CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2021

Contents

9:00 Classes

A pair of fractal curves (Ben, $\boxed{\text{MTW}}$ΘF)

"Fractal" is one of those words that comes up a lot in popular math; usually the definition is something like "a shape that looks like a part of that same shape." In this class, we'll start by investigating a pair of fractal curves[1] and then try to peer, just a little bit, into a more general picture.

The two particular curves, the Cantor–Lebesgue Staircase and Minkowski's ? Function, both arise when we switch between different ways of writing real numbers—in base two or base three, or as continued fractions. The more general picture should provide some insight into what would happen if we chose other perspective still. Best of all, the Cantor-Lebesgue Staircase is a very useful function to see at least once, because it is a particularly fun counterexample!

Class format: Interactive Lecture

Prerequisites: The second day will make the most sense if you've seen continued fractions before—but as long as you're willing to take my word on one or two things, the entire course should not have any prerequisites.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Recommended	Smörgåsbord Discrete analysis

The pirate game (Ben, MTW$\boxed{\Theta\text{F}}$)

In modern political science and economics, one of the major mathematical tools for analyzing and solving problems is game theory. Real, serious people use game theory to study real, serious problems.

That's not really what this course is about. We'll first cover some of the first basics of game theory, to make sure that everyone is on the same page when it comes time to think about some very unserious, entertaining, fun games. These games include such deep and relevant questions as: What should you do when you're captured by the bears? How many pirates have to die while dividing up treasure? How can the JCs best resolve their differences through the noble art of Nerf gun warfare?

Nevertheless, if you've heard a bit about game theory and want to get a grasp on the basics, this class should help familiarize you with some of the terminology and give you a good starting point for later investigations.

Class format: Interactive Lecture, then problem solving

[1]Truth in advertising!

Prerequisites: None!

Chilies	Class Actions	Themes (click for info)
🌶️	🤔 👥 HW Recommended	Smörgåsbord Games

Euclidean geometry beyond Euclid (*Yuval Wigderson*, MTW ΘF)
The poet Edna St. Vincent Millay wrote a sonnet[2] called "Euclid alone has looked on Beauty bare". In many ways, I agree with the sentiment—Euclidean geometry is one of the most beautiful subjects in mathematics, full of astonishing results and miraculous proofs. But I don't think it's fair to say that Euclid has a monopoly on its beauty; there's a ton of beauty in Euclidean geometry that Euclid didn't know about!

This class will be about those things: questions that Euclid might have asked (but didn't), and theorems that Euclid might have proved (but didn't). In many cases, we'll see that some simple-looking problems in Euclidean geometry lie on the forefront of modern mathematics, more than 2000 years after Euclid almost posed them (but didn't). For instance, it turns out that to *really* understand a basic question about counting points and lines in the plane, you probably need to understand ℓ-adic étale intersection cohomology and perverse sheaves.

Unfortunately, I have no idea what ℓ-adic étale intersection cohomology and perverse sheaves are, so we won't really get into that. But I plan to discuss a number of different topics in not-quite-Euclidean geometry, show you some of the surprising connections to other areas of mathematics like number theory and probability, and tell you about several conjectures that are still unproven, even though they could be as old as Euclid (but aren't).

Class format: Lecture on a virtual whiteboard, without breakout rooms.

Prerequisites: None

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Optional	Smörgåsbord Discrete analysis Rigid shapes

Algorithms on your phone (*Agustin Garcia*, MTW ΘF)
Your phone can do a whole lot of things, but how? Turns out, not all of it is hocus pocus and machine learning wizardry. In this class, we'll focus on *music processing* algorithms. We'll learn about the Fourier Transform and use it to identify frequencies in a signal. Then we'll look at applications, like Shazam's music detection algorithm and, time permitting, automatic beat detection.

Class format: The class will be conducted in a single Zoom room (lecture style). I will screen share as I hand-write on a notes app, and occasionally share other media/ code.

