CLASS DESCRIPTIONS—WEEK 1, MATHCAMP 2021

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Class action descriptions

Different classes will have you engage with the content in different ways. The following actions are meant to help you get a sense for what you might expect to do in each class.

If an action is selected, this means that the teacher is intentionally building this activity into your learning experience. An action may be listed as optional if the teacher is planning to offer space for this activity, but will not expect everyone to participate in it.

- **Active Listening:** You should be ready to do this for every class. May include listening to someone present information, reading things on slides, taking notes, and having opportunities to ask or respond to questions.
- 📖 **Reading:** Reading and interpreting a document for yourself (problem statements, paragraphs, etc.). This does not include anything that will be explained out loud at the same time, such as lecture notes or slides.
- 💬 **Speaking:** Sharing things out loud with others in the class. Ideas, problem solutions, etc.
- ✏️ **Writing:** Drawing or typing your thoughts during the class. Proofs, responses to questions, shared whiteboard, etc.
- 🤔 **Problem solving:** Thinking about how to solve problems that have been given to you.
- 👥 **Collaborating:** Working with others in a small group to accomplish a task.
- 🚣 **Other activitying:** Anything not included in the previous categories: playing a game, using a piece of software (besides Zoom), building a craft, etc.

Day 0 Colloquium

The Icosian game (Misha, Day 0 (Saturday, July 3) at 10:25)
Can a knight visit all 64 squares of a chessboard in 63 jumps, then come back to the start? What if we ask the same question for a 4×4 board? What if we're instead walking around the vertices of a dodecahedron?

In this colloquium, we will figure out when the answer to such a question is definitely "yes", and when it is definitely "no". In between, there will be a disturbingly large range of cases where we can only say "I don't know". But that's okay, because I'll also explain why, if you could always solve this problem easily, then you'd be able to win a million dollars, steal billions of dollars, and break all of mathematics as we know it.

This colloquium will start at 🌶️ and end at 🌶️🌶️🌶️🌶️.

9:00 Classes

Better sleep through modeling (*Olivia Walch*, MTWΘF)
What happens if we come up with math models to describe the phenomena of human sleep and circadian rhythms? What if we hook these models up to people's wearables and run them on phones? What if we use the models to compute what people should do during the day to sleep better at night? (Will people accept math as their sleep savior??)

Class format: Slides, mostly lecture

Prerequisites: Must have experience sleeping

Chilies	Class Actions	Themes (click for info)
🌶️	HW Recommended	Smörgåsbord Math in real life

Better sheep through modeling (J-Lo, MTWΘF)
Sheep need food, but there is not always enough food to go around. How do individuals (like sheep) respond to a lack of resources? Will they become aggressive and greedy, or learn to cooperate? If they settle on a solution, will they keep their word or betray each other's trust?

Evolutionary game theory is the study of how strategies and behaviors change over time. By making some simple assumptions about how individual sheep make decisions, we will be able to make predictions about how the entire herd changes.

Class format: The class will start with a game: you will be a sheep trying to collect as much food as possible. This will be followed by some whole-group review and discussion of this game, where participants will have the opportunity to share their thoughts if they wish to. The class will gradually transition to more of a lecture format as we begin to define some tools that can be used to study games of this kind.

Prerequisites: Must be willing to be a sheep

Chilies	Class Actions	Themes (click for info)
🌶️	🚣 (Optional: 💬) HW Recommended	Smörgåsbord Games Math in real life

Insert geometry joke here (Zoe, MTWΘF)
Looking at drawings, Pac-man's universe, or even the ground we walk on, many things appear "flat". Looking at space locally only tells us so much about the fundamental properties of the space itself. In this class we will look at an overview of different geometries and the ways of thinking of surfaces. We will especially look at ways of visualizing spaces and how they connect to problems one might encounter in various areas of math.

It can be extremely valuable to consider problems in spaces where they naturally reside. Mathematically, if we have a problem that loops in on itself, say considering words with two letters, we might want to consider solving that problem on a torus. If we need to differentiate the parity of an object, maybe it lives in a Möbius strip? For example, if we are looking to accurately represent what we see around us in a drawing on a piece of paper, we are using the properties of projective space!

Class format: The class will mostly be lectures with interruptions of hands on activities that campers will be recommended to participate in

Prerequisites: Linear algebra, if you're taking the week 1 Introduction to linear algebra at the same time that will be enough!

