (Ben, Thursday–Friday)
It is well-known that the unit interval in the real line (that is, the interval $[0, 1] = \{x : 0 \leq x \leq 1\}$) has the same cardinality as the unit square in the plane (the set $[0, 1] \times [0, 1]$, that is). But if you look at the usual ways of showing that these are the same size, you'll find that the function you construct is not continuous everywhere. (For example, if you construct this bijection by "interleaving decimals", you'll need to make some arbitrary choices for numbers that have multiple decimal expansions. This doesn't matter much for whether it's a bijection, but it matters a LOT for whether it's continuous!)

In some ways this makes sense. Such a continuous bijection would sort of say that our spaces not only had the same *size* (which we knew) but also the same *shape*, which seems sillier. After all, a line looks different than a plane; in particular it seems like a curve should have a length, but not an area.

But some curves are too long to have a length, and instead fill up two dimensions of space. This course will show one construction of such a curve, which was one of the early examples of a fractal in mathematics. Nowadays, Hilbert's Space-Filling Curve can be found helping companies such as Google store multi-dimensional data (e.g. a map) in memory (which usually "looks like" something single-dimensional, more or less).

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: Familiarity with uniform convergence (Introduction to analysis will be enough to follow this)

Cluster: Real analysis.

Quantum mechanics. (*Andrew Guo*, Tuesday–Friday)
Quantum mechanics is weird and nobody understands it[1]. But with the right mathematical tools, almost anybody can understand it well enough to use it! In this course, we will learn about the weird phenomenon of electrons that behave as if they can exist in two diametrically opposed states at the same time—i.e. in a quantum "superposition". We'll then see how superposition allows computers that run on quantum mechanics to perform computations in parallel, which gives them the power to solve certain problems more efficiently than any computer built on ordinary, classical mechanics (such as the one you're currently reading this on). Along the way, we will attempt to axiomatize quantum mechanics in the language of linear algebra and construct an example of a quantum algorithm with an exponential quantum speed-up.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Linear Algebra

Ramanujan graphs, quaternions, and number theory. (*Dan Gulotta*, Tuesday–Friday)
Suppose you are designing a computer network. You would like to design the network so that information can be sent efficiently between any pair of computers. In theory, you could directly connect every pair of computers, but this could get very expensive. So in practice, each computer will only have a direct connection to a few others.

There are various ways of measuring the efficiency of a network. One of these is the *spectral gap*. Define a matrix A so that $A_{ij} = 1$ if computers i and j are connected, and $A_{ij} = 0$ otherwise. If each computer is connected to k others, then one of the eigenvalues of A will be k. The network is called a *Ramanujan graph* if all of the other eigenvalues have absolute value at most $2\sqrt{k-1}$. (This is, in a certain precise sense, almost the smallest that the eigenvalues can be.)

I will show how to use ideas from number theory to construct Ramanujan graphs. I will describe some surprising connections between these graphs and some seemingly very different kinds of objects.

[1]Richard P. Feynman, 1965

In particular, the fact that these graphs are Ramanujan is a consequence of the Ramanujan–Petersson conjecture, an important theorem in the theory of modular forms.

See more at the course website: `https://people.maths.ox.ac.uk/gulotta/mc20.html`

Chilis:

Homework: Recommended.

Prerequisites: Linear algebra, basic number theory (modular arithmetic), basic graph theory (just the definition of a graph). Knowing some group theory might be helpful.

Cluster: Graph theory.

Weierstrass approximation. (Neeraja, Tuesday–Wednesday)
If you're familiar with Taylor series, you already know that an infinitely differentiable function can be approximated uniformly by polynomials. But infinitely differentiable functions are a relatively small subclass of functions, and it's natural to ask if this very useful property could be extended to a larger class. It turns out, as Weierstrass first proved in 1885, that all continuous functions defined on a closed interval can be approximated uniformly by polynomials. To place this result in context, note that the class of continuous functions contains many "monsters", such as the Weierstrass function, which is everywhere continuous and nowhere differentiable! In this class we'll give a constructive proof, due to Bernstein, of the Weierstrass approximation theorem.

Chilis:

Homework: Recommended.

Prerequisites: Epsilon-delta definition of continuity, uniform continuity (if you'd like to take this class but don't know what uniform continuity is, please talk to me!)

Cluster: Real analysis.

Noon Classes

A Rubik's cube-based approach to group theory. (Alan & Dennis, Monday–Friday)
The Rubik's cube is a very mathematical object! For example, if U, D, L, R, F, B denotes turning the up, down, left, right, front, back face 90 degrees clockwise (respectively), then we can write $R^4 =$ Identity to mean that doing R four times brings us back to where we started. We can also write more complicated equations, such as $(R^2U^2)^6 =$ Identity.

In this class, we'll introduce some fundamental concepts in group theory through the Rubik's cube. By studying the structure of the Rubik's cube group, we will naturally be led to ideas such as group actions, permutations, symmetries, invariants, commutators, and semidirect products. With these and other tools, we will explain why certain states, such as having one flipped edge, are unreachable and derive the number of possible positions a Rubik's cube has.

In addition to using the Rubik's cube to motivate concepts in group theory, we'll also use group theory to help us understand the cube. We'll see how to use commutators to solve the cube. One benefit of this approach is that you can understand every move you are making—you don't have to memorize any magic "algorithms". We'll also see how permutations play an important role in both blindfolded cubing and fewest moves competitions.