Prerequisites: Integration and Euler's Identity. Familiarity with asymptotic (big O) notation may help you appreciate the content more but is not necessary.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Optional	Smörgåsbord CS & algorithms Math in real life

[2] https://www.poetryfoundation.org/poems/148566/euclid-alone-has-looked-on-beauty-bare

Functions of a complex variable (1 of 2) (Mark, $\boxed{\text{MTW}\Theta\text{F}}$)
Spectacular (and unexpected) things happen in calculus when you allow the variable (now to be called $z = x + iy$ instead of x) to take on complex values. For example, functions that are "differentiable" in a region of the complex plane now automatically have power series expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both inside and outside math. For example, functions of a complex variable were used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if a and b are positive integers with $\gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, \ldots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann Hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet's theorem is certainly beyond the scope of this class and heat conduction probably is too, but we'll prove a major theorem due to Liouville that 1) leads to a proof of the so-called "Fundamental Theorem of Algebra", which states that any nonconstant polynomial (with real or even complex coefficients) has a root in the complex numbers; I believe Jorge will give the details in a week 4 class and 2) is vital for the study of "elliptic functions", which have two independent complex periods, and which may be the topic of a week 5 class. Meanwhile, we should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and venturing boldly forth into the complex plane! This class runs for two weeks, but it should be worth it. (If you can take only the first week, you'll still get to see a good bit of interesting material, including one or two of the things mentioned above.)

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: Multivariable calculus (the week 1 crash course will, by definition, be enough; you should have some comfort with partial derivatives and with line integrals, preferably including Green's theorem—but the week 1 course may not get to Green's theorem, in which case we'll cover it some time in the first week of this class)

Required for: Functions of a complex variable (2 of 2) (W3)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	HW Recommended	Analysis

Model theory (Aaron, $\boxed{\text{MTW}\Theta\text{F}}$)
At Mathcamp, we encounter loads of different mathematical widgets. There are groups, graphs, posets, tosets, rings, fields, vector spaces, and more. That's a lot to keep track of, but with model theory, we can view all of these as examples of the same phenomenon.

We'll tie all these together with a nice logical framework. We'll give general definitions of "mathematical structures," "axiom systems," and "proofs".

Then we'll use those definitions to construct some Alice-in-Wonderlandishly weird examples. A theorem that makes structures big, a theorem that makes structures small, infinite natural numbers, infinitesimal reals, and tiny universes of set theory that can fit in your (countably infinite) pocket.

Class format: I'll lecture during the class block. The homework will go over extra examples in a bunch of areas of math, and at least one big theorem will be proved in homework.

Prerequisites: Either group theory, graph theory, or ring theory (there are a lot of contexts where we can find model-theory examples, but it's important to have at least one you're comfortable with)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	🤔 HW Required	Algebraic structures

Representations of symmetric groups (Samantha, MTWΘF)
A representation of a group G is a homomorphism $\phi : G \to GL_n(\mathbb{C})$. In this class, we'll focus on representations of symmetric groups S_n, as their representations are particularly nice. You'll learn what it means for a representation to be irreducible, and why irreducible representations can be thought of as the building blocks for all representations. We'll also find all of the irreducible representations of S_n.

Class format: Lecture. There will be some recommended homework problems.

Prerequisites: intro group theory and intro linear algebra

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️🌶️	(Optional: 🤔 👥) HW Recommended	Algebraic Combinatorics Algebraic structures

10:10 Classes

Hilbert's 3rd problem (*Steve Schweber*, MTWΘF)
One very simple way to show that two shapes have the same size is to cut one into a few pieces and then rearrange those pieces to form a copy of the other (just straight-line cuts and only finitely many pieces—no shenanigans allowed, we're not doing set theory!). We can show that this always works for polygons: given polygons P and Q with the same area, we can always cut P into finitely many pieces using straight-line cuts and then reassemble those pieces into a copy of Q.

This raises a natural question: what about polyhedra? In 1900, Hilbert listed the question of whether two polyhedra of equal volume can always be "decomposed into each other" as one of 23 problems he thought would guide mathematical research in the coming century. (Granted, this one turned out to be a bit easier than expected—it was solved a year later by Hilbert's own student, Dehn.) The solution to the problem is a beautiful application of abstract algebra ...and a little pinch of the axiom of choice (OK fine we may be doing a teeny bit of set theory).

