CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2020

Contents

9 am Classes

Bairely complete (Ben, Tuesday–Friday)
It's well-known that the real numbers are uncountable, due to the elegant diagonalization argument of Georg Cantor. It's also well-known that the Cantor set has the same cardinality as the real numbers. Can you write the reals as a union of countably many Cantor sets? You might be tempted to reach for measure theory here (which studies the lengths of sets), but measure theory will not help us here: there are Cantor sets with positive length, so infinitely many of them can still cover an infinitely long line.

There are some functions that are continuous, but nowhere differentiable. How abundant are these "Weierstrass functions"? Is the set of these "small" or "large"? What do we even mean by "small" and "large" in this context?

One generalization of Cantor's diagonalization argument is the Baire[1] Category[2] Theorem. This gives one possible answer to the question of what we mean by "small" and "large" here, and lets us figure out whether the set of nowhere differentiable functions is "small" or "large" in this sense.

It also lets us answer a lot of other questions. Can you find a function that's continuous at every irrational, and discontinuous at every rational? If you've done that, can you do the reverse, finding a function continuous at each rational number and discontinuous at each irrational? There are some functions whose derivatives are not everywhere continuous—but can we at least say that derivatives are "usually" continuous, that is, continuous on a "large" set? Or are there functions whose derivative is discontinuous on a "large" set?

Chilis:

Homework: Recommended.

Prerequisites: Some exposure to epsilon-delta and uniform convergence. Topology might help you orient yourself in the first days of this course, but is not necessary.

Cluster: Real analysis.

Congruences of Bernoulli numbers and zeta values (Eric, Tuesday–Friday)
The Riemann zeta function is a wonderful thing that packages many of the mysteries surrounding prime numbers into the form of an analytic function. Among the many interesting things about this function are its values at integers: there is a precise formula for $\zeta(2n)$ in terms of powers of 2, π, some factorials, and a Bernoulli number. You may be familiar with the equality $\zeta(2) = \pi^2/6$. In the 1850s Kummer found a deep connection between these special values of ζ and the arithmetic of the

[1]Not that kind of bear.
[2]Not that kind of category.

integers, allowing him to prove some cases of Fermat's last theorem by understanding congruences of these special values.

We'll start the class by learning what Bernoulli numbers are and relating them to special values of ζ (we will be pretty cavalier about issues of convergence in this part). With that in hand we'll develop the theory of integration on $\mathbb{Z}/p^k\mathbb{Z}$ in order to give a clean proof of the Kummer congruences: if $n \equiv m \mod (p-1)$ then $B_n/n \equiv B_m/m \mod p$. (In fact we'll extend this statement to similar congruences mod p^k.) Secretly what we'll be doing is showing that there is a p-adic Riemann zeta function built by interpolating special values of the usual ζ function.

A note on format: I'd like to try and have this class run using in-class worksheets and mini lectures instead of regular lecturing, so expect that for the first few days at least. If it seems like that is not working well we may switch to a more traditional lecture format.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Modular arithmetic at the level of knowing which elements of $\mathbb{Z}/n\mathbb{Z}$ are invertible. You should be happy with Fermat's little theorem ($x^{p-1} \equiv 1 \mod p$ if $x \not\equiv 0 \mod p$). Having seen a formal power series before would be nice, maybe at the level of knowing the power series expansion of e^x. Knowing the definition of integrals through Riemann sums is not necessary but many things will make much more sense if you do.

Cluster: Number theory.

Geometric programming (Misha, Tuesday–Friday)
Geometric programming is a moderately obscure kind of optimization problem. Maybe you've heard of linear programming; it's a tiny bit like that but completely different.

It is about solving problems using the AM–GM inequality. A classic easy example: "If you have 40 feet of fence, what's the largest area you can fence off?"

When these problems have more variables and more constraints, there are multiple ways to apply AM–GM, and this leads to a beautiful duality theory that's a distant cousin of the LP dual and the Lagrangian dual. (If these words made no sense, good; one of my goals in this class is to show you how dual problems arise in optimization.) Geometric programming is a fun way to dip your toes into operations research without much background required.

