WEEK 5 CLASS PROPOSALS, MATHCAMP 2020

Contents

Alan's Classes

Complex analysis (🌶🌶🌶, Alan, 3–5 days)
We'll define what a contour integral in the complex plane is, and prove a nonempty subset of the following fundamental theorems from complex analysis: Cauchy's integral theorem, Cauchy's integral formula, analyticity of holomorphic functions, residue theorem.

Homework: Recommended.

Prerequisites: Multivariable calculus, specifically contour integrals (a.k.a. line integrals) and Green's theorem. Uniform convergence.

Dirac delta function (🌶🌶, Alan, 2 days)
The Dirac delta function, a.k.a. the unit impulse function, is the "function" which satisfies

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

This may seem like nonsense, but this function shows up naturally in many physical problems.

In this class, we'll talk about the theory of distributions (note that "distribution" has many different meanings in mathematics), which will allow us to describe the delta function rigorously and make sense of statements such as $\frac{d^2}{dx^2}|x| = 2\delta(x)$. In fact, we'll learn how to differentiate *any* function. Then we'll see some applications of all this.

Homework: Recommended.

Prerequisites: Single variable calculus, integration by parts.

Discrete Fourier transform (🌶🌶, Alan, 2 days)
We'll define the discrete Fourier transform. This is the Fourier transform on $\mathbb{Z}/N\mathbb{Z}$, which is the group $\{0, 1, \ldots, N-1\}$ with addition modulo N. We'll see an efficient algorithm to compute the discrete Fourier transform in $O(N \log N)$ time, called the fast Fourier transform. This is one of the most important algorithms in digital signal processing.

Homework: Recommended.

Prerequisites: Complex numbers, $e^{ix} = \cos x + i \sin x$, run-time efficiency of an algorithm.

Parseval's identity (🌶🌶🌶, Alan, 2 days)
The focus of this class will be a rigorous proof of Parseval's identity for Fourier series. The identity can be thought of as an analogue of the Pythagorean theorem. It says that if $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ has Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$, then $\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |a_n|^2$.

First we will use the Fejér kernel and convolutions to show that trigonometric polynomials are dense in the space of continuous functions on $[0, 1]$. Then we will use this to prove Parseval's identity. Actually, without too much extra work, we can also use trigonometric polynomials to prove the Weierstrass approximation theorem, which states that (regular) polynomials are dense in the space of continuous functions on $[0, 1]$. This proof will be different from the one that Neeraja gave in her Week 2 class.

Homework: Recommended.

Prerequisites: Fourier series, Hermitian inner product, orthogonal projection, uniform convergence, convolution.

Shannon's interpolation formula (🌶🌶, Alan, 2 days)
Suppose there is a function $f : \mathbb{R} \to \mathbb{C}$, and you only know $f(x) = 0$ for all $x \in \mathbb{Z}$. Can you determine f? No, because any function of the form $\sin n\pi x$ (with $n \in \mathbb{Z}$) fits this description. In other words, you could have a function of frequency 0, or 1/2, or 1, or 3/2, etc. However, if you also know that f has only frequencies lower than 1/2, then there *is* enough information to conclude that f is the zero function. (By "f has only frequencies lower than 1/2," we mean that $\widehat{f}(\xi) = 0$ for all $|\xi| \geq \frac{1}{2}$, where $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx$ is the Fourier transform.)

In general, if you know the values of $f : \mathbb{R} \to \mathbb{C}$ on $\mathbb{Z}$, and f has only frequencies lower than 1/2, then f is uniquely determined. The formula to reconstruct f from its values on $\mathbb{Z}$ is called Shannon's interpolation formula, and is important in signal processing and information theory. In this class, we will prove this formula by first proving another identity in Fourier analysis called the Poisson summation formula.

Homework: Recommended.

Prerequisites: Know basic properties of Fourier series and Fourier transform (my Week 3 class is enough).

APURVA'S CLASSES

How Riemann *finally* understood the logarithms (🌶🌶, Apurva, 2 days)
Logarithms are hard to define for complex numbers. (If you came to Jon's talk you know this all tpo well.) Euler settled the question by saying that the logarithm is a multi-valued function. But functions aren't allowed to be multi-valued! What's going on?

Riemann realized that the way to fix this is by not thinking about functions but instead studying graphs of functions. This led to the definition of a Riemann surface and resulted in the creation of half a dozen new branches of mathematics.

In this class, we will see how Riemann fixed the multi-valued logarithm problem and prove that an elliptic curve is a torus.

Homework: Optional.

Prerequisites: You should know the polar decomposition of complex numbers and Euler's identity.

How to glue donuts (🌶🌶, Apurva, 2 days)
Mathematicians routinely encounter higher dimensional geometric objects. But our brains are incapable of imagining anything in higher dimensions. One way we circumvent this obstacle is by breaking up complex higher dimensional objects into simpler lower dimensional ones, like donuts.

In this class, we'll see how a donut can be a powerful tool in the hands of a mathematician and learn how to visualize four dimensional objects.

Homework: Optional.

Prerequisites: None.

Ben's Classes

Cantor's leaky tent (🌶🌶–🌶🌶🌶, Ben, 2 days)
One of the notorious counterexamples in point-set topology is called "Cantor's Leaky Tent" or the "Knaster–Kuratowski Fan." This space is connected! But there's one particular point that, when removed, makes the space *totally disconnected.* In this class, we'll go over all of these terms, put up our tent, and prove that it does exactly what it's supposed to.

Homework: Recommended.

Prerequisites: Some point-set topology. Knowing, or at least being willing to accept, the Baire Category Theorem.

The Stone–Čech Compactification (🌶🌶🌶–🌶🌶🌶🌶, Ben, 2–3 days)
Compactness is a smallness condition for topological spaces. For instance, any continuous function from a compact space to the real numbers is bounded (this manifests itself in the Extreme Value Theorem of calculus, for example). But a lot of spaces that we like are not compact; the real numbers aren't. If you just add "points at $\pm\infty$" to the reals, though, you do end up with a new compact space that contains a copy of the reals. Such an object is called a "compactification of the reals."

So, we see that we can make a space "smaller" (in the sense of becoming compact) by making it "bigger" (in the sense of literally adding points). How big can compactifications be? Can they be arbitrarily large, or is there a "largest compactification?" And what would that mean, anyways?

In this course we'll use the idea of universal properties to motivate what "largest compactification" ought to mean, and then discuss how we might try to build such a thing. Hopefully, we'll then see how to turn this vague idea into a proof!

Homework: Recommended.

Prerequisites: Point-set topology including Tychonoff's Theorem. Category theory is not necessary (though it won't hurt).

Which things are the rationals? (🌶🌶–🌶🌶🌶, Ben, 3 days)
Do you know what the rationals look like, as a topological space? Can you recognize them in different guises? For example, which of the following spaces are homeomorphic to the rationals?

- $\mathbb{R}$, the real numbers?
- $\mathbb{Z}$, the integers?
- $\mathbb{Q}^2$, the space of rational points in the plane?
- The algebraic numbers (that is, the real numbers which are solutions to polynomial equations)?
- The Cantor set?