Class format: Mostly lecture, still working on getting hardware together but I suspect document camera or similar.

Prerequisites: None.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️🌶️	HW Recommended	Rigid shapes

Introduction to graph theory (Marisa, MTWΘF)
A *graph* is a mathematical object with a bunch of things (called "vertices"), some of which have connections between them (called "edges"). You could argue that just about anything is a graph. And you could extrapolate, perhaps, that graph theory is the most important subject in all of mathematics.

But all jokes aside: it's a branch of math in which we get to ask—and sometimes answer—lots of interesting questions right away, even without building up too much machinery. For example: in Misha's colloquium from Day 0 about the Icosian Game, we got right to "finding a Hamiltonian cycle in a graph." This week will have a similar flavor to the first half of Misha's colloquium: we'll be building up some vocabulary as we investigate matchings, planar graphs, colorings, and lots more.

Class format: We'll spend most our time collaborating as a whole group, with lots of opportunities for you to chime in (out loud or in the chat).

Prerequisites: None

Required for: Graph colorings (W3); The probabilistic method (W4); Evolution of random graphs (W4)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	(Optional: 💬 🤔) HW Recommended	Graph Theory

Sit down and (don't) solve SAT? (Zoe, M TWΘF)
How can we tell how hard a problem is? There are a lot of hard problems, but are they just problems that a human can't do? Or that might take a computer billions of years to solve? A lot of modern society depends on the fact that factoring integers is hard; it is how we can do all e-commerce. If this seems intriguing, we can all sit together and look at SAT or satisfiability and how it relates to all sorts of other problems!

We will see an overview of some basic complexity class definitions and why it's interesting to try to classify problems by their complexity. The class will mainly focus on getting in as many surprising SAT reductions as we can because why not.

Class format: Lectures and lots of example problems and problems to work on

Prerequisites: None

Chilies	Class Actions	Themes (click for info)
🌶️	✏️ 🤔 👥 (Optional: 💬 🚣) HW Optional	CS & algorithms

The special theory of relativity (Jorge, M TWΘF)
What happens to the classical laws of motion as we get closer and closer to the speed of light? A very young Einstein (in his early 20s) asked this question at the beginning of the 20th century and came to groundbreaking conclusions, that supported the development of some very interesting mathematics.

In this class, we will cover the mathematical description of space-time (i.e you'll get to learn why time is called the "fourth dimension" and how it is NOT really independent from space!) We will study the correct transformation laws of space-time between two observers in relative motion at speeds in the order of the speed of light–also known as Lorentz transformations. From these, seemingly-paradoxical consequences arise: the simultaneity of events, the length of objects and the very perception of time are seen as different between observers moving at different speeds!

We will also study the algebraic structure of four-vectors, which are needed to describe physical quantities in a way consistent with Lorentz transformations, as well as the index notation used to describe tensors. This will be great preparation to continue studying the theory of General Relativity (see class "The Schwarzchild solution") on Week 3!

Class format: Lecture based with individual problem solving in class, if time allows.

Prerequisites: The algebra of vectors and matrices, especially writing systems of linear equations in matrix form. Some familiarity with differential equations can help students obtain a greater appreciation of the theory.

Required for: The Schwarzschild solution (W3)

Chilies	Class Actions	Themes (click for info)
🌶🌶🌶	(Optional: 🤔) HW Recommended	Math in real life

Topology through Morse theory (*Kayla Wright*, M $\boxed{\text{TW}\Theta\text{F}}$)
Many mathematicians joke that a donut and a coffee mug are "the same." In this class, we will flip a donut on its side, and show this is true using a chocolate icing function. We will also talk about some basic topology and on top of this, we will develop general techniques to show that defining special functions from a topological space to $\mathbb{R}$ can completely determine a space up to continuous deformation.

Class format: Interactive lecture. I plan to share my iPad stream and write live! Maybe I can assign some readings beforehand if students are interested.

Prerequisites: Group Theory

Required for: Using the Cantor set to classify (infinite) surfaces (W3)

Chilies	Class Actions	Themes (click for info)
🌶🌶🌶	HW Optional	Squishy shapes

12:10 Classes

Combinatorial species (Linus, $\boxed{\text{MTW}\Theta\text{F}}$)
Permutations are a species. You know how to count how many permutations there are.