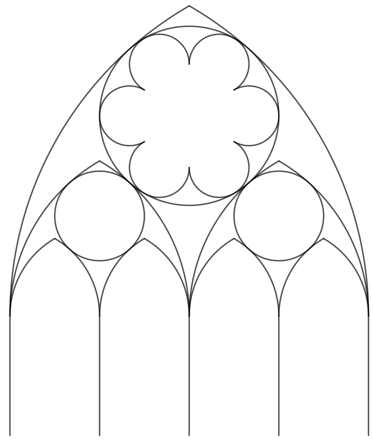
Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: Mostly none; I will spend a bit of time on one day of the class talking about partial derivatives. If you're not comfortable with those, this shouldn't affect your enjoyment of the rest of the class.

Gothic windows (Kinga, Tuesday)
You probably have already heard about the cathedral Notre Dame in Paris. Maybe you've even visited it. When I was in Paris a few years ago, one of the things I remembered most were its huge, beautiful rose windows. But that's not the only building that has memorable windows! There are many more.

In this class, we'll take a closer look at gothic traceries. What were some frequently used patterns and shapes? You'll learn how to construct a few of the most common ones using only a compass and a ruler, but there are also some that can't be constructed this way. We'll calculate lengths and talk about geometry. There will be lots of pictures!



Chilis: 🌶

Homework: Optional.

Prerequisites: None

Regular expressions and generating functions (Linus, Tuesday–Friday)
To cheat at Mathcamp's famed week 4 puzzle hunt, I use regular expressions. For example, if I know a puzzle answer uses the letters d, u, c, and k in that order, I can type the regular expression "*d*u*c*k*" into `onelook.com` to get a list of all English words it could be.

To count anything, e.g. the number of domino tilings of a $4 \times n$ rectangle, I use generating functions, a magical tool in combinatorics.

Learn how regular expressions and generating functions are the same thing, and use them together to instantly solve a bunch of problems like:

- "What's the most chicken nuggets I can't order if they come in 5-piece and 8-piece boxes?"
- "Why do rational numbers have repeating decimals?"
- Problem 5c on this year's Qualifying Quiz

NOTE: The first two days of this class will cover regular expressions and finite automata, which overlaps with Mia's class last week.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: none

Cluster: Counting things.

Spectral graph theory (Ania, Wednesday–Friday)
When you think about graphs, you probably imagine some connected dots. However, graphs can be also represented as matrices! Spectral graph theory is a way of turning problems about graphs into linear algebra by associating a matrix to a graph (called the adjacency matrix) and studying its eigenvalues. In this class, we'll see applications of this method to prove some cool facts about graphs!

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Intro linear algebra, basics of graph theory

Cluster: Graph theory.

10 am Classes

Extremal set theory: intersecting families (Neeraja, Monday–Friday)
In an n-element set, what is the largest number of subsets of which no one subset contains any other? What is the largest number of k-element subsets of which every pairwise intersection is nonempty? What if every pairwise intersection must have size exactly t? This course will answer some of these questions! We'll prove some classical results about families of subsets of $\{1, 2, 3, \ldots, n\}$ which intersect or fail to intersect each other in specific ways. Possible results include Sperner's theorem, Dilworth's theorem, Erdős-Ko-Rado theorem and Bollobás' two families theorem.

Chilis:

Homework: Recommended.

Prerequisites: Set notation (union, intersection, complement) and proof by induction.

Cluster: Counting things.

FUNdamental groups and friends: an introduction to topological invariants (Katharine, Monday–Friday)
In topology, we often consider shapes ("spaces") up to some idea of equivalence (that is, we consider some spaces to be "the same"). This can make it difficult to tell spaces apart! A topological invariant is a machine that takes in topological spaces and spits out something more understandable—for example, a number or a group. If two spaces are "the same" then they will result in the same output. (However, if two spaces are not "the same" then we might still get the same output.) We'll look at a bunch of examples: the Euler characteristic, scissors congruence, fundamental groups, and homology groups (very roughly).

