CLASS DESCRIPTIONS—WEEK 4, MATHCAMP 2020

Contents

9 am Classes

(Relatively) prime complex numbers (Emily, Monday–Friday)
What does it even mean for a complex number to be prime? Well first, we must restrict ourselves to some sort of "complex integers," specifically the Gaussian integers $\mathbb{Z}[i]$. There are multiple different categories of primes in this ring, and once we understand them all we can do lots of interesting things, such as computing prime factorizations!

Now what does it mean for two Gaussian integers θ and η to be relatively prime? It means exactly what you would expect based on the regular old integers: $\gcd(\theta, \eta) = 1$. The more interesting question to ask is "given $\theta \in \mathbb{Z}[i]$, what is the number of Gaussian integers relatively prime to and less than θ?" But wait, how do we know when a complex number is less than another?? Can I even ask if $-4 + 3i < 3 - 5i$??? It turns out we can answer this more interesting question without even talking about "less than"; instead, it comes down to understanding quotient rings in $\mathbb{Z}[i]$, and defining something called the Euler phi function.

We won't stop at just the Gaussian integers; we can explore an entirely different type of complex integers, the Eisenstein integers $\mathbb{Z}[\rho]$ where $\rho = e^{2\pi i/3}$. All the ideas we used to understand primes and the Euler phi function for the Gaussian integers extend *fairly* nicely to the Eisenstein integers!

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Ring theory, number theory, and familiarity with arithmetic in the complex numbers. Talk to me if you are unsure if you have the required background.

Cluster: Number theory.

Complexity theory (Linus, Monday–Friday)
Finding elegant proofs of theorems is hard. Playing chess is hard. Solving Sudoku puzzles is hard. But which is hardest?

The answer is chess.

It turns out that if you can solve an arbitrary $n \times n$ Sudoku puzzle quickly, then you can also find a short proof of an arbitrary theorem quickly (as long as one exists). So, Sudoku is at least as hard as doing math research.

However, for quite involved reasons, mathematicians believe that being able to solve Sudoku puzzles does *not* enable you to solve a chess problem quickly. If you manage to prove them correct, then you will become famous overnight. On the contrary, chess is known to be exactly as hard as e.g. Portal 2.

If an all-powerful alien descended to earth claiming that chess is a win for the first player, then how would it convince us? Just beating us over and over again wouldn't be convincing—that'd only prove the alien is good at chess, not that its strategy is optimal. But it turns out there is a way the alien could do it, using polynomials over finite fields.

Topics: P vs. NP vs. PSPACE vs. EXP, and a host of others; oracles; zero-knowledge proofs; how not to prove $\mathsf{P} \neq \mathsf{NP}$; and more! (Though not the alien chess thing. It's too hard.)

Note: despite the frivolous nature of this blurb, this class will be about the theory of computational hardness, not about fun games like Portal 2.

Chilis: 🌶🌶 → 🌶🌶🌶

Homework: Recommended.

Prerequisites: Know what an algorithm is. Grasp intuitively the cliff between reading a 400-page book and reading all possible 400-page books. It'd be cool if you were reasonably familiar with big-O notation.

Cluster: Math and computers.

The John Conway hour (Mira & Misha, Monday–Friday)
The John Conway extravaganza continues! This week, we'll be talking about one of Conway's favorite topics: games. As before, all the days are independent (although the last three days can also be taken as a unit).

- Day 1 (Misha, 🌶🌶): PRIMEGAME. Find out why the finite sequence
$$\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, 55$$
is actually a machine that generates prime numbers!
- Day 2 (Misha, 🌶🌶): Conway's Game of Life. A game in which cells in a grid evolve according to simple rules that can create complex patterns from simple starting conditions. This cellular life might reproduce to fill the plane, die off, or even perform arbitrary calculations.
- Day 3 (Mira, 🌶🌶🌶): Numbers and Games. All numbers are secretly games, though not all games are numbers.
- Day 4 (Mira, 🌶🌶): Games and Codes. All impartial games are secretly error-correcting codes (another area in which Conway was famous), though not all codes are games.
- Day 5 (Mira, 🌶): Games Conway played. John Conway invented several games you can actually play, including Phutball (Philosopher's Football) and Sprouts. Also, every time he came to visit Mathcamp, he would challenge campers to play Dots and Boxes—a game he did not invent, but almost always won. I'll tell you a little bit of the math behind these games, and also some Conway stories.