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	🤔 🚣 (Optional: 📖 ✏️ 💬 👥) HW Recommended	Squishy shapes

Introduction to group theory (Samantha, $\boxed{\text{MTW}\Theta\text{F}}$)
A group is a set of items together with a way for them to "interact" with each other. For example, one could take the real numbers, which interact with each other via adding. Or you could take the set of symmetries of a square, where interactions occur by composing the symmetries. In this class, we'll cover the basics about groups- defining them, interesting properties that certain groups have, mapping between groups, manipulating groups, and so forth.

Note: Groups are the fundamental object in algebra, and will be foundational for a lot of the classes at Mathcamp; as such, homework will be required so that you may get used to interacting with groups and proving results about them. Most of class time will be spent on lectures, but I may also give some in-class work time for the homework, if time permits.

Class format: Most of class time will be spent on lectures, but I may also give some in-class work time for the homework if time permits. Work time would be done in break out rooms so you could work with your classmates, if you'd like!

Prerequisites: None.

Required for: Representations of symmetric groups (W2); Topology through Morse theory (W2); Dirichlet's class number formula (W2); Finite fields and how to find them (W3); Lights, camera, group actions! (W3); Kleinian groups and fractals (W3); Archers at the ready! (W4); Finite Fourier analysis (W4)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	🤔 (Optional: 👥) HW Required	Algebraic structures

Sparsest cut (Alan, $\boxed{\text{MTW}\Theta\text{F}}$)
The sparsest cut problem asks the following: Given a (finite simple undirected) graph G, find a subset S of the vertices that minimizes

$$\frac{\text{number of edges with one endpoint in } S \text{ and one endpoint in } S^c}{\min(\text{number of vertices in S, number of vertices in } S^c)}.$$

The point of the denominator is to try to balance the sizes of S and S^c. You can think of sparsest cut as a discrete analogue of the isoperimetric problem.

Is there an efficient way to find the minimum? The short answer is "probably not," but fortunately, there is an efficient way to approximate the solution: if G has N vertices, then there is an algorithm that gives you a subset S which does not necessarily attain the minimum, but it will not be too far off. In particular, it will be within a factor of $\log N$ of the minimum.

In this class, we will introduce the algorithm and then discuss why this algorithm produces good approximations. A key part of the proof is connecting the sparsest cut problem to a fundamental question about discrete geometric spaces. If there is time, we will briefly mention how the discrete 5-dimensional Heisenberg group plays a role in a better (but more difficult) approximation algorithm for sparsest cut. These are just some of the many examples of the connections between theoretical computer science and metric geometry.

Class format: I will give lectures using Google Jamboard, and I will screen-share Jamboard from my tablet. If you'd like, you can open up Jamboard in your browser to browse through previous slides.

Prerequisites: There are no official prerequisites for this class. In particular, you do not need to have any background in theoretical computer science or graph theory. We will introduce many simple but new ideas rather quickly, which is why this is a 3-chili class.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	HW Recommended	CS & algorithms Discrete analysis Graph Theory

Topics in number theory (Misha, $\boxed{\text{MTW}\Theta\text{F}}$)
This is a class about exploring two things: number theory, and the various ways an online class can be taught. More precisely,

(1) On Monday, I will guide you through proving a theorem about Pascal's triangle modulo a prime p through a series of problems you can work on alone or in groups.
(2) Tuesday will be about how to compute the GCD of two numbers - in theory, and in the computer algebra system Mathematica. (Using Mathematica yourself is optional.)
(3) On Wednesday, I will give a more standard lecture about the Chinese remainder theorem—and, as a bonus, Lagrange interpolation.
(4) On Thursday, we'll look at some math competition problems about unique factorization, and solve them together in class.
(5) Friday's class is a slide-based lecture about several things we can prove (ending with quadratic reciprocity, if you've heard of this infamous theorem) by looking at the same product in two different ways.

These classes will be mostly independent, but they will generally be easier to follow if you go to all of them, and Wednesday's class is specifically a prerequisite for Friday's.

Class format: It will vary. See above for details!

Prerequisites: None

Required for: Factoring large prime numbers (W3); What are your numbers worth? or, the part of algebraic number theory we can actually do (W3)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	🤔 (Optional: 💬 🤔 👥 🚣) HW Optional	Smörgåsbord Number theory

10:10 Classes

How to count primes (Viv, $\boxed{\text{MTW}\Theta\text{F}}$)
How many primes are there?