Chilis:

Homework: Recommended.

Prerequisites: Proof techniques, some group theory (definition of a group, some examples). A little point-set topology (definition of a space, continuous maps of spaces) is helpful, but not required.

Cluster: Topology.

Fourier analysis (Alan, Monday–Friday)
Around 1800, the French mathematician Jean-Baptiste Joseph Fourier accompanied Napoleon through Egypt. Egypt was very hot, and Fourier became interested in heat, so he developed Fourier series to solve the differential equation known as the "heat equation." (This is a story I heard from Elias Stein, the mathematician who taught me Fourier analysis.)

The central idea of Fourier series is to decompose a periodic function into pure oscillations (i.e. sine waves):

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

This is what our ears do when we listen to music; it explains why the C-sharp of a piano sounds different from same C-sharp of a violin. (In class, we'll see this with some demonstrations using the software Audacity.)

Fourier analysis has wide applications to other areas, including signal processing (e.g., wireless communication), number theory (e.g., Dirichlet's theorem on primes in arithmetic progressions), quantum mechanics (e.g., the Heisenberg uncertainty principle, which Neeraja will cover in Week 4), and Boolean functions (as in Tim!'s Week 1 class).

In this class, we will learn how to find the Fourier series of any periodic function, prove some basic properties, and see how this can be used to solve differential equations. We will also look at the Fourier transform, which is an analogue of Fourier series for functions which are not periodic. With the remaining time, we'll discuss some of the many applications.

Chilis:

Homework: Recommended.

Prerequisites: Single variable calculus: know integration by parts and what a partial derivative is

Cluster: Real analysis.

Required for: Uncertainty principle (W4)

How not to prove the Continuum Hypothesis (week 1 of 2) (Susan, Monday–Friday)
When Cantor did his pioneering work in set theory he discovered that there are infinitely many different sizes of infinity—in particular, the real numbers are larger than the natural numbers. However, the obvious follow-up question—whether there are sizes of infinity in between—went unresolved for more than fifty years. This question become known as the Continuum Hypothesis.

In this class, we will explore what it means for a subset of the real numbers to be "small" or "large." We'll explore the mysteries of Cantor's middle-thirds set and discuss why logicians like to think of the real numbers as a tree rather than a line.

Finally, we'll discuss an alternative to the Axiom of Choice called the Axiom of Determinacy. This axiom allows us to express many questions about subsets of the real numbers in terms of two-player games, and to prove a result that looks an awful lot like the Continuum Hypothesis.

Chilis:

Homework: Recommended.

Prerequisites: None

Representation theory of finite groups (week 1 of 2) (Mark, Monday–Friday)
It turns out that you can learn a lot about a group by studying homomorphisms from it to groups of linear transformations (if you prefer, groups of matrices). Such a homomorphism is called a representation of the group; representations of groups have been used widely in areas ranging from quantum chemistry and particle physics to the famous classification of all finite simple groups. For example, Burnside, who was one of the pioneers in this area along with Frobenius and Schur, used representation theory to show that the order of any finite simple group that is not cyclic must have at least three distinct prime factors. (The smallest example of such a group, the alternating group A_5 of order $60 = 2^2 \cdot 3 \cdot 5$, is important in understanding the unsolvability of quintic equations by radicals.) We may not get that far, but you'll definitely see some unexpected, beautiful, and important facts about finite groups in this class, along with proofs of most or all of them. With any luck, the first week of the class will get you to the point of understanding character tables, which are relatively small, square tables of numbers that encode all the information about the representations of particular finite groups; these results are quite elegant and very worthwhile, even if you go no further. In the second week, the chili level may ramp up a bit (from about $\pi + 0.4$ to a true 4) as we start introducing techniques from elsewhere in algebra (such as algebraic integers, tensor products, and possibly modules) to get more sophisticated information.

Chilis:

Homework: Recommended.

Prerequisites: Linear algebra, group theory, and general comfort with abstraction.

Cluster: Group theory.