Domino tilings of width-2 rectangles, is a species. You may or may not know how to count them.

If you breed them, you get the species "Permutations next to domino tilings." Or if you breed them a different way you get "Permutations *of* domino tilings" or "Domino tilings, where each tile is a permutation." Do you know how to count those? How about doing it in literally one line of algebra?

The theory of combinatorial species shows us how to associate each species to a "generating function," an algebraic cheat code for solving combinatorics problems. A beautiful correspondence transports us between algebraic operations on generating functions, and different types of breeding. We'll use this to count some elusive types of objects, and also, to give 2-line proofs of a few bijections that would probably take a whole class period to prove normally.

Class format: I'll lecture some amount (more or less depending on the day) and then break y'all into breakout rooms. I'll have some problems for you to solve, as well as time for camper-guided exploration and discovery of new species.

Prerequisites: If you already know how to use generating functions, you'll be a little bored in day 1. Knowing how to differentiate functions like xe^x is nice, but it's totally OK to just type it into WolframAlpha.

Chilies	Class Actions	Themes (click for info)
🌶🌶🌶	🤔 👥 HW Optional	Algebraic Combinatorics

Dirichlet's class number formula (Viv, $\boxed{\text{MTW}\Theta\text{F}}$)
Here's a fun fact:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Some people think of this fact as a fact about the Taylor expansion of arctangent. But I think of it as a very deep fact about the function $x^2 + y^2$, or in other words, a specific example of Dirichlet's wonderful class number formula.

A *binary quadratic form* is a function

$$f(x, y) = ax^2 + bxy + cy^2,$$

with $a, b, c \in \mathbb{Z}$, which has *discriminant* $d = b^2 - 4ac$. The *class number* of a given discriminant d is the number of equivalence classes of binary quadratic forms with discriminant d under a certain group action. Dirichlet's proof of his Class Number Formula is a truly beautiful argument, with ideas ranging from group theory to clever averaging to, at one point, the area of an ellipse. Come explore one of my favorite proofs of all time!

Class format: Lecture; I'll be sharing a tablet screen and writing.

Prerequisites: Group Theory; if you haven't seen group actions, talk to me! It is also helpful if you have seen the Chinese Remainder Theorem, and 2-by-2 matrices (how they multiply, and how they act on 2-dimensional vectors).

Chilies	Class Actions	Themes (click for info)
🌶🌶🌶🌶	HW Recommended	Number theory

Introduction to analysis (Alan & Charlotte, $\boxed{\text{MTW}\Theta\text{F}}$)
This class is a rigorous introduction to limits and related concepts in calculus. Consider the following questions:

(1) Every calculus student knows that $\frac{d}{dx}(f + g) = f' + g'$. Is it also true that $\frac{d}{dx}\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n'$?
(2) Every calculus student knows that $a + b = b + a$. Is it also true that you can rearrange terms in an infinite series without changing its sum?

Sometimes, things are not as they seem. For example, the answer to the second question is a resounding "no." The Riemann rearrangement theorem, which we will see, states that we can rearrange the terms in infinite series such as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ so that the sum converges to π, e, or whatever we want!

To help us study the questions above and many other ones, the key tool we'll use is the "epsilon-delta definition" of a limit. This concept can be hard to work with at first, so we will study many examples and look at related notions, such as uniform convergence. Being comfortable reasoning with limits is central to the field of mathematical analysis, and will open the door to some very exciting mathematics.

Class format: Mostly lecture-based. We'll spend some time each class in breakout rooms discussing problems.

Prerequisites: a calculus class of some kind

Required for: The inverse and implicit function theorems (W4); Nowhere differentiable but continuous functions are everywhere! (W4)

Chilies	Class Actions	Themes (click for info)
🌶🌶	💬 🤔 👥 HW Required	Analysis

Introduction to ring theory (Susan, $\boxed{\text{MTW}\Theta\text{F}}$)
Ring theory is a beautiful field of mathematics. We cut ourselves loose from our usual number systems—the complexes, the reals, the rationals, the integers, and just work with ...stuff. Stuff that you can add. And multiply. Rings are structures in which addition and multiplication exist and act

as they "should." Polynomials, power series, matrices, real-valued functions on a set—wherever you have some way of defining an addition and a multiplication, you've got a ring.