Oh, wait, some of those questions are really hard! Some of them we can deal with easily: the reals and the Cantor set, for example, are uncountable. The integers, on the other hand, are "discrete." Both of those let us tell that these things are not the rationals. But those don't let us do anything about the other two. These don't even let us figure out whether the sets $(0,1)\cap\mathbb{Q}$ and $[0,1]\cap\mathbb{Q}$ are homeomorphic!

In this course, we'll learn which things are the rationals, and which things are Cantor sets[1]. These questions are answered by a theorem of Sierpinski and a theorem of Brouwer.

Homework: Recommended.

Prerequisites: Some kind of point-set topology, and some introductory group theory.

[1] Wait, where do Cantor sets come into it? Well, we'll see that in the course!

Dennis's Classes

Homotopy colimits (🌶🌶🌶–🌶🌶🌶🌶, Dennis, 2–3 days)
Let's take the circle S^1, and crush it down to a point. Now let's do it twice. Nothing changed, right? We just get the boring point as a result.

Well, let's look into this further. Crushing the circle down to a point, homotopically it's the same as gluing in a disk, so the circle is now filled in. After all, the disk is trivial homotopically; it's contractible! (This is exactly like with the fundamental group). If we do this process twice, we now have *two* disks, glued to the circle. This gives us an upper hemisphere and a lower hemisphere, glued at the equator, in other words S^2, the sphere! The sphere is definitely *not* a point. What happened?

The first case is a "strict" pushout, while the second is what's called a "homotopy" pushout. The second case is actually much better: for one, it remembers that we tried "crushing the circle to the point" *twice*, while the first one doesn't! Secondly, the second construction is a homotopy invariant! That means if you replace S^1 with a different, but homotopy equivalent, space, and perform the construction again, you'll get something homotopic to S^2 back, while the first one fails.

In this class, we'll go over some important constructions—like gluing, quotienting, taking special subspaces—of topological spaces. We'll also observe how they aren't great! They are almost *never* invariant under homotopy! We'll then go over the general process of correcting this, making everything "homotopical", or squishier, than it was before.

Homework: Optional.

Prerequisites: Knowledge of topological spaces, homotopy, some gluing of topological spaces.

The matrix exponential and Jordan normal form (🌶🌶–🌶🌶🌶🌶, Dennis, 2–5 days)
You've heard of the exponential function. You might have heard that the exponential function helps us solve differential equations, especially of the form $y' = ay$. What if you have not just one equation, but a whole system of them? What do you do then?

Well, first of all, if they are all linear, you'd probably think to use linear algebra; namely to write the system as one matrix equation $x' = Ax$, where now x is a vector and A is a matrix. How do we solve this? If you guessed, as in the above case, that we should use "e^{At}", whatever that means, you'd be right!!

In this class we'll explore exactly what this matrix exponential is and how it helps us solve differential equations. Along the way we'll need the Jordan normal form, which is a generalization of diagonalization, that at least puts matrices in upper triangular form (not necessarily diagonal form). But it's enough for us to actually *compute* the matrix exponential!

If we have time, we'll also go over how the matrix exponential ties together the Lie group GL_n and its Lie algebra, as well as all the other matrix groups. (Don't worry if you have no idea what this means!)

Homework: Optional.

Prerequisites: Basic linear algebra (familiarity with diagonalization), some calculus 2. For the proof of Jordan normal form only, we'll need some knowledge of polynomial rings.

Emily's Classes

Block designs (🌶🌶–🌶🌶🌶, Emily, 2 days)
Suppose you are running a scrabble tournament and 13 people show up to play. You wish to structure the tournament so that each game consists of four people, and each pair of people plays against each other in some game exactly once. Is this structure possible? If so, how many games must be played?

Now suppose the year is 1850 and you are Thomas Kirkman. You wish to solve the following problem: "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."

Both of these problems can be understood using block designs: a set together with a collection of subsets that satisfy some certain conditions. We will explore some properties of block designs and how we can construct them; this will involve some combinatorics and in some cases projective planes.

Homework: Optional.

Prerequisites: None. Some basic linear algebra may be helpful at a few points, but it is not required.

Everything is a permutation group (🌶🌶–🌶🌶🌶, Emily, 2 days)
Every group is isomorphic to a subgroup of a symmetric group, meaning that the elements of every group can be represented by permutations. This is Cayley's theorem, and it is pretty straightforward to prove. However, it is not always easy to find what these permutations are. This requires coset enumeration, which deals with counting the number of cosets of a given subgroup H of a group G. We will explore an algorithm which counts these cosets, and as a result yields the permutation representation of a group and its order (when finite).

Homework: Optional.

Prerequisites: Group theory.

My favorite integrals (🌶, Emily, 1 day)
We will take a look at a few of my favorite integrals (TBA so that you don't go trying to solve them right now!). Why are these mysterious integrals my favorites? The reason is that they look deceptively simple, yet require the use of multiple techniques and some creative thinking. Hopefully we don't get stuck in an infinite loop of u-substitutions!

Homework: None.

Prerequisites: Single variable calculus.

Tridiagonal symmetric matrices, the golden ratio, and Pascal's triangle (🌶🌶🌶, Emily, 2 days)
Tridiagonal symmetric matrices are a type of Toeplitz matrix, which is a matrix in which every diagonal descending from left to right is constant. We will study a specific family of these matrices, namely $n \times n$ matrices with ones on the superdiagonal and subdiagonal and zeroes elsewhere:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Now where do the golden ratio and Pascal's triangle come in? It turns out that for certain values of n, the golden ratio (and its friends) appear as eigenvalues, and Pascal's triangle can tell us what the characteristic polynomials will look like! We will explore and prove these phenomena using a combination of linear algebra, trigonometry, and combinatorics.

Homework: None.

Prerequisites: Linear algebra (should know characteristic polynomials and eigenvalues), comfortable with summation notation and $\binom{n}{k}$ related to Pascal's triangle.

Eric's Classes

How to ask questions (🌶, Eric, 1 day)
In this class you will learn about asking questions and also ask questions, though possibly not in that order. You will have the opportunity to learn practical wisdom on how to ask questions in a mathematical context and how to be intentional about your question asking.

Your homework will be to ask questions, in this class and others.

Homework: Required.

Prerequisites: None.

Intersecting curves using linear algebra (🌶🌶, Eric, 1 day)
All you need is some basic linear algebra in order to prove the following cool fact: given 5 "generic points" in the plane, you can always find a degree 2 curve which passes through those points, and that curve will be unique. Using nothing but some counting of polynomials and the inclusion-exclusion formula for dimensions of intersections of vector spaces, we'll prove this and a host of other enumerative statements from algebraic geometry. You'll also learn a bit about the idea of genericity in algebraic geometry and how to think about it in this context.

This class will be IBL in that rather than doing a ton of lecturing I will prepare a worksheet which guides you through the material, so be prepared to spend 45–50 minutes working in small groups (which you can choose) if you come to this class.

Homework: Optional.

Prerequisites: Dimensions of vector spaces to the point where you're comfortable with the statement $\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W)$ for subspaces U, W of a vector space V.

Irreducibility of polynomials (🌶🌶–🌶🌶🌶, Eric, 1–2 days)
We'll survey a bunch of strategies for proving that (integral) polynomials are irreducible. Things we might touch on include: reduction mod p, substitution techniques, Eisenstein's criterion, Newton polygons, Cohn's criterion, and many more. We'll use these techniques to show that whole classes of polynomials are irreducible (cyclotomic, truncated exponentials), and try to characterize in general when these techniques will/won't apply.