Well, OK, infinitely many, but how many primes are there up to 100? 1000? 1000000? x? What kind of answer am I even looking for here?

In 1896, de la Vallée Poussin and Hadamard independently proved the Prime Number Theorem, which says that the number of primes up to x is $\frac{x}{\ln x}(1+o(1))$. We won't prove the Prime Number Theorem, but we will understand what the statement means, and we'll build up some fundamental tools in analytic number theory to allow us to prove something close.

Class format: Lecture; I'll be sharing a tablet screen and writing, with (hopefully) daily notes.

Prerequisites: Comfort with single-variable calculus (specifically the integral test for convergent series, differentiation, integration, integration by parts)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Recommended	Discrete analysis Number theory

Incidence combinatorics (Aaron, $\boxed{\text{MTW}\Theta\text{F}}$)

> *"My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances."* — Paul Erdős

In 1946, Paul Erdős asked two simple questions: If you place n points in the plane, and then measure the distances between every pair of points, what is the minimum number of distinct distances you can get? What is the maximum number of pairs of points that can be exactly distance 1 from each other?

The Distinct Distances Problem served as a challenge for decades, with mathematicians inventing a whole new field of mathematics – incidence combinatorics – to create a better lower bound every few years. Finally in 2015, Guth and Katz won a bunch of prizes for almost solving it. Meanwhile, the Unit Distances Problem is wide open—only one improvement has been made since 1946.

We will work together to reinvent this philosophy of counting, prove basically the best-known bounds on the Unit Distances Problem, and learn how you can count all kinds of things with just points on lines and curves.

Class format: IBL: This class will consist almost entirely of solving problems. Other than a brief intro each day, and some opportunities to present work, we will be in breakout rooms, working on the same problems.

Prerequisites: We will briefly use some probability theory (it will be good if you understand the phrase "linearity of expectation"). I'll also assume some graph theory, though I'll try to review all definitions.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️🌶️	📖 🤔 👥 (Optional: ✏️ 💬) HW Recommended	Discrete analysis Graph Theory

Introduction to quantum computing (Jorge, $\boxed{\text{MTW}\Theta\text{F}}$)

In this class, we will introduce what quantum computing is, and we will accomplish two main goals in order to do that. First, we introduce the basic unit of quantum information–the qubit. We will study the electron spin as a physical example where qubits can be created experimentally, and then we will introduce some math in order to represent and manipulate these units of information. (Spoiler: the math has a lot to do with vectors and matrices!)

The second goal is introducing the quantum teleportation algorithm. This is the procedure by which we can transport quantum information between two different people, with the only requisites of sharing an entangled qubit (I'll teach you what "entangled" means) and your usual Internet connection so you can send over 2 classical bits of information. Why so much trouble just to send information, you may ask? Well, as we shall see, measuring quantum information in any way tends to destroy it (you can say qubits are very shy), so this algorithm is kind of a big deal for quantum computers to work.

Class format: I'll be using my tablet as a whiteboard, and will lecture as we go. Questions are encouraged at any time. Will assign a couple of problems at the end of each lecture.

Prerequisites: Don't be intimidated by the description–no quantum mechanics or physics will be needed! We'll only need familiarity with how to add and subtract vectors, either numerically (when given the value of the components) or graphically (if the vectors are drawn on a plane). Knowing how to operate with matrices is a plus.

Chilies	Class Actions	Themes (click for info)
🌶🌶	HW Recommended	CS & algorithms
		Math in real life

Mathcamp crash course (Assaf, $\boxed{\text{MTW}\Theta\text{F}}$)
Math is useless unless it is properly communicated. Most of math communication happens through a toolbox of terminology and proof techniques that provide us with a backbone to understand and talk about mathematics. These proof techniques are often taken for granted in textbooks, math classes (even at Mathcamp!) and lectures. This class is designed to introduce fundamental proof techniques and writing skills in order to make the rest of the wonderful world of mathematics more accessible.

This class will cover direct proofs from axioms, proofs using negation, proofs with complicated logical structure, induction proofs, and proofs using cardinality and the pigeonhole principle. If you are unfamiliar with these proof techniques, then this class is *highly recommended* for you. If you have heard of these techniques, but would like to practice using them, this class is also right for you.