Noon Classes

Classifying complex semisimple Lie algebras (Kayla, Monday–Friday)
In this class, we will be dipping our toes into the vast subject of Lie Theory! We will give some motivation why Lie theory is the intersection of all of mathematics and focus on Lie algebras. The goal of the week will be describing the structure of Lie algebras through showing that their eigenspaces have the beautiful combinatorial structure of a root system.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: You should be comfortable with linear algebra, specifically eigenspaces, linear transformations, actions of vector spaces. Also having an understanding of basic abstract algebra topics such as groups, conjugacy, definition of an algebra would be great. Lastly, knowing some point set topology would be great: definition of a topology, basic topological constructions, more to come.

Geometry of lattices (*J-Lo*, Monday–Friday)
You are standing at an intersection in the town of Skewvillle. Like in many towns, Skewvillle has two sets of streets, each set consisting of evenly spaced parallel lines. Unlike in most towns however, the two sets of streets are nowhere near perpendicular, and the distance between adjacent intersections is somewhat absurd (see Figure 1).

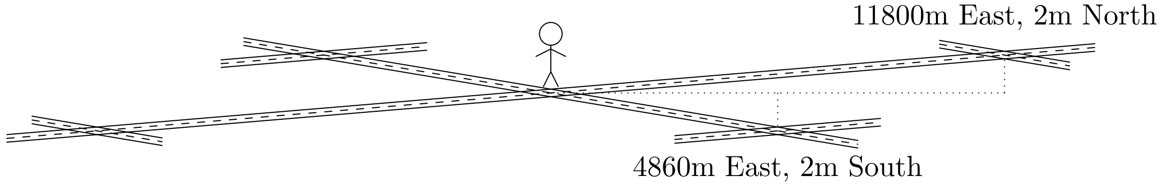


Figure 1. You and adjacent intersections (not to scale).

Your friend claims to be at an intersection that's only 52 meters away from you. If you have to stay on the roads,[3] how far will you need to travel in order to meet your friend?

Lattices are what you get when you take linear algebra and try to make it discrete. In some lattices, like the intersections in Skewvillle, "actual distance" (*as the crow flies*) and "step distance" (*follow the roads*) can be very different. We will prove as much as we can about the relationship between these two notions of distance, including two important theorems by Minkowski, a complete classification of all 2-d lattices, and an introduction to lattice basis reduction. We'll end with a bit of cryptography: finding your friend in a 200-dimensional version of Skewvillle may be so hard that the problem could save internet security as we know it today from quantum destruction.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Linear algebra: you should know how to interpret a matrix as a linear map, be comfortable with multiplying matrices, and understand what the determinant of a linear map means geometrically. (This class did not make it onto the Prerequisites chart, so if you wanted to take this course but didn't take linear algebra, I or any of the mentors would be happy to get you up to speed.)

Cluster: Algebra and geometry.

[3]The law in Skewvillle requires you to stay exactly in the middle of the road at all times; no cutting corners.

Grammatical group generation (Eric, Monday–Tuesday)
Do you like silly word games[4]? Normal subgroups and presentations of groups[5] got you down? Come to this *extremely* light-hearted romp through the world of grammatically generated groups!

In this class, *based on a real actual published math paper*, we will use group theory to understand how many homophones and anagrams the English language has. If you think this sounds silly, it's because it is silly. But we'll do it anyways. Be prepared for terrible jokes and words you will never see used in any other context.

Chilis:

Homework: Optional.

Prerequisites: It would be nice if you've seen normal subgroups and quotient groups before. If you're not super comfortable with them that is great! This class is a very gentle way to get better thinking about them.

Cluster: Group theory.

Information theory (Mira, Monday–Friday)
In 1948, Claude Shannon published a paper called "A Mathematical Theory of Communication." By the time the paper came out as a book in 1949, its name had changed to "The Mathematical Theory of Communication." It took only a year for people to realize that what Shannon had invented was *the* theory—now usually called information theory.