Homework: Recommended.

Prerequisites: Basic modular arithmetic.

The lemma at the heart of my thesis (🌶🌶🌶, Eric, 1 day)
In the words of my thesis advisor "mathematics is not about proving theorems, it's about proving lemmas." I'll tell you the story (and prove the lemma!) of the lemma at the heart of my thesis. (Spoilers: it's a lemma about the structure of quotient rings of $\mathbb{Z}[\zeta]$ where ζ is a root of unity.)

Homework: None.

Prerequisites: You should know what a quotient ring is.

Kayla's Classes

Dimer models in cluster algebras (🌶🌶–🌶🌶🌶, Kayla, 1–3 days)
What exactly is a cluster algebra? Loosely, a cluster algebra is an algebra generated by a special set of variables called "cluster variables" that satisfy something called the binomial exchange relation. In

this class, we will be looking at something called the F-polynomial and how we can model this with a dimer model on a graph (aka perfect matchings in a graph).

Homework: Recommended.

Prerequisites: Some comfort with abstract algebra (knowing the definition of an algebra through generators).

Introduction to Coxeter groups (🌶–🌶🌶, Kayla, 1–3 days)
I want to begin by defining Coxeter systems and state the classification of Coxeter systems. Then I want the campers to play with a bunch of examples (I have about 10 in mind as of right now that would be good for them to work through). We will start with looking at what the Coxeter groups of simple Coxeter graphs are and work our way up to more technical examples like reflection groups and potentially discuss Weyl groups of roots systems. Time permitting and if there is interest, I would like to talk about some of the combinatorics of Coxeter groups introducing Bruhat order and weak order.

Homework: Optional.

Prerequisites: Group theory.

Intro to Combinatorial Topology (🌶🌶–🌶🌶🌶, Kayla, 1 day)
Do you like combinatorics? Do you like topology? Ever wondered if there is any intersection between the two areas? It turns out that we can make topological spaces out of poset structures! Learn about what these topological spaces look like when we discuss the property of being shellable.

Homework: None.

Prerequisites: Having some familiarity with topology is nice! Also having seen posets and Hasse diagrams would be good.

Posets and the Möbius Function (🌶–🌶🌶, Kayla, 1–3 days)
Give an example-heavy introduction to posets and lattices in algebraic combinatorics following https://arxiv.org/pdf/1409.2562.pdf section 4. We will discuss types of posets and where they arise in math!

Homework: Optional.

Prerequisites: None.

Linus's Classes

Matrix completion (🌶🌶🌶, Linus, 1 day)
Can you find the pattern and fill in the question marks in the following matrix??

$$\begin{pmatrix} 10 & 10 & ? & 4 \\ 6 & 10 & 9 & ? \\ ? & 8 & 6 & 5 \\ 3 & 7 & 7 & 9 \end{pmatrix}$$

Did you get it? The numbers represent how much (columns) me, my boyfriend, and my roommates enjoy (rows) Nichijou, Dark Souls, The Lobster, and League of Legends. I haven't seen The Lobster, so if you figure out that ?, please let me know.

Well... okay. We can't hope to solve this exactly. But with enough $\mathfrak{B}$ig $\mathfrak{D}$ata, and a dose of linear algebra, we can find a good approximation. It's machine learning!

Homework: None.

Prerequisites: Linear algebra: you should be able to define rank, and prove that if $A : \mathbb{R}^m \to \mathbb{R}^k$ and $B : \mathbb{R}^k \to \mathbb{R}^n$ are linear maps then BA has rank at most k.

Perceptron (🌶🌶, Linus, 1 day)
For the third year in a row, I refuse to teach neural networks at Mathcamp. (There'd be almost nothing I can prove.)

But I'll skirt the edges, by teaching the simplest unit inside a neural network: a perceptron. These babies solve the following problem: given points in some n-dimensional space labeled $+$ and $-$, how can we efficiently find a hyperplane separating all the $+$ from all the $-$ (assuming one exists)?

(What if only *most* of the $+$ and $-$ obey the rule, but there are some outliers? What if, instead of a hyperplane, a more complicated boundary (e.g. a polynomial) separates the $+$ from the $-$?)

Homework: None.

Prerequisites: Vectors, dot products.

Marisa's Classes

King chicken theorems (🌶, Marisa, 1 day)
Chickens are incredibly cruel creatures. Whenever you put a bunch of them together, they will form a pecking order. Perhaps "order" is an exaggeration: the chickens will go around pecking whichever chickens they deem to be weaker than themselves, and whenever chickens encounter one another, it's a peck-or-be-pecked situation. Imagine you're a farmer, and you're observing the behavior of your chickens. You would like to assign blame to the meanest chicken. Is it always possible to identify the meanest chicken? Can there be two equally mean chickens? Are there pecking orders in which all the chickens are equally mean?

Homework: None.

Prerequisites: None.

Mark's Classes

A tour of Hensel's world (🌶🌶🌶, Mark, 1 day)
In one of Euler's less celebrated papers, he started with the formula for the sum of a geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and substituted 2 for x to arrive at the apparently nonsensical formula

$$1 + 2 + 4 + 8 + \cdots = -1.$$

More than a hundred years later, Hensel described a number system in which this formula is perfectly correct. That system and its relatives (for each of which 2 is replaced by a different prime number p), the p-adic numbers, are important in modern mathematics; we'll have a quick look around this strange "world".

Homework: None.

Prerequisites: Some experience with the idea of convergent series.

Counting, involutions, and a theorem of Fermat (🌶🌶, Mark, 1 day)
Involutions are mathematical objects, especially functions, that are their own inverses. Involutions show up with some regularity in combinatorial proofs; in this class we'll see how to use counting and

an involution, but no "number theory" in the usual sense, to prove a famous theorem of Fermat about primes as sums of squares. (Actually, although Fermat stated the theorem, it's uncertain whether he had a proof.) If you haven't seen why every prime $p \equiv 1 \pmod 4$ is the sum of two squares, or if you would like to see a relatively recent (Heath-Brown 1984, Zagier 1990), highly non-standard proof of this fact, do come!

Homework: None.

Prerequisites: None.

Cyclotomic polynomials and Migotti's theorem (🌶🌶–🌶🌶🌶, Mark, 1–2 days)
The cyclotomic polynomials form an interesting family of polynomials with integer coefficients, whose roots are complex roots of unity. Looking at the first few of these polynomials leads to a natural conjecture about their coefficients. However, after the first hundred or so cases keep confirming the conjectured pattern, eventually it breaks down. In this class we'll prove a theorem due to Migotti, which sheds some light on what is going on, and in particular on why the conjecture finally fails just when it does.

Homework: None.

Prerequisites: Some experience with complex numbers, preferably including roots of unity; some experience with polynomials.

Exploring the Catalan numbers (🌶🌶, Mark, 1 day)
What's the next number in the sequence $1, 2, 5, 14, \ldots$? If this were an "intelligence test" for middle or high schoolers, the answer might be 41; that's the number that continues the pattern in which every number is one less than three times the previous number. If the sequence gives the answer to some combinatorial question, though, the answer is more likely to be 42. We'll look at a few questions that do give rise to this sequence (with 42), and we'll see that the sequence is given by an elegant formula, for which we'll see a lovely combinatorial proof. If time permits, we may also look at an alternate proof using generating functions.