Chilis: 🌶 → 🌶🌶🌶

Homework: None.

Prerequisites: None. Days 3–5 may have some overlap with Tim!'s Combinatorial Game Theory class, but neither class depends on the other.

The Kakeya needle problem, projective geometry, and fractal dimensions (Alan, Monday–Friday)
Let's go through the three topics in the course title.

(1) The Kakeya needle problem asks the following question: Suppose you have a unit line segment (a "needle") in the plane and you'd like to rotate it 180 degrees, so that it points in the opposite direction. What is the area of the smallest region you can do this in? This problem can be solved with elementary geometric techniques, and the answer may not be what you expect!
(2) The real projective plane is like the Euclidean plane, except parallel lines intersect at "points at infinity." In this space, there is a magic trick. If you wave your wand and say the magic words

("point-line duality"), then all the points will transform into lines and vice versa. Furthermore, any theorem about points and lines that was true will still remain true!

(3) We can generalize the notion of "dimension" to talk about s-dimensional sets for any nonnegative real number s. This allows us to better understand sets such as fractals. For example, the Koch snowflake (look it up!) has infinite length and zero area, and turns out to be neither 1-dimensional nor 2-dimensional. It is actually $\log_3 4$-dimensional!

For the first few days, we will discuss these three topics independently of each other. Then we will see the surprising connections they have to each other, as well as to the Kakeya conjecture, a famous unsolved problem in analysis.

Chilis: 

Homework: Optional.

Prerequisites: None

Cluster: Real analysis.

Uncertainty principle (Neeraja, Monday–Friday)
A physicist is stopped for speeding. "No, officer, I don't know how fast I was going. But I do know exactly where I am." That's a reference to Heisenberg's Uncertainty Principle, which you might have seen stated in something like the following form in a physics class:

$$\Delta x \cdot \Delta p \geq \frac{h}{2\pi}.$$

Here Δx is the error in the measurement of the position of a particle, Δp is the error in the measurement of its momentum and h is Planck's constant. One way to actually witness this principle in action is to listen for "interference beats" while tuning a musical instrument. If two notes are almost in tune, i.e. the difference in frequency is very small, then we hear fewer interference beats, which are pulsations of loudness, per second. So the smaller the frequency difference, the more time we need to observe it.

The mathematical version of the uncertainty principle involves the Fourier transform, which is a map that sends certain well-behaved functions $f : \mathbb{R} \to \mathbb{C}$ to other functions of the same form. Heuristically, the uncertainty principle says that the "spread" of a function and its Fourier transform are inversely proportional. If most of the mass of the original function is clustered tightly in one area, the mass of the Fourier transform of the function must be spread out more widely. In this class, we'll define the Fourier transform and use some of its properties to prove the uncertainty principle. We'll also discuss the interpretation of the uncertainty principle in quantum mechanics and in the musical context mentioned above.

Chilis: 

Homework: Recommended.

Prerequisites: Single-variable calculus (change of variables and integration by parts), complex numbers (Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$). Although not strictly necessary, prior exposure to Fourier series or the Fourier transform (for example in Alan's class in Week 3!) will be very helpful.

Cluster: Real analysis.

10 am Classes

Brooks' theorem blues (Misha, Monday–Friday)
(...and reds, and purples, and oranges, until we get to Δ colors...)

Brooks' theorem says that a graph with maximum degree Δ has chromatic number at most Δ as well, except for two cases: odd cycles, and complete graphs. In this class, we'll see several proofs of this theorem and its extensions, with detours to topics such as Kempe chains, list coloring, and complexity theory.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Some graph theory: you should be able to understand and prove the statement that any graph with maximum degree Δ has chromatic number at most $\Delta + 1$. (One worse than Brooks' theorem.)

Cluster: Graph theory.

How not to prove the Continuum Hypothesis (week 2 of 2) (Susan, Monday–Friday)
This is a continuation of last week's class!

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: How not to prove the Continuum Hypothesis (week 1 of 2)

Representation theory of finite groups (week 2 of 2) (Mark, Monday–Friday)
This is a continuation of last week's class with the same name. If you would like to join, please check with me first; we can talk through what you may need to catch up on, and I can also give you access to the notes and the problem sets from last week.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Linear algebra, group theory, general comfort with abstraction, and week 1 of the class.

