CLASS DESCRIPTIONS—WEEK 4, MATHCAMP 2021

Contents

9:00 Classes

Archers at the ready! (Zoe, $\boxed{\text{MTW}\Theta\text{F}}$)
In homological algebra, it is easy to get sick of drawing arrows. Between the fleets of arrows and all the chains, cochains, cycles, boundaries, and torsion, it is hard to tell whether we are studying math or some medieval fantasy realm. While looking at homological algebra we will also motivate the topological origins while showing why we want to take these mechanisms further in more algebraic contexts. Warning: commutative diagrams will be everywhere.

Class format: This will be a lecture style class

Prerequisites: Group theory (maybe some field theory and ring theory too)

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️🌶️	📖 🚣 (Optional: ✏️ 💬 🤔) HW Recommended	Algebraic structures

Causal inference: how can we tell if X causes Y? (*Mira Bernstein*, $\boxed{\text{MTW}\Theta\text{F}}$)
You probably know that correlation does not imply causation. But usually, whether we are interested in basic science, medicine, or social policy, it's causation that we actually care about. So how **can** we tell whether X causes Y?

Statisticians have developed a number of *ad hoc* approaches to this question over the years — most famously, randomized controlled trials (RCTs). But sometimes randomization isn't possible: for example, you can't do an RCT to determine whether smoking causes cancer. It is only in the last few decades that a unifying mathematical framework for causal inference has emerged. Using the formalism of structural causal models, a researcher can tell exactly which quantities they need to measure, which variables they need to manipulate, and what causal conclusions they can draw from the available data. In addition to being extremely useful, structural causal models are also fun to play with: you basically get to solve lots of mini-puzzles involving directed graphs.

Warning: Math-wise, this class is 🌶️, but it might be more like a 🌶️🌶️🌶️ class in terms of how much it taxes your brain: you'll need to learn a whole new way of formalizing real-world problems, which can be quite challenging. However, it'll tax your brain somewhat differently than the way math classes usually do.

Homework Recommended: In the homework, you actually get play with data in R and implement the things we discussed in class. (You don't have to know any R: I'll provide a quick tutorial with everything you need.)

Class format: Mix of interactive lecture and IBL

Prerequisites: Comfort with Bayes' Theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$. If you're not familiar with this formula and its applications, don't worry – it's very elementary and you can easily look it up! For example, if you read the explanation at https://brilliant.org/wiki/bayes-theorem/ and work out some of the examples given there, that should be enough.

Chilies	Class Actions	Themes (click for info)
🌶️	🤔 👥 🚣 (Optional: ✏️ 💬) HW Recommended	Math in real life

The 17 worlds of planar ants (*Dror Bar-Natan*, $\boxed{\text{MT}}$WΘF)
My goal is to get you hooked, captured and unreleased until you find all 17 in real life, around you.

We all know that the plane can be filled in different periodic manners: floor tiles are often square but sometimes hexagonal, bricks are often laid in an interlaced pattern, fabrics often carry interesting patterns. A little less known is that there are precisely 17 symmetry patterns for tiling the plane; not one more, not one less. It is even less known how easy these 17 are to identify in the patterns around you, how fun it is, how common some are, and how rare some others seem to be.

Gotta Catch 'Em All!

Class format: Frontal yet conversational lessons on a shared zoom whiteboard on which everyone will write.

Prerequisites: None

Chilies	Class Actions	Themes (click for info)
🌶️	✏️ (Optional: 💬) HW Recommended	Smörgåsbord Symmetries

The derivative as a linear transformation (Alan, $\boxed{\text{MT}}$WΘF)
Suppose we have a function $\mathbf{f}: \mathbb{R}^3 \to \mathbb{R}^2$. We can write $\mathbf{f}(x, y, z) = (g(x, y, z), h(x, y, z))$. In multivariable calculus, we learn about partial derivatives of $\mathbf{f}$. There are 6 different partial derivatives, which we can arrange into a matrix:

$$\begin{pmatrix} \partial_x g(x,y,z) & \partial_y g(x,y,z) & \partial_z g(x,y,z) \\ \partial_x h(x,y,z) & \partial_y h(x,y,z) & \partial_z h(x,y,z) \end{pmatrix} \tag{1}$$

If all the partial derivatives are continuous, then the matrix above is called the *total derivative* (or just *derivative*) of $\mathbf{f}$ and is denoted $\mathbf{f}'(x, y, z)$.

Arranging all the partial derivatives this way is not just for notational convenience. A 2×3 matrix represents a linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$. In this class, we will see how to think of the derivatve of a $\mathbb{R}^m \to \mathbb{R}^n$ map as a linear transformation, and we will use this point of view to prove and interpret results such as the chain rule (in both single-variable and multivariable calculus).