Homework: None.

Prerequisites: None, but at the very end generating functions and some calculus might make an appearance.

Multiplicative functions (🌶🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.

Homework: Optional.

Prerequisites: No fear of summation notation; a bit of number theory. (Group theory is *not* required.)

Perfect numbers (🌶, Mark, 1 day)
Do you love 6 (a big number!) and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called perfect, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a

particular form, called Mersenne primes—a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Homework: None.

Prerequisites: None.

Quadratic reciprocity (🌶🌶🌶, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is q a square modulo p?"
(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Homework: Recommended.

Prerequisites: Some basic number theory (Fermat's Little Theorem should be enough).

Simplicity itself: A_n and the "other" A_n. (🌶🌶🌶, Mark, 2 days)
The monster group (of order roughly $8 \cdot 10^{53}$) gets a lot of "press", but it's not the largest finite simple group; it's the largest *exceptional* finite simple group. (Reminder: a simple group is one which has no normal subgroups other than the two "trivial" ones; by using homomorphisms, all finite groups can be "built up" from finite simple groups. The complete classification of finite simple groups was a monumental effort that was completed successfully not far into our new millennium.). What about the unexceptional finite simple groups? They come in infinite families, and in this class we'll look in some detail at two of those families: the alternating groups A_n and one class of groups of "Lie type", related to matrices over finite fields. (If you haven't seen finite fields, think "integers mod p" for a prime p.) By the way, the simplicity of the alternating groups plays a crucial role in the proof that in general, polynomial equations of degree 5 and up cannot be solved by radicals (there is no "quintic formula"). We'll prove that A_n is indeed simple for $n \geq 5$, and we should be able to prove simplicity for the other class of groups also, at least for 2×2 matrices.

Homework: None.

Prerequisites: Some group theory and linear algebra. Familiarity with finite fields would help a bit, but is not needed.

The Prüfer Correspondence (🌶🌶, Mark, 1 day)
Suppose you have n points around a circle, with every pair of points connected by a line segment. (If you like, you have the complete graph K_n). Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments, but in only one way. (This tree will be a spanning tree for K_n). How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll go through a combinatorial proof that is especially cool.

Homework: None.

Prerequisites: None.

The Riemann zeta function (🌶🌶–🌶🌶🌶, Mark, 2–3 days)
Many highly qualified people believe that the most important open question in pure mathematics is the Riemann hypothesis, a conjecture about the zeros of the Riemann zeta function. Having been stated in 1859, the conjecture has outlived not only Riemann and his contemporaries, but a few generations of mathematicians beyond, and not for lack of effort! So what's the zeta function, and what's the conjecture? By the end of this class you should have a pretty good idea. You'll also have seen a variety of related cool things, such as the probability that a "random" positive integer is not divisible by a perfect square (beyond 1) and the reason that $-691/2730$ is a useful and interesting number.

Homework: Optional.

Prerequisites: None.

Wedderburn's theorem (🌶🌶🌶, Mark, 1 day)
You may well have seen the quaternions, which form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$ with basis $1, i, j, k$ and multiplication rules $i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$. Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Homework: None.

Prerequisites: Some group theory; knowing what the words "ring" and "field" mean. Familiarity with complex roots of unity would help.

Mia's Classes

Extreme extremal graph theory (🌶🌶–🌶🌶🌶, Mia, 1 day)
A typical question in extremal graph theory asks, given a graph G with n vertices, how many edges does G need to guarantee that H is a subgraph? But what if I want not one graph H, but MANY? What if I want ALL of the cycles C_k, up to some fixed k? This class will look at a delightful proof of Bondy's theorem, which gives conditions that guarantee not one cycle, but all of them.

Homework: None.

Prerequisites: Graph theory.

Extremal graph theory: the reprise (🌶🌶🌶, Mia, 2–3 days)
A natural extension of the motivational question above is: Given a graph G with n vertices, how many edges does G need to guarantee that H is a minor? A topological minor? The first fact we will prove is that every graph of average degree at least 2^{r-2} has a K_r minor. In fact, we can do much better; a recent theorem by Kostochka says that an average degree of at least $cr\sqrt{\log r}$ is sufficient. Unfortunately, topological minors are a little trickier to guarantee and we'll prove that every graph of average degree at least cr^2 has a K_r topological minor. If there is time, we'll look at a beautiful proof by Thomassen which says that, rather counterintuitively, we can force a K_r minor simply by raising the girth.

Homework: Optional.

Prerequisites: None.

The Sylow theorems (, Mia, 2–3 days)
Suppose I give you a mystery group and all I tell you about it is its order. What can you tell me? A surprising amount, actually! For example, if I tell you that a group has order 77, you can tell me that it has exactly one subgroup of order 11 and exactly one of order 7. In fact, you can even tell me that it is an Abelian group, isomorphic to $\mathbb{Z}_7 \oplus \mathbb{Z}_{11}$. All you need are the Sylow theorems.

In this class, we'll learn not only how to perform this group sleuthing, but also why it works. We'll start by developing two extremely useful tools for partitioning groups, cosets and conjugacy classes, which give deep insights into the structure of a group. Then, we'll move on to prove Lagrange's theorem, which states that the order of a subgroup divides the order of a group, and its partial converse, the Sylow theorems. Throughout the class, we'll look at what fascinating facts we can deduce about our mystery groups.

Homework: Optional.

Prerequisites: Group theory.

Mira's Classes

No, you can't just vote your conscience (, Mira, 1 day)

> *No one pretends that democracy is perfect or all-wise. Indeed it has been said that democracy is the worst form of Government except for all those other forms that have been tried from time to time. . .*
>
> —*Winston Churchill*

You may have heard of Arrow's Theorem, which says that if you want your voting system to satisfy certain reasonable-sounding conditions, then your only option is a dictatorship. But this class is *not* about Arrow's Theorem, because Arrow's Theorem is not depressing enough: its definition of a voting system is so restrictive that it barely ever applies in practice.

The Gibbard–Satterthwaite theorem is less famous, but I think much more depressing. Suppose a voting system satisfies two very simple criteria:

(a) If candidate A is the top choice of all voters, then A wins;
(b) The system is not a dictatorship.

The Gibbard–Satterthwaite theorem says that, if there are more than two candidates, then any such system is vulnerable to strategic voting: there is at least one voter who can obtain better results by voting dishonestly than by voting honestly. In other words, democracy can *always* be gamed.

In this class, you won't necessarily learn how, but you'll learn why.

Homework: None.

Prerequisites: None.

The mathematics of polygamy (and bankruptcy) (, Mira, 1 day)
Here is a passage from the Mishnah, the 2nd century codex of Jewish law:

> *A man has three wives. According to their prenuptial agreements, when he dies one of them should get 100 [silver pieces], one should get 200, and one should get 300.*
>
> *If his total estate is 100, the wives split it equally.*
>
> *If the estate is 200, then the first wife gets 50 and the other two get 75 each.*
>
> *If the estate is 300, then the first wife gets 50, the second one gets 100, and the third one gets 150.*
>
> *Similarly, any joint investment with three unequal initial contributions should be divided up in the same way.*

For 1800 years, this passage had baffled scholars: what could possibly be the logic behind the Mishnah's totally different ways of distributing the estate in the three cases? Then, in 1985, a pair of mathematical economists produced a beautifully simple explanation based on ideas from game theory. They showed that for any number of creditors and for any estate size, there is a unique distribution that satisfies certain criteria, and it turns out to be exactly the distribution proposed in the Mishnah. The proof is very cool, based on an analogy with a simple physical system. See if you can figure out this ancient puzzle for yourself, or come to class and find out.

