A Shady Summation

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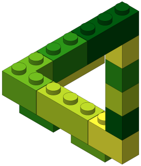

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The summation

This class is about a single sum:

$$\sum_{n=1}^{\infty} \frac{\lceil \pi n \rceil}{7^n} = \frac{\lceil \pi \rceil}{7} + \frac{\lceil 2\pi \rceil}{7^2} + \frac{\lceil 3\pi \rceil}{7^3} + \cdots$$

Let's use the most advanced techniques to evaluate this sum and try to understand it.

WolframAlpha

sum ceiling(pi n)/7^n

What WolframAlpha says

Input interpretation:

$$\sum 7^{-n} \lceil n\pi \rceil$$

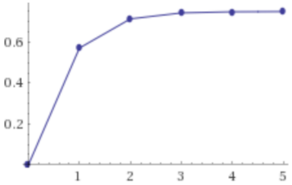
$\lceil x \rceil$ is the ceiling function

Approximated sum: Fewer digits More digits

$$\sum_{n=0}^{\infty} 7^{-n} \lceil n\pi \rceil \approx$$

0.75000020237785889082362121988516270775099104437498836327311377764177985660314430431801494843816592083121

Partial sums: More terms Show points



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Questions

Looking at that approximation, there is a natural guess to make: obviously, the exact value $\frac{926\,485}{1\,235\,313}$.

Mathematica says...this is pretty good!

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In[3]:= N[Sum[Ceiling[Pi n]/7^n, {n, 1, 100}] - 926485/1235313]
Out[3]= -1.64117 × 10^-83
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Questions we still have:

1. Why was it so close to $\frac{3}{4}$?
2. Is it actually $\frac{926\,485}{1\,235\,313}$?
3. If not, is it equal to anything nice?

Where does $\frac{3}{4}$ come from?

Things we know about π: it is a bit bigger than 3.

For small values of n, then, we expect to get $\lceil \pi n \rceil = 3n + 1$.

We have

$$\sum_{n=1}^{\infty} \frac{3n+1}{7^n} = \sum_{n=1}^{\infty} \frac{1}{7^n} + 3 \sum_{n=1}^{\infty} \frac{n}{7^n} = \frac{1}{6} + 3 \cdot \frac{7}{36} = \frac{3}{4}.$$

Of course this is not going to give us the correct terms. But the first wrong term is when $n = 8$:

$$\frac{3n+1}{7^n} = \frac{25}{7^8} \qquad \frac{\lceil \pi n \rceil}{7^n} = \frac{26}{7^8}$$

This is going to be most of the error, and $\frac{1}{7^8}$ is only about $0.000000173\ldots$.

Aside: computing $\sum_{n=1}^{\infty} \frac{n}{7^n}$

Write $\frac{n}{7^n}$ as a sum:

$$\sum_{n=1}^{\infty} \frac{n}{7^n} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{7^n}.$$

Swap the order of summations:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{7^n} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{7^n}.$$

Use the geometric series formula... twice:

$$\sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{7^n} = \sum_{k=1}^{\infty} \frac{1/7^k}{1-\frac{1}{7}} = \frac{\frac{1/7^1}{1-\frac{1}{7}}}{1-\frac{1}{7}} = \frac{7}{36}.$$

The error in $\frac{926\,485}{1\,235\,313}$

What's up with our second guess, $\frac{926\,485}{1\,235\,313}$?

We know a bit more about π: we have $\pi \approx \frac{22}{7}$. Computing the sum

$$\sum_{n=1}^{\infty} \frac{\left\lceil \frac{22}{7} n \right\rceil}{7^n}$$

is much more annoying, but works the same way as our previous sum, and gives $\frac{926\,485}{1\,235\,313}$ as an answer.

What is the error here? Well, a third, much better approximation to π is $\frac{355}{113}$.

The first time $\left\lceil \frac{22}{7} n \right\rceil$ and $\left\lceil \frac{355}{113} n \right\rceil$ disagree is when $n = 113$. This contributes most of the error: $7^{-113} \approx 3.19 \times 10^{-96}$.

The answer is not rational

Theorem

The value of $\displaystyle\sum_{n=1}^{\infty} \frac{\lceil \pi n \rceil}{7^n}$ *is irrational.*

I will only sketch the proof, leaving out a few details. Assumptions:

1. There is no end to approximations of π like $\frac{22}{7}, \frac{355}{113}, \frac{104\,348}{33\,215}, \ldots$.
2. Consecutive approximations have a nice relationship: e.g., $\lceil \frac{22}{7} n \rceil$ and $\lceil \frac{355}{113} n \rceil$ first disagree at $n = 113$.

These are both true for all irrational numbers; nothing special about π.

Rational numbers are hard to approximate!

Rational numbers are hard to approximate! Here is a number line of the rational numbers in $[0, 1]$ with denominator at most 20:

There are gaps around fractions with small denominator, like 1 or $\frac{1}{3}$ or $\frac{3}{4}$.

Algebraically: if $\frac{p}{q} \neq \frac{3}{4}$, then $\left|\frac{3}{4} - \frac{p}{q}\right| = \frac{|3q-4p|}{4q} \geq \frac{1}{4q}$.

Our sum is easy to approximate!

Our sum is easy to approximate: it is within about 7^{-8} of $\frac{3}{4}$.

If $\left|\frac{3}{4} - \frac{p}{q}\right| \approx 7^{-8}$, then q is at least about $\frac{7^8}{4}$.

There's more: our sum is within about 7^{-113} of $\frac{926\,485}{1\,235\,313}$.

If $\left|\frac{926\,485}{1\,235\,313} - \frac{p}{q}\right| \approx 7^{-113}$, then q is at least about $\frac{7^{113}}{1\,235\,313}$.

These lower bounds keep going. The next one will be about $\frac{7^{33\,215}}{7^{113}}$, by comparing the sum with $\frac{355}{113}$ to the better approximation $\pi \approx \frac{104\,348}{33\,215}$.

Our sum cannot be rational: it can be approximated by rational numbers much better than any rational number ever could!

Further reading

These slides: https://tinyurl.com/shadysum

The inspiration for the shady sum: a paper called *Strange Series and High-Precision Fraud* by Borwein and Borwein.

More about approximating π (and other numbers):
http://www.ams.org/publicoutreach/feature-column/fcarc-irrational1