Somehow, in throwing away the numbers that gave us our initial intuition about how addition and multiplication should work, we are left with a tool that is immensely powerful. Ring theory is the backbone of fields such as algebraic geometry, representation theory, homological algebra, and Galois theory.

This class will be a quick introduction to some of the basics of ring theory. We will cover the ring axioms, homomorphisms of rings, quotient rings, and several important examples and counterexamples.

Class format: Interactive lecture with problem sets designed to substantially boost understanding.

Prerequisites: None

Required for: Finite fields and how to find them (W3); Noncommutative ring theory (1 of 2) (W3)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	(Optional: 💬) HW Required	Algebraic structures

Problem solving: geometric transformations (Misha, MTWΘF)
In this class, we will learn how to use geometric transformations to solve math competition problems. The following topics will be covered:

(1) Translation and central symmetry (Monday)
(2) Rotation and reflection (Tuesday)
(3) Similarity and spiral similarity (Wednesday)
(4) Inversion (Thursday)
(5) On Friday, we'll see problems of all types mixed together.

In class, we will learn about how to use these transformations, and how to spot when they can be used, by solving problems together. There will be problems left to solve on your own. You won't need to solve these to keep up with the class, but you should, because solving problems on your own is critical to learning problem-solving.

Class format: Extra-interactive lecture: I will often ask for suggestions for how to begin solving a problem from you, and follow those whenever possible.

Prerequisites: The equivalent of a high school geometry class.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	🤔 (Optional: 💬) HW Recommended	Rigid shapes Symmetries

Colloquium (3:00–4:00)

How Math Invented a New Way to Fight Infectious Disease (*Po-Shen Loh*, MTWΘF)
Last Spring, a team of math and science people joined forces with some great non math and science people to develop a way to use network theory, smartphones, and anonymous epidemiological data to fight COVID. One year on, they discovered a categorically new way to fight disease, which in theory could have a significant impact on the COVID pandemic now, and on many future pandemics. They are now collaborating with several teams of researchers to empirically study deployments in practice. Its origins come from math, game theory, and computer science. This became the NOVID app, which is fundamentally different from every other pandemic app (and which resolves deep issues with "contact tracing apps").

Functionally, it gives you an anonymous radar that tells you how "far" away COVID has just struck. "Far" is measured by counting physical relationships (https://novid.org).

The simple idea flips the incentives. Previous approaches focused on controlling you after you had already been exposed to the virus, preemptively removing you from society because you were suspected of being infected. This new tool lets you see incoming disease before you're exposed, to defend yourself just in time. This uniquely aligns incentives so that even if people do what is selfishly best for themselves (self-defense), they end up contributing to the good of the whole.

Class action descriptions

Different classes will have you engage with the content in different ways. The following actions are meant to help you get a sense for what you might expect to do in each class.

If an action is selected, this means that the teacher is intentionally building this activity into your learning experience. An action may be listed as optional if the teacher is planning to offer space for this activity, but will not expect everyone to participate in it.

- **Active Listening:** You should be ready to do this for every class. May include listening to someone present information, reading things on slides, taking notes, and having opportunities to ask or respond to questions.
- 📖 **Reading:** Reading and interpreting a document for yourself (problem statements, paragraphs, etc.). This does not include anything that will be explained out loud at the same time, such as lecture notes or slides.
- 💬 **Speaking:** Sharing things out loud with others in the class. Ideas, problem solutions, etc.
- ✏️ **Writing:** Drawing or typing your thoughts during the class. Proofs, responses to questions, shared whiteboard, etc.
- 🤔 **Problem solving:** Thinking about how to solve problems that have been given to you.
- 👥 **Collaborating:** Working with others in a small group to accomplish a task.
- 🚣 **Other activitying:** Anything not included in the previous categories: playing a game, using a piece of software (besides Zoom), building a craft, etc.