Homework: None.

Prerequisites: None.

The redundancy of English (🌶, Mira, 1 day)
NWSFLSH: NGLSH S RDNDNT! (BT DN'T TLL YR NGLSH TCHR SD THT...)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting, or to have a conversation in a noisy (Zoom) room. The redundancy is a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

But can we quantify exactly *how* redundant English is? In other words, how much information is conveyed by a single letter of English text, relative to how much could theoretically be conveyed? We will answer this question in the way that Claude Shannon, the father of information theory, originally answered it: by (1) generating a bunch of amusing gibberish; (2) playing a word game that I call Shannon's Hangman, and using it as a way of communicating with our imaginary identical twins.

Homework: None.

Prerequisites: The definition of information, which was covered on Day 1 of my Week 3 Information Theory class. If you were not in that class, you can take a look at the slides for Day 1, and DM me if you have any questions. The definition is pretty straightforward, and you don't need to know anything else from that class to enjoy this one.

Misha's Classes

Avoiding arithmetic triples (🌶🌶, Misha, 1 day)
Three-term arithmetic progressions like $1, 2, 3$ or maybe $89, 97, 105$ are the worst. If you hate them as much as I do, you might be on board with my plan to "fix" the number line, get rid of some natural numbers, and avoid all these arithmetic triples.

You might be less on board with my plan if it turns out that my strategy is to keep only the numbers

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

and get rid of every number which isn't a power of 2.

Is there a less radical solution? Come to this class and find out!

Homework: None.

Prerequisites: None.

Computing trig functions by hand (🌶🌶, Misha, 1 day)
When you learn about trig functions, you typically memorize a few of their values (for $30°$ or $45°$, say) and if you want to know any of the other values, you get pointed to a calculator.

Has that ever seemed unsatisfying to you? If so, take this class, in which we'll see that finding some of these values is as easy as solving polynomials, and approximating all of them is as easy as multiplication. If time allows, we'll learn how to compute inverse trig functions, and also how to quickly find lots of digits of π.

Homework: None.

Prerequisites: Be familiar with the formula $e^{ix} = \cos x + i \sin x$.

Factorials and quadratic reciprocity (🌶🌶, Misha, 1 day)
Have you been scared of the proof (and also the statement) of quadratic reciprocity?

I can't help you be less scared of the statement. But I'll show you a very non-scary proof.

We'll warm up by proving Fermat's little theorem, Wilson's theorem and Euler's criterion by looking at $(p-1)!$ mod p in a few different ways. Finally, we'll look at a product of something just a *tiny* bit more complicated, and discover why knowing that p has a square root mod q tells us whether q has a square root mod p.

Terms and conditions apply. Void if p and q are not prime. Results may vary depending on residue mod 4. Consult a specialist if one of your primes is even.

Homework: None.

Prerequisites: Modular arithmetic: you should be comfortable with the claim that a has an inverse modulo m if and only if $\gcd(a, m) = 1$.

How not to solve linear inequalities (🌶🌶, Misha, 1 day)
If I give you a system of equations

$$2x + 3y - 5z = 1$$
$$7x - 2y + 6z = 2$$

you can find a solution you like by eliminating the variables, one at a time, until you get to a solution. That's boring, and efficient.

We're going to do the same thing to a system of inequalities. This will be exciting but horribly horribly inefficient. I'll explain why it was worth doing anyway.

Homework: None.

Prerequisites: A bit of linear algebra: we will have some matrices, we will multiply them on the right by column vectors, on the left by row vectors, and sometimes do both.

Random walks and electric networks (🌶🌶🌶, Misha, 1 day)
In this class, I will tell you a few basic rules about how voltage and current in an electric network behave.

Then, we'll see that some properties of a random walk on such a network follow the same rules—and prove that any two objects that follow these rules must be the same object.

You don't need to have taken my Markov Chains class in week 2. If you did take that class, you will see completely new things about random walks, so you don't need to worry about being bored.

Homework: None.

Prerequisites: None.

Neeraja's Classes

A very brief introduction to de Rham cohomology (🌶🌶🌶🌶, Neeraja, 2 days)
This is an extension of the "Integration on manifolds" class from week 1. Recall that we showed in week 1 that $d(d\omega) = 0$ for any differential form ω on $\mathbb{R}^n$. In this class we'll consider the following related question (actually, the converse): if $d\eta = 0$ for an n-form η, then is it always the case that $\eta = d\omega$ for some $(n-1)$-form ω? Studying this question will lead us to define the de Rham cohomology groups, which express rich topological information about a manifold. In this class we'll prove the Poincaré

lemma, which answers the above question in the affirmative if η is an n-form on a star-shaped domain in $\mathbb{R}^n$.

Homework: Recommended.

Prerequisites: Integration on manifolds from week 1 (but we'll do lots of review!) Note: group theory is NOT required.

Complex dynamics: Julia sets and the Mandelbrot set (🌶🌶, Neeraja, 1 day)
If $p(z)$ is a polynomial, the sequence of *iterates* is the sequence

$$p(z), p(p(z)), p(p(p(z))), p(p(p(p(z)))), \ldots$$

for a fixed complex number z. For what values of z does this sequence converge? Diverge? For what values of z is it periodic? These questions led mathematicians Julia and Fatou to define certain sets, one of which is the filled Julia set, the set of all z for which the sequence of iterates is bounded. In this class, we'll draw some pictures and study some properties of filled Julia sets. In the process, we'll also come across the Mandelbrot set, which has been called "the most fascinating and complicated subset of the complex plane."

Homework: None.

Prerequisites: Complex numbers (taking the modulus, writing a complex number in polar form).

Stirling's formula (🌶🌶, Neeraja, 1–2 days)
Stirling's formula gives an asymptotic estimate for $n!$, i.e. an approximation for $n!$ as n gets large. The formula first arose from correspondence between Stirling and de Moivre in the 1720s, when de Moivre used a version of the formula to discover what was essentially the central limit theorem in probability! (These days the central limit theorem is usually proved without using Stirling's formula.) In this class, we'll prove Stirling's formula and use it to solve some fun problems, such as showing the recurrence of the random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$.

Homework: Optional.

Prerequisites: Single-variable calculus (integration by parts).

Pesto's Classes

de Bruijn sequences (🌶🌶, Pesto, 2 days)
De Bruijn sequences are sequences containing all possible sequences of a given length exactly once as a subsequence, like 0111010001 for 0/1 sequences of length 3. We'll describe their properties (🌶), prove their existence (🌶🌶), and talk about their use in Sanskrit poetry (🌶).

Homework: Optional.

Prerequisites: None

Finding the center (🌶🌶🌶, Pesto, 1 day)
Given n points in the plane, how can we find the center of the smallest circle containing them if:

(1) Programmer time is the main constraint;
(2) Worst-case runtime is the main constraint;
(3) Average-case runtime is the main constraint?