Here are some problems that can assess your knowledge of proof writing:

(1) Negate the following sentence without using any negative words ("no", "not", etc.): "*If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel.*"
(2) Given two sets of real numbers A and B, we say that A dominates B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two disjoint, nonempty sets A and B such that A dominates B and B dominates A.
(3) Prove that there are infinitely many prime numbers.
(4) Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
(5) Define rigorously what it means for a function to be increasing.
(6) What is wrong with the following argument (aside from the fact that the claim is false)? On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

    Proof. We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. □

(7) Mathcampers can message each other privately on Slack over the course of camp. Prove that there are two campers who messaged the same number of people throughout camp.

If you would not be comfortable writing down proofs or presenting your solutions to these problems, then you can probably benefit from this crash course. If you found this list of questions intimidating or didn't know how to begin thinking about some of them, then you should *definitely* take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Note that this class is *not* focused on exposing mathematical background, so if you'd like to brush up on the foundations of graph theory, number theory, etc., this is not where we will do that.

Class format: Lecture: handwritten notes on shared screen. Homework: proof-writing exercises to be submitted for feedback.

Prerequisites: None!

Chilies	Class Actions	Themes (click for info)
🌶️	✏️ HW Required	Smörgåsbord

Multivariable calculus crash course (Mark, $\boxed{\text{MTW}\Theta\text{F}}$)
In real life, interesting quantities usually depend on several variables (such as the coordinates of a point, the time, the temperature, the number of campers in the room, the real and imaginary parts of a complex number, …). Because of this, "ordinary"(single-variable) calculus often isn't enough to solve practical problems. In this class, we'll quickly go through the basics of calculus for functions of several variables. As time permits, we'll look at some nice applications, such as: If you're in the desert and you want to cool off as quickly as possible, how do you decide what direction to go in? What is the total area under a bell curve? What force fields are consistent with conservation of energy?

One reason, and maybe the best reason, to take this crash course right now rather than waiting until you encounter the material naturally after BC calculus and/or in college, is to be able to take the course on functions of a complex variable (which have many amazing features) that starts in week two.

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: Basic knowledge of single-variable calculus (both differentiation and integration)

Required for: Functions of a complex variable (1 of 2) (W2); PDEs part 1: Laplace's equation (W4)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	HW Recommended	Analysis

12:10 Classes

A Combinatorial Proof of the Jacobi Triple Product Identity (Gabrielle, $\boxed{\text{MTW}\Theta\text{F}}$)
For all q and z for which the following sums and products makes sense, we have the following equality, called the Jacobi Triple Product Identity:

$$\prod_{i=0}^{\infty}(1+zq^i)\prod_{j=0}^{\infty}(1+z^{-1}q^{j+1})\prod_{k=0}^{\infty}(1-q^{k+1}) = \sum_{\ell=-\infty}^{\infty} z^{\ell}q^{\ell(\ell-1)/2}$$

Yikes, but it's useful (we will see an application!). Even more yikes, we're going to prove this equality using partitions, where a partition of a nonnegative integer n is a way of writing n as a sum of positive integers, e.g. $4+1+1+1$ is a partition of 7. Come learn about the shocking and beautiful interplay between combinatorics and analysis. (The intercast of this interplay includes partitions, Young diagrams, and generating functions.)

Class format: Days 1-4: Lecture < 30 minutes, IBL for the rest of class time; Day 5: Lecture

Prerequisites: comfort with infinite series (be able to sum a geometric series, but we will ignore questions of convergence) and limits (not much deeper than an intuitive notion); if you are worried about it, talk to me!

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️🌶️	📖 (Optional: 👥) HW Recommended	Algebraic Combinatorics

Continued fractions (Ben, $\boxed{\text{MTW}\Theta\text{F}}$)
One common, and quite good, approximation for π is $\frac{22}{7}$, or $3+\frac{1}{7}$. A slightly better one is

$$\frac{333}{106} = 3 + \frac{1}{7+\frac{1}{15}},$$

and a *much* better one is

$$\frac{355}{113} = 3 + \frac{1}{7+\frac{1}{16}}.$$

(It's accurate to six decimal places—if we approximate π by $\frac{3141}{1000}$, it's accurate to only three places despite having a relatively *huge* denominator, which you'd expect to let us get closer.)