Cluster: Group theory.

So you like them triangles? (Dennis, Monday–Friday)
Are (topological) spaces weird? Haunting? Scary? These great untamed beasts of manifold shapes and forms have terrorized us for too long. Luckily for us, the humble triangle (simplex) has vowed to help us understand these horrific monsters. But how?

Here's the idea: we "triangulate" a space by cutting it up into simplices. Once we've done so, we can perform calculations, such as find a space's Euler characteristic, or more generally, its cohomology groups, which are most importantly *not spaces*! Instead they are objects of algebra, so we can use our favorite equation solving techniques to help us.

In this course, we'll go over what simplicial complexes (or delta complexes, or simplicial sets; there are many variants) are and how to use them to compute cohomology of a space. We'll focus on example calculations and using them to deduce things about some basic spaces. Using our tools, we can formally prove some cool results like Brouwer's fixed point theorem and the Jordan curve theorem, two theorems that are intuitively pretty obvious, but mathematically difficult. We'll also go over some of the main themes of algebraic topology, connecting our construction to the fundamental (or homotopy) groups.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Some linear algebra, and perhaps some group theory. Calculating kernels and quotients will be important.

Cluster: Topology.

Solving equations with origami (Eric, Monday–Friday)
Put a piece of paper in front of you. Mark two points on it. Pretend that your paper is a piece of the complex plane, and that your two marked points are 0 and 1. Which other points in the plane

can you construct by folding your paper and marking where your existing points fold to? If you allow arbitrary folds you can hit anything, but what if we restrict ourselves to folds we can "line up" using our existing points and lines?

We'll be able to answer this question precisely by developing a system of axioms for one-fold origami and analyzing their algebraic potential. Along our journey to understanding the limits of the algebraic power of origami we'll travel the world (Japan to India to Austria to Italy and back), gently encounter some flavours of math from abstract algebra and algebraic geometry, and employ a truly wonderful piece of 19th century mathematics to solve equations by shining lasers on turtles. If that's not enough for you, one of the historical characters we'll encounter has possibly the greatest name in mathematics: Margherita Piazzola Beloch (arguably the first person to really understand the algebraic power of origami).

Classes will be a mix of lectures and paper folding activities, so come prepared with a stack of paper you can fold!

Chilis:

Homework: Recommended.

Prerequisites: You should be comfortable with the concept of the dimension of a vector space. You do not need any prior knowledge of origami to follow this course.

Cluster: Algebra and geometry.

Noon Classes

Combinatorial game theory (Tim!, Tuesday–Friday)
Ania graffitied Mathcampus! She put 7 gnomes in the amphitheater, 10 aliens in the formal garden, and 12 bears on the President's lawn! You and a friend decide to clean it up, as a game. You alternate turns; on your turn you pick a location, and remove any number of pieces of graffiti (at least one) from (only) that location. You'd each like to be the one to remove the last of the graffiti, because that person will get all the credit and glory. Should you go first or second? What's your strategy?

The solution to this game is beautiful and surprising. And there are many other games with a similar flavor: these are called impartial combinatorial games. In fact, some of you played such a game against me in Week 1 relays! To solve these games, you need two tools. Part of the fun is figuring them out, so in this class, you will do that! Plus, we will see many examples of games, from the easy to the still-unsolved.

Chilis:

Homework: Recommended.

Prerequisites: None.

Connections to category theory (Katharine, Tuesday–Friday)
Imagine yourself in a 9am group theory class. Your teacher defines the direct product of groups, $G \times H$, as the set of all pairs of an element of G and an element of H, with componentwise multiplication. "Huh," you think to yourself, "this sure seems like a Cartesian product of sets, but with groups". At 10am you go to a graph theory class where your teacher tells you about the tensor products of graphs. This, too, feels strangely familiar. After lunch, you're in a topology class where your teacher defines a product space. "What??" you think, "Am I trapped in some sort of weird time loop? Are the mentors so tired they could only write one set of course notes between them?"