These have different answers!

Homework: None.

Prerequisites: Understand the statement "An algorithm runs in time $O(n^2)$".

Flows (🌶🌶–🌶🌶🌶, Pesto, 2–4 days)

(1) We can color the vertices, edges, or faces of a planar graph, but nonplanar graphs don't have any faces to color, which makes them feel left out. We'll talk about how to color them anyway.
(2) A bear appears at the main entrance to the dorms. Many Campers Scatter Panickedly, trying to escape through the other doors. How should we evacuate?
(3) Hall's Marriage Lemma: if each of a bunch of boys has a certain set of girls they want to marry and vice versa, can everyone be happily married off?

These questions have in common that they can be approached as questions about *flows*, a way of assigning numbers to the edges of a digraph so that every vertex has the same sum coming in and going out. We'll see how.

Homework: Recommended.

Prerequisites: Having seen a proof of Hall's Marriage Lemma before will help: 🌶🌶🌶 if you haven't, 🌶🌶 if you have.

Multi-Coefficient Solving of Problems (🌶🌶🌶🌶, Pesto, 4 days)
Polynomials are a frequent topic of Olympiad-style competitions, since there are many and interesting problems using them. For instance, if $a_1, \ldots, a_n$ are distinct real numbers, find a closed-form expression for

$$\sum_{1\le i\le n} \prod_{1\le j\le n, j\neq i} \frac{a_i + a_j}{a_i - a_j}.$$

Don't see the polynomials? Come to class and find them.

If you took my class last year and want to take this one, let me know and I'll make it mostly disjoint.

Homework: Required.

Prerequisites: Linear algebra: be able to write at least two bases for the vector space of polynomials of degree at most 3 in x.

Planar graphs (🌶🌶, Pesto, 1–2 days)
A planar graph is one you can draw in the plane with no edges crossing. Is the number of regions formed the same no matter how you draw a planar graph in the plane? (Yes.) How many distinct ways can you draw a planar graph in the plane? If you want to draw a graph in the plane, do you have to plan ahead, or can you do it greedily? (You can be mostly greedy.) Do curved edges help, or can you draw every planar graph with straight-line edges? What graphs can be drawn in the plane?

Homework: Optional.

Prerequisites: None.

Voting theory 101 (🌶, Pesto, 1 day)
"The only fair voting system is a dictatorship". What properties would make a voting system "fair'? What sorts of (non-dictatorship) voting systems are pretty good, even if they're not "fair"?

Homework: None.

Prerequisites: None.

Susan's Classes

Continued fraction expansions and e (, Susan, 3 days)
The continued fraction expansion of e is

$$1+\cfrac{1}{0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{\ddots}}}}}}}}}}}}}}$$

Okay, but seriously, though, why?!?! Turns out we can find a simple, beautiful answer if we're willing to do a little integration. Or maybe a bit more than a little? No previous experience with continued fractions necessary. Come ready to get your hands dirty—it's gonna be a good time!

Homework: None.

Prerequisites: None.

More Determinacy! (, Susan, 2 days)
In How not to prove the Continuum Hypothesis, we *might* prove the consistency of analytic determinacy. We *might* show that AD implies that every subset of $\mathbb{R}$ is measurable. We definitely won't have time to do both, and we might not do either. If you want to see how these results (or more likely just one of them) work, come to this class!

Homework: None.

Prerequisites: How not to prove the Continuum Hypothesis.

Skolem's paradox (, Susan, 2 days)
Holy Axiomatizations, Batman! A ninja has snuck into the Museum of Real Numbers and stolen all but countably many of them. You, the curator, have a huge exhibition tomorrow. What are you going to do? Why, it's simple! You'll use the Lowenheim–Skolem theorem to build a countable model of set theory, complete with the real numbers. From inside the museum, no one will be able to tell that it's countable. To keep real number ninjas from interfering in your life, come to our class.

Homework: Recommended.

Prerequisites: None.

Tychonoff's theorem (, Susan, 2 days)
If you take the product of finitely many compact topological spaces, you get a compact topological space. What happens if you take an infinite product? Obviously you still get a compact space. Except… maybe not so obviously? This result is due to Andrey Nikolayevich Tikhonov, and turns out to be equivalent to the axiom of choice! This could be a good class for you if you have ever asked yourself the following questions:

- Why does the extreme value theorem for calculus work?
- What does an infinte-dimensional cube look like?
- What do you get when you cross a topologist with a logician?
- Wait... did Susan just spell Tychonoff's theorem with two F's, and then immediately spell Tikhonov's name with a V?

Even if you haven't ever asked yourself these questions, come to this class to find out the answers!

Homework: Recommended.

Prerequisites: None.

Tim!'s Classes

Calculus without calculus (🌶🌶, Tim!, 1–4 days)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Lizka is 5 cubits tall and Eric is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Lizka's head to the top of Eric's head that touches the ground in the middle. What is the shortest length of string you can use?
- Joanna rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters away along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest proportion of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Recommended.

Prerequisites: Some calculus will be useful for context, but we won't actually use calculus (that's the point).

Discrete derivatives (🌶🌶, Tim!, 1–4 days)
Usually, we define the derivative of f to be the limit of $\frac{f(x+h)-f(x)}{h}$ as h goes to 0. But suppose we're feeling lazy, and instead of taking a limit we just plug in $h = 1$ and call it a day. The thing we get is kind of a janky derivative: it's definitely not a derivative, but it acts sort of like one. It has its own version of the power rule, the product rule, and integration by parts, and it even prefers a different value of e. We'll take an expedition into this bizarre parallel universe. If we have three or four days, we'll apply what we find to problems in our own universe: we'll talk about Stirling numbers, and we'll solve difference equations and other problems involving sequences.

Homework: Optional.

Prerequisites: Calculus (derivatives).

Dynamic programming (🌶🌶, Tim!, 1–3 days)
Dynamic programming is a reliable way to get efficient algorithms for all sorts of problems. Here are a few such problems:

- If you have k eggs and an n story building, find a strategy to determine the highest floor of the building you can drop an egg from without it breaking.
- If you have a bunch of text that you would like to typeset (for instance, in LaTeX), how should you break the text into lines so that it is the most aesthetically pleasing?
- Given a collection of items each with a given integer weight and value, find a set of items that weighs at most 100 kilograms and has the maximum possible value.
- Given a level of Super Mario Bros., determine whether it is possible to beat the level (noting that any part of the level that scrolls off-screen gets reset).

Come find some quick algorithms!

Homework: Recommended.

Prerequisites: None.

Infinitesimal calculus (Tim!, 3–4 days)

If you've learned the definition of *continuous function*, you may have learned that a function f is continuous if an infintely small change in x results in an infinitely small change in $f(x)$. This is a pretty good definition: it's short, and you can picture it on a graph, and you can see the connection to more geometric descriptions ("a function is continuous if you can draw its graph without lifting your pen"). It's also how many of the pioneers of calculus thought about the subject.

But if you take a proof-based calculus class, you might see this definition instead: *A function f is continuous at c if for all $\epsilon > 0$, there is a $\delta > 0$ such that for all x with $|x - c| < \delta$, we have that $|f(x) - f(c)| < \epsilon$.* What an ugly definition! To be sure, it's correct, and is often useful, but nevertheless it's clunky and counterintuitive. Why would any class use it instead of the "infinitely small" definition? The problem is that there is *no such thing* as an infinitely small (or *infinitesimal*) real number.