Class format: lectures on Jamboard

Prerequisites: You should know the definition of the derivative from single-variable calculus. (You do need to know any multivariable calculus. Furthermore, this class does not overlap with Mark's multivariable class from Week 1.) You should be comfortable with matrix multiplication.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Recommended	Smörgåsbord Analysis

The inverse and implicit function theorems (Alan, MT$\boxed{\text{W}\Theta\text{F}}$)
If a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{f}'(\mathbf{x}) \neq 0$ for all $\mathbf{x}$, then the function $\mathbf{f}$ is invertible. In this class, we will look at a generalization of this to higher dimensions called the *inverse function theorem*: "If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a function such that $\mathbf{f}'$ is continuous and $\det \mathbf{f}'(\mathbf{x}_0) \neq 0$, then $\mathbf{f}$ is locally a $\mathcal{C}^1$ homeomorphism near $\mathbf{x}_0$." (We will explain the precise meaning of this in class.)

It turns out the higher-dimensional situation is much harder than the one-dimensional situation. To understand the proof of the inverse function theorem, we will need tools such as the total derivative, linear algebra, and the Banach fixed-point theorem. We will also see a corollary of the inverse function theorem called the implicit function theorem, which allows us to describe solutions to system of equations as $\mathcal{C}^1$ submanifolds of Euclidean space.

Class format: lectures on Jamboard

Prerequisites: Week 2 "Introduction to analysis" and Week 4 "The derivative as a linear transformation" (or the equivalent to these classes)

Chilies	Class Actions	Themes (click for info)
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The probabilistic method (Mia, MT$\boxed{\text{W}\Theta\text{F}}$)
Consider a tournament consisting of n soccer teams, each of which plays every other team precisely once. Suppose that for any set of k teams in the tournament, there is some team that beat all of them in tournament play. For a fixed k, how big does n have to be for such a tournament to exist? Does such an n always exist?

A simple enough question to pose – but one that remained unsettled until 1963, when Erdős introduced and used probabilistic and nonconstructive methods to provide a (remarkably elegant!) proof that such tournaments exist! This class will illustrate several such examples of problems in combinatorics that are incredibly difficult to resolve with normal constructive methods, yet have elegant and short probabilistic proofs.

Class format: Interactive lecture with occasional breakout rooms for problem-solving.

Prerequisites: Basic graph theory

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	🤔 HW Recommended	Smörgåsbord Discrete analysis Graph Theory

Trail mix (Mark, $\boxed{\text{MTW}\Theta\text{F}}$)
Is your mathematical hike getting a little too strenuous? Would you like to relax a bit with a class that offers an unrelated topic every day, so you can pick and choose which days to attend, and that does not expect you to do homework? If so, some Trail Mix may be just what you need to regain energy. Individual descriptions of the five topics follow.

Trail Mix Day 1: The Prüfer Correspondence.

Suppose you have n points around a circle, with every pair of points connected by a line segment. (If you like, you have the complete graph K_n.) Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments, but in only one way. (This tree will be a spanning tree for K_n.) How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll find a combinatorial proof that is especially cool.

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Prerequisites: None.

Trail Mix Day 2: Cyclotomic Polynomials and Migotti's Theorem.

The cyclotomic polynomials form an interesting family of polynomials with integer coefficients, whose roots are complex roots of unity. Looking at the first few of these polynomials leads to a natural conjecture about their coefficients. However, after the first hundred or so cases keep confirming the conjectured pattern, eventually it breaks down. In this class we'll prove a theorem due to Migotti, which sheds some light on what is going on, and in particular on why the conjecture finally fails just when it does.

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Prerequisites: Some experience with complex numbers, preferably including complex roots of unity; some experience with polynomials.

Trail Mix Day 3: Exploring the Catalan numbers.

What's the next number in the sequence $1, 2, 5, 14, \ldots$? If this were an "intelligence test" for middle or high schoolers, the answer might be 41 ; that's the number that continues the pattern in which every number is one less than three times the previous number. If the sequence gives the answer to some combinatorial question, though, the answer is more likely to be 42 . We'll look at a few questions that do give rise to this sequence (with 42), and we'll see that the sequence is given by an elegant formula, for which we'll see a lovely combinatorial proof.

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Prerequisites: None

Trail Mix Day 4: Intersection Madness.

When you intersect two ellipses, you can get four points, right? So why can't you get four points when you intersect two circles? Well, actually you can, and what's more, two of the four points are always in the same places! If this seems paradoxical (and, I hope, interesting), wait until we start intersecting two cubic curves (given by polynomial equations of degree 3). There's a "paradox" there too, first pointed out by the Swiss mathematician Cramer in a letter to Euler, and the resolution of that paradox leads to a "magic" property of the nine intersection points. If time permits, we'll see how that property (known as the Cayley-Bacharach theorem) gives elegant proofs of Pascal's hexagon theorem and of the existence of a group law on a cubic curve.

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Prerequisites: None, although a little