All sorts of communication channels existed in Shannon's day: telegraph, telephone, radio, and TV, not to mention plain old human writing and speech. Shannon's insight was that all these different media could be analyzed within a single mathematical framework: the transmission of *information*, a concept that could be defined mathematically. Shannon showed that any channel—even a very noisy one, with lots of errors and distortion—has a certain rate at which it can transmit information virtually error-free. Anything up to that rate is possible, at least in theory; anything beyond it is hopeless.

Shannon's paper was the mathematical foundation of the digital revolution: every digital device that you've ever used runs on information theory just as surely as it runs on electricity. But the basic framework of information theory is actually quite elementary. In this course, I hope to let you discover a lot of it on your own—while solving some really fun problems along the way. To begin with, we'll have to define exactly what we mean by "information"; for this we'll need some probability theory, which we'll pick up as we go. We'll prove Shannon's Noiseless Coding Theorem, and while we may not get to the full proof of the Noisy Coding Theorem (aka the Channel Capacity Theorem), we'll definitely get far enough that you'll understand the statement and the intuition behind the proof.

Chilis:

Homework: Recommended.

Prerequisites: Basics of probability theory: discrete random variables, expected value, joint and conditional probabilities, Bayes' Rule. *You can still take the class if you are not solid on these concepts: we won't review them in class, but you can learn them through the homework and ask me about them at TAU. However, in this case, you should consider the class to be homework-required!*

Cluster: Probability and statistics.

Let's reverse-engineer photoshop (*Olivia Walch*, Thursday–Friday)
In this two-day class, we're going to try to reverse engineer the math behind image editing software as best we can. We'll talk about color channels, pixel operations, the difference between raster and

[4] But not stupid word games, we don't do stupid things here.

[5] Not to be confused with presentations about groups, as in Katharine's week 1 class.

vector formats, how to make nice looking strokes with Bézier curves, and image compression. At the end of the class, everyone will be required[6] to draw me a beautiful picture with what they've learned.

Chilis: 🌶

Homework: Recommended.

Prerequisites: None.

Math and literature (Yuval, Wednesday)
Many Mathcampers (including me!) love reading, but we often think that reading and doing math are fundamentally different things. Though this is sometimes the case, there are many instances in which math and literature are inextricably related. In this class, we'll explore some incredible pieces of literature and discuss some of the math that went into their creation.

Among the amazing feats we'll see are the following.

- How one author wrote a book of sonnets that contains more poetry than the rest of humanity has ever produced, combined.
- How a nearly 200-year old conjecture due to Euler was eventually disproven, and how this disproof led to one of the most remarkable novels of the 20th century.
- How the plot of a book can correspond to the steps of a proof by contradiction, including how the big plot twist at the end yields the contradiction.

Note: Though I will provide some of the literature for you to read, doing so is totally optional. In particular, feel free to come even if you aren't comfortable reading in English!

Chilis: 🌶

Homework: None.

Prerequisites: None

The John Conway hour (Pesto & Tim!, Monday–Friday)
John Conway was a great mathematician who proved deep and important theorems, while also managing to work in a variety of fields. Incredibly, on top of this, he had a penchant for making math entertaining and accessible. Even more incredibly, every year, for many years, he spent a whole week at Mathcamp, teaching classes on his favorite topics and playing games with campers. Conway died this year from COVID-19. Mira, Misha, Pesto, and Tim! are teaching two weeks of classes in his memory. We all have memories from Conway's visits and classes that we would like to share with you.

The topics of Conway's classes were always "NTBA"—*not* to be announced. When you showed up to his class, you wouldn't know what he was about to talk about, and sometimes he wouldn't either (but he always made it exciting). We will bend this traditional structure a little bit; this week, Pesto and Tim! will be talking about:

- Monday: Rational Tangles
- Tuesday: Wallpaper Groups
- Wednesday: Conway's Soldiers
- Thursday: Doomsday Algorithm (for the day of the week)
- Friday: Look-and-Say Sequence

The days are independent; you can show up to one without having been to the others.

Chilis: 🌶 → 🌶🌶

Homework: None.

Prerequisites: None except for day 5, for which you should know what an eigenvalue is.

[6]not required, but it would still be nice