Most proof-based calculus classes usually throw in the towel on infinitesimals at this point and haul out ϵ and δ instead. But not us. We'll just add some infinitesimal numbers to the real numbers to get the *hyperreal numbers*. And we'll get to have nice definitions like the one at the start of this blurb. We'll go through the process of defining the hyperreals. Then, we'll visit some of the the highlights of a calculus class, with proofs that are correct and often much simpler than the standard ones, but which nevertheless are alien and bizarre.

You see, when you start playing with with the fundamental building blocks of reality, things can start going totally bananapants. And perhaps we'll come to understand why most calculus classes shy away from infinitesimals.

Homework: Recommended.

Prerequisites: Some calculus.

Juggling torches in a tar pit (Tim!, 1 day)

Juggling is different from math. Math takes time and deep contemplation. Juggling happens quickly and requires coordination.

Juggling is the same as math. It takes practice. There are rules: What comes up must go down. You can only hold one thing in one hand at one time. One might wonder, mathematically, which juggling patterns are possible to juggle.

Terry Pratchett quote, but every "magic" is replaced with "math":

"There's a lot of loose thinking about math. People go around talking about mystic harmonies and cosmic balances and unicorns, all of which is to real math what a glove puppet is to the Royal Shakespeare Company.

"Real math is the hand around the bandsaw, the thrown spark in the powder keg, the dimension-warp linking you straight into the heart of a star, the flaming sword that burns all the way down to the pommel. Sooner juggle torches in a tar pit than mess with real math. Sooner lie down in front of a thousand elephants."

Homework: None.

Prerequisites: None.

The puzzle of the superstitious basketball player (🌶🌶, Tim!, 1 day)
Here's one of my favorite math puzzles. It's from Mike Donner, and it was published on FiveThirtyEight.

> A basketball player is in the gym practicing free throws. He makes his first shot, then misses his second. This player tends to get inside his own head a little bit, so this isn't good news. Specifically, the probability he hits any subsequent shot is equal to the overall percentage of shots that he's made thus far. (His neuroses are very exacting.) His coach, who knows his psychological tendency and saw the first two shots, leaves the gym and doesn't see the next 96 shots. The coach returns, and sees the player make shot No. 99. What is the probability, from the coach's point of view, that he makes shot No. 100?

I remember solving it. I had to do a bit of tedious calculation to arrive at the final answer. And when I saw the answer, I was astounded. It was so simple. I thought I was done with the puzzle, but really I was just beginning. Such a simple answer had to have a simple explanation, right? There are in fact a few simple explanations, each more satisfying than the previous.

In the end, I will make the following claim: even if we accept the scenario described by the puzzle, the basketball player's view of the world is totally wrong, and he is probably just superstitious. Perhaps there is a lesson here that we can take back with us to our real lives.

Homework: None.

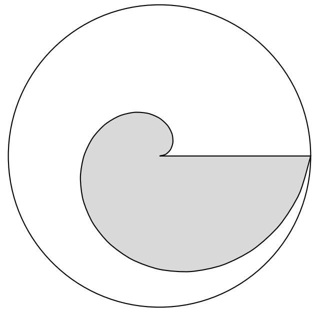
Prerequisites: None, but we'll spoil the answer to the puzzle pretty early in the class, so if you'd like to think about the puzzle yourself (which I wholly recommend), do it beforehand!

Yuval's Classes

Ancient Greek calculus (🌶–🌶🌶, Yuval, 1–2 days)
If you've ever seen a formal construction of the real numbers, you've probably heard of Dedekind cuts, named after the 19th-century German mathematician Richard Dedekind. However, he really doesn't deserve all the credit: 2000 years earlier, a Greek mathematician named Eudoxus of Cnidus came up with more or less the same definition. In my opinion, Eudoxus is the most important mathematician you've never heard of.

Even more, Eudoxus used his understanding of the real numbers to do what is essentially calculus; for instance, he was the first person to rigorously compute the volume of a cone. However, the mantle of ancient calculus was really picked up by Eudoxus's biggest fanboy, Archimedes. In my opinion, Archimedes is probably the most important mathematician you *have* heard of.

Building off of Eudoxus, Archimedes did some truly mind-blowing things. He computed the area of an arbitrary parabolic segment. He computed the volume and surface area of a sphere. He computed approximations of π. Perhaps most amazingly, he determined the area inside the following region, now called the *Archimedes spiral*.



If you've never seen this before, try it yourself—what fraction of the area of the circle is enclosed by the spiral? Even with modern integration techniques, the answer is not so easy to determine.

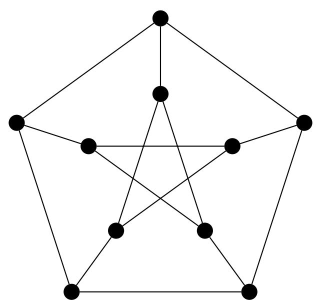
In this class, we'll get a sampling of ancient Greek proto-calculus. We'll start with Eudoxus's definition of the real numbers and we'll learn the "method of exhaustion", which was the proof technique he used to do calculus (it's more or less just evaluating an integral as a limit of Riemann sums). Then we'll move on to Archimedes and watch him do his magic, and we'll finish with his absolutely gorgeous argument for computing the area of a spiral.

Homework: Optional.

Prerequisites: Having seen integrals and Dedekind cuts will be helpful, but not necessary.

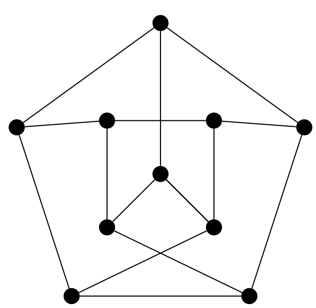
Crossing numbers (, Yuval, 2–3 days)

We really like drawing graphs in the plane. For instance, here's a drawing of the Petersen graph.



Sometimes, when we draw graphs in the plane, some of the edges cross, which is a real bummer. Even worse, this is often unavoidable—if a graph is non-planar, then we will *always* have a crossing, no matter how we do our drawing.

Nevertheless, we can still try to do better. For instance, here's a different drawing of the Petersen graph:



As you can see, this drawing has only two crossings, which is better than the five crossings we had earlier. As it turns out, two crossings is the best we can do for the Petersen graph: its *crossing number* is 2.

Somewhat surprisingly, studying crossing numbers is an enormously fruitful activity. In this class, we'll prove a fundamental result about crossing numbers, and then use this result to say many interesting things about apparently unrelated areas of math. For instance, we'll use this to attack one of my favorite open problems in all of math: what is the largest number of unit distances that can exist among n points in the plane?

Homework: Recommended.

Prerequisites: Basic graph theory: you should know what it means for a graph to be planar. A bit of probability will be helpful but not required.

The Erdős–Stone–Simonovits theorem (🌶🌶🌶🌶, Yuval, 2 days)
In Mia's Extremal graph theory class, you saw a statement but not a proof of the Erdős–Stone–Simonovits theorem, a theorem so important that it has been rightly called the fundamental theorem of extremal graph theory. It's also one of the coolest theorems I know.