There are many wild and wonderful parts of math that are peculiar to their particular fields. (Fortunately, we don't actually write only one set of course notes, so you can learn all about them!) There are also many constructions that are wonderful in part because they pop up repeatedly across so many fields. Category theory gives us language to precisely define these kinds of constructions in a way that can implemented in a multitude of settings. We'll focus on the basic language of categories and universal properties. Universal properties are a way of defining mathematical objects by what they **do** rather than how they're built.

Because we're focusing on drawing connections between different areas of math, it will definitely help to have seen a few different areas. I'll talk about groups sets, topological spaces, graphs, vector spaces, and more. However, it's not crucial that you are familiar with every single example. As long as you are comfortable hearing some things you don't understand, you should feel free to skip a prereq or two.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Groups and group homomorphisms, sets and set maps, at least 1 of: topological spaces, graphs, vector spaces, rings

Extremal graph theory (Mia, Tuesday–Friday)

Do you like to live on the *edge*? Are you an *edge-maximal*[1] sort of mathematician? Well, I've got a class for you. In extremal graph theory, we'll consider what happens when we take graphs to their max.

Let H be your favorite graph. We'll consider the following question: Given a graph G with n vertices, what is the minimum number of edges you need to guarantee that H is a subgraph? This question will lead us to the proof of Mantel's theorem, Turán's theorem, and finally, the statement of the Erdős–Stone–Simonovits theorem, which gives beautiful bound on the number of edges required. However, the E–S–S theorem fails to give useful information for one major class of graphs. Whomp. Not to be deterred, we'll spend the last day and a half looking at partial fixes for that failure!

If you came to my class during Sneak Peek, there will be some overlap with the first day, but all subsequent days will be new material.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Graph theory. Familiarity with AM–GM and Jensen's inequality will be helpful but not required.

Cluster: Graph theory.

Fair squares (mod p) (Maya, Tuesday–Thursday)

A number a is a square mod p if there is some x with $x^2 \equiv a \bmod p$. For example, 2 is a square mod 7, because $2 \equiv 16 = 4^2 \bmod 7$.

The distribution of squares mod p seems arbitrary, and in this class, we will show that it is in fact very similar to a random distribution. For example, knowing that a is a square mod p tells us almost nothing about whether $a + 1$ is. Along the way, we will build up machinery to count Pythagorean triples, solutions to the equation $x^2 + y^2 \equiv z^2 \bmod p$.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: Comfortable working with multiplicative inverses mod p

Cluster: Number theory.

Functions you can't integrate (Ben, Tuesday–Friday)

In AP calculus, it always seems as if differentiation is a lot easier than integration. In particular, for

[1] Sorry, couldn't resist the pun.

all of our old friends like sine, cosine, e^x and so forth, we can take their derivatives and write them down in terms of other old friends by following some simple rules. Integration, on the other hand, has a lot more "tricks" and weird techniques.[2]

Here, we'll explore these difficulties in integration, and prove that some easy-to-write-down functions, such as e^{x^2}, don't have an easy-to-write-down integral. To show this, we won't be doing any sort of analysis: there will be no ϵs or δs in this course. Instead, we'll be using the tools of ring theory to study this question. Along the way, we'll see a very nice way to describe the "functions we can write down" or the so-called "elementary functions" in terms of field extensions.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Introductory ring theory, some linear algebra, and knowing the product rule for derivatives.

What the continuum *cannot* be (*Steve Schweber*, Friday)

One of the oldest questions in set theory is exactly how big the set of real numbers is. We know that $2^{\aleph_0}$ is uncountable, but can we narrow that down at all? In particular, is there any set whose cardinality is strictly between $\aleph_0$ and $2^{\aleph_0}$?

It is now known that very little can be proved about $2^{\aleph_0}$ from ZFC (the usual axioms of set theory) alone. However, there is one important thing which can be proved outright: there are certain types of uncountable infinity which the cardinality of $\mathbb{R}$ *cannot* be. In this class we'll examine exactly what ZFC can prove about $2^{\aleph_0}$.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Understand the notation "$2^{\aleph_0}$" and have a vague notion of what an ordinal is—basically, be happy with the "big number line"

$$0, 1, 2, 3, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega\cdot 2, \ldots, \omega\cdot 3, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots\ldots\ldots\ldots$$

[2]Though, as a good analyst, I should tell you: *way* more functions are integrable than differentiable. Pretty much any function that a sane person can write down is integrable. Almost no functions are differentiable. Fun fact!