Chilis:

Homework: Optional.

Prerequisites: None officially. Some knowledge of basic group theory would help, but is not necessary. It would be helpful if you have a Rubik's cube and know how to solve it. If you would like to learn, here is a YouTube tutorial made by reputable speedcubers: `https://www.youtube.com/watch?v=1t1OL2zN0LQ`. Or (shameless self-promotion ahead) you could check out Alan's website: `http://learn2cube.com/beginners/intro`. Ask us if you have any questions!

Cluster: Group theory.

Cantor, Fourier, and the first uncountable ordinal. (Ben, Monday–Friday)
Where did set theory come from? The usual answer is that it was "motivated by Cantor's work in real analysis," to quote Wikipedia. Further investigation reveals that it was due to Cantor's investigations of trigonometric series, in particular, which motivated these discoveries. Such series were widely studied in the 19th century, due to the work of Fourier on the heat equation, but many basic questions about them remained open in the 1860s.

One of these questions is natural: Can a function be represented as a trigonometric series in more than one way? Some partial work had been done on this question before Cantor, but the general problem was wide open.

In this course, we will work through these partial results to catch up to the state of the art in 1870, when Cantor was working on precisely this question. We will then follow Cantor as he answers this question, and as he takes the first steps on the road to set theory.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Knowledge of limits, series, and integrals. Having encountered uniform convergence would be helpful.

Cluster: Real analysis.

Introduction to ring theory. (Eric, Monday–Friday)
A ring is a set of objects that you can "add" and "multiply." Many examples of rings come from two sources, arithmetic (rings whose elements are like numbers) and geometry (rings whose elements are like functions). We'll explore many familiar concepts (like modular arithmetic, prime factorization, division with remainder), what they mean in these two worlds, and the varied and interesting ways in which they can break down.

We'll start by recontextualizing how we think of the 4 arithmetic operations $(+, -, \times, \div)$, and figuring out how to make "modular arithmetic" work in any setting where we can "add" and "multiply." Then we'll work through a hierarchy of rings with extra structure, seeing how various nice properties die out as we relax how much extra structure we have, but also keeping some things alive by expanding our idea of what prime factorizations should be.

Homework is listed as required, but only 1 or 2 problems a day will be required. The required problems will be discussed at the start of each class to set up the day's lecture. Plenty of non-required problems will be available to explore examples and gain familiarity with concepts.

Chilis: 🌶🌶🌶

Homework: Required.

Prerequisites: None; in particular we'll run without group theory. Some examples will assume familiarity with other areas of math, hopefully with enough examples that everyone gets to interact with things they know.

Cluster: Algebra and geometry.

Required for: (Relatively) prime complex numbers (W4); Functions you can't integrate (W4)

Modeling computation. (Mia, Monday–Friday)
How does one mathematically model a computer? Well, computers are really large, complicated, and gnarly, so instead, mathematicians work with idealized computers, called *computational models*. This class examines three important computational models, exploring the capacities and limitations of each. We'll start by examining deterministic finite automata, studying the computations they can execute and building clever DFA's to perform the computations of our choosing. From there, we will study nondeterministic finite automata and then pushdown automata. With these three models in hand, some natural questions arise: Are there computations that one model can perform but not the others? Are there computations that none can perform?

To make this more concrete, we turn to . . . slackbot. As you may have noticed, slackbot performs the following *very important* operation, given a string of character, it determines if the string contains the word "yucca" in it (and then responds appropriately). This is a computation! One question you might ask is, can we build automata, of our given type, that determine if a string contains the word "yucca"? What about determining if the string contains an even number of "yucca"'s? Or what about determining if the string contains the same number of "yucca"'s as "cactus"'s? Come to this class and find out!
Chilis: 🌶🌶
Homework: Recommended.
Prerequisites: None
Cluster: Math and computers.

The Plünnecke–Ruzsa inequality. (Milan, Thursday–Friday)
You probably know how to add integers, but what about sets of integers? A natural definition for the sum of two sets of integers A and B is

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

What can we do with this? Given any set of integers A, it is not too hard to prove that $|A+A| \geq 2|A|-1$. Furthermore, we have equality exactly when A is an arithmetic progression. What about when we are close to equality? Intuitively, A should still have some arithmetic structure. The Plünnecke–Ruzsa inequality tells us that in some sense our structure is still preserved when we iterate adding A to itself many times. In this class, we will develop the necessary tools to understand the Plünnecke–Ruzsa inequality precisely, and then we will see a clever recent elementary proof.
Chilis: 🌶🌶🌶
Homework: Optional.
Prerequisites: None

Wallis and his product. (*Jon Tannenhauser*, Monday–Wednesday)
John Wallis (1616–1703) published what is essentially the infinite product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$$

in his 1656 treatise *Arithmetica infinitorum*. His work mixed suggestive analogy, intuitive leaps, and sheer moxie—and sparked reactions ranging from awe to skepticism to spluttering rage. We'll trace Wallis's argument (day 1; 🌶🌶) and discuss how and why it touched off a battle in a long-running 17^{th}-century war over infinitesimals, which was actually—although no one knew it at the time—a struggle over the foundations of calculus (day 2; 🌶). Then we'll look at a recent (October 2015) derivation of the Wallis product via the quantum mechanics of the hydrogen atom (day 3; 🌶🌶🌶🌶)!
Chilis: 🌶 → 🌶🌶🌶🌶
Homework: Recommended.
Prerequisites: None.