In this class, we'll prove the Erdős–Stone–Simonovits theorem. To prove it, we'll need to take a surprising detour to the world of extremal *hypergraph* theory, which is like extremal graph theory, except way harder (and more hyper). For instance, Mantel's theorem from 1907 is the simplest and oldest result in extremal graph theory. The corresponding result for hypergraphs was conjectured by Turán in 1961 but is still wide open, and Erdős offered $500 for a proof of this conjecture.

Nevertheless, we'll be able to prove *just* enough about extremal hypergraph theory to deduce the Erdős–Stone–Simonovits theorem. So get hype!(rgraph)

Homework: Recommended.

Prerequisites: Extremal graph theory. If you know some graph theory but didn't attend Mia's class, come talk to me; you'll probably be able to follow the class, but it'll be hard to really understand what's going on if you've never seen any sort of extremal arguments before.

The mathematics of Jamboard-sharing (🌶🌶🌶, Yuval, 1 day)
During Week 1 relays, you may have encountered a question like the following.

> Linus has n campers in his linear algebra class, and they will work in groups of k. Linus wants to assign a Jamboard page to each of the $\binom{n}{k}$ possible k-tuples of campers. Since the number of total Jamboard pages is limited, he doesn't want to be wasteful, and may assign one page to more than one k-tuple. However, he wants to ensure that disjoint sets of campers get distinct pages. For example, if $k = 4$, then the 4-tuple {Alicia, Ben, Cody, Diana} will get a different page from the 4-tuple {Evan, Frank, Grace, Harini}, though {Alicia, Ben, Cody, Diana} may get the same page as {Alicia, Cody, Evan, Grace}.
>
> What is the smallest number of Jamboard pages Linus can use?

Though it may not look like it, this problem is actually a famous question in graph theory, first asked by Kneser in 1956. It took more than twenty years to resolve, until finally Lovász did so in 1978.

Despite the simplicity of the problem, Lovász's result is one of the most important turning points in 20th-century combinatorics. The reason is that although the question deals with elementary concepts like subsets and intersections, his proof used heavy topological machinery. In fact, it was this proof that begin the field of topological combinatorics, which is a major research area today, and which uses techniques from topology to attack questions that don't appear to involve any topology at all.

In this class, we'll see a beautiful simplification of Lovász's argument, due to Greene from 2002. Along the way, we'll explore the Borsuk–Ulam theorem, which is the main topological tool we'll need for the proof.

Note: Kayla is proposing a class about combinatorial topology. Despite the nearly identical names, combinatorial topology has *nothing* to do with topological combinatorics, or with this class. Sometimes mathematicians are just kinda bad at naming things.

Homework: Recommended.

Prerequisites: It would be helpful to know what the n-dimensional sphere S^n is, and what it means for a subset of S^n to be open or closed.

The uncertainty principle: redux (🌶🌶🌶🌶, Yuval, 1 day)
In Neeraja's Week 4 class, you saw/will see the standard proof of Heisenberg's uncertainty principle. This proof uses a number of important properties of the Fourier transform, such as the fact that it converts differentiation to multiplication by a variable (i.e. that $\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$), as well as important analytic techniques, such as integration by parts.

In this class, we won't do any of that. Instead, we'll derive Heisenberg's uncertainty principle from some very basic facts about the Fourier transform, together with a bunch of applications of the Cauchy–Schwarz inequality. This same technique will also allow us to prove a number of other uncertainty principles which hold in much greater generality.

Homework: Optional.

Prerequisites: You should be comfortable with the Cauchy–Schwarz inequality for integrals, which says that

$$\int f(x)g(x)\,\mathrm{d}x \le \left(\int f(x)^2\,\mathrm{d}x\right)^{1/2} \left(\int g(x)^2\,\mathrm{d}x\right)^{1/2}.$$

In theory, the only background in Fourier theory you'll need is that the Fourier transform is defined by $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x$. So congrats, you now have the prereqs for this class! However, to believe that this is an interesting and important thing to study, it will very much help if you attended Alan's Fourier analysis class and/or Neeraja's Uncertainty principle class.

CO-TAUGHT CLASSES

Many Counterexamples, Some Pathology (🌶–🌶🌶🌶🌶, Some staff, 1–5 days)
Do you want to see your instructors talk about awful stuff? In this class, various teachers will make reasonable-sounding statements and then tell you why they were wrong.

Some of the MYRIAD, CONFUSING, STRANGE POSSIBILITIES are listed below!

- Come hear from Ben, about bad things in analysis!
- Katharine is happy to do pathological spaces.
- Linus will show you a combinatorial geometry conjecture 'the answer to this problem is exactly $n+1$,' and then how the answer is really, really not $n+1$.
- Mira will state statistics that will stupefy you.
- Misha will destroy the hopes and dreams of economists studying the Internet.
- Susan will show you a ring that does a truly terrible thing.

Homework: Recommended.

Prerequisites: None.

CAMPER CLASSES

Functional completeness (🌶🌶, Erin, 1 day)
Let's say you're an engineer and you're tasked with making some machine. You call up the electronics company that makes your components, and it turns out your budget only allows you one mass-produced

circuit! Is there one circuit that can accomplish anything you could need to make? As it turns out, there are. These circuits are called functionally complete: logic gates or sets of logic gates that, on their own, can make any circuit, from calculators to robots to computers. Not all gates are cut and dried, either. The gate that takes 5 inputs and outputs whether the number of 1s is triangular is complete, but the same with 4 inputs isn't. But how do we find these elusive elements? We can use something called Post's Criterion. Come to the class to learn about it, prove it, and ensure we pick the right component!

Homework: None.

Prerequisites: None.

The Hilbert cube (🌶🌶🌶🌶, Harini, 1 day)
You're probably familiar with a one dimensional cube—the closed unit interval. And two dimensional cubes are also easy—unit squares. Three dimensional cubes are normal, and maybe you've seen hypercubes somewhere or the other. That's fine and good, because these are all finite dimensional. What happens when we try to take an infinite dimensional cube? How do things work there? This is called the Hilbert Cube, and it turns out that generalizing properties of finite dimensional cubes to it can be done, but with a little bit of cleverness. For example, we CAN define distance in this cube! Not only that, but when we do so, we can find a copy of ANY sufficiently small metric space inside it. And even more surprisingly, we can actually write down how to find this copy! Come to my class to find out how to do this!

Homework: None.

Prerequisites: None.

Dominant eigenvalues and directed graphs (🌶🌶🌶, Yuyuan, 1 day)
Suppose we have the following vector equation:

$$Ax + b = x$$

for some positive invertible matrix A and nonnegative vector b, does there exist a non-negative solution for x? This question can be answered with the help of the Perron–Frobenius theorem, which states that for an irreducible matrices, the dominant eigenvalue (i.e. the eigenvalue with the greatest magnitude) is real and positive, and its corresponding eigenvector has all positive entries. This theorem has many practical applications, such as in the fields of population modeling, statistical mechanics, and economic modeling. However, most proofs of this theorem require a lot of linear algebraic machinery. In this class, we will see a proof of this theorem that does not involve lots of algebraic manipulations; instead, we will assemble a proof through constructing directed graphs from matrices and converting the process of matrix multiplication into a process on graphs.

Homework: None.

Prerequisites: Familiarity with directed graphs, big-O notation, and eigenvalues.