These "very close" approximations all come from continued fractions, which you might also have heard about in the golden ratio, φ, which is given by

$$\varphi = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}} = \frac{1+\sqrt{5}}{2}.$$

More generally, if we take an eventually periodic continued fraction, such as

$$3 + \frac{1}{4+\frac{1}{5+\frac{1}{4+\frac{1}{5+\frac{1}{4+\frac{1}{5+\dots}}}}}}$$

we can show that it's a number of the form $\frac{a+\sqrt{d}}{b}$ for some integers a, b, d.

But what about the other way around? If we have a real number of the form $\frac{a+\sqrt{d}}{b}$, what can we say about its continued fraction expansion?

This course will aim to answer this question, and also provide a more general introduction to continued fraction expansions.

Class format: The course will be primarily lecture-based, with homework problems building intuition for the concepts and exploring related topics

Prerequisites: None! Some past exposure to the idea of a limit won't hurt, but isn't strictly necessary

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Recommended	Discrete analysis

Cryptography and how to break it (Linus, $\boxed{\text{MTW}\Theta\text{F}}$)
In a typical cryptography class, you learn cryptosystems like Diffie-Hellman that let two people communicate securely even if all their messages are open to the public. And, yes, we will do that. (Contents: Diffie-Hellman, RSA, Yau circuit evaluation, mayyybe elliptic curves.)

But more importantly. Just as there exist smart mathematicians trying to make the internet secure, there exist smart mathematicians trying to learn your credit card details. And from their battle springs cool math. So in this class, we'll look at a ton of attacks that bust open modern cryptosystems when they are implemented even slightly incorrectly.

Class format: Lecture

Prerequisites: Modular arithmetic, enough to have $a^p = a \mod p$ deep in your heart.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	HW Recommended	CS & algorithms Number theory

Introduction to linear algebra (Emily, $\boxed{\mathrm{MTW\Theta F}}$)

Linear algebra is a fundamental area of mathematics that deals with vectors, matrices, and linear systems, but really it is so much more than that. It provides us with the building blocks to better understand so many areas of applied and pure math, from geometry to homological algebra to engineering. Every mathematician learns linear algebra at some point!

Some topics that we will explore: vector spaces, dimension, matrices, linear transformations, rank, and as many other topics as we can fit in at a 2-chili pace. We will approach topics from both computational (doing examples) and theoretical (doing proofs) perspectives, so some comfortability with abstract concepts would be good to have. Homework is required in the sense that you should definitely attempt it to gain a fuller understanding, but it will be neither collected nor graded.

Class format: You can expect about half of our class-time to be spent in a lecture format, and the other half spent doing IBL style worksheets.

Prerequisites: None officially, but taking the Mathcamp crash course at the same time would be a good idea if you are new to abstraction.

Required for: Representations of symmetric groups (W2); The calculus of variations (W3); The Schwarzschild solution (W3); The derivative as a linear transformation (W4); The fundamental theorem of algebra and its many proofs (W4); Finite Fourier analysis (W4)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	📖 🤔 (Optional: 💬 👥) HW Required	Algebraic structures

Kakeya sets over finite fields (Charlotte, $\boxed{\mathrm{MTW\Theta F}}$)

Imagine you have a needle on a table in front of you. (For our mathematical purposes, we assume said needle has zero width.) You would like to move the needle along the table so that it ends up rotated 180 degrees.

Certainly rotating through an appropriately sized circle would do the trick, but can we rotate the needle using less area? It turns out the answer to this question is yes - to an extreme degree. We can rotate it using arbitrarily small area—that is, through as small a region as we would like!

This probably surprising and counterintuitive fact (that we'll prove in class) leads to one of the most important and still unsolved conjectures in harmonic analysis: the Kakeya conjecture. In an effort to shed some light on this conjecture, mathematicians analyzed the analogous problem in the setting of finite fields. In the finite field setting, a shockingly simple and elegant proof was found using ... polynomials! In this class we'll discover the power of polynomials while proving the finite field Kakeya conjecture.

Class format: I'll be taking notes on a tablet during class. You'll be listening, and asking or answering questions if you'd like to!

Prerequisites: It would be helpful, but not necessary, to be comfortable with the definition of a vector space and its dimension.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	(Optional: ✏️ 💬 🤔) HW Recommended	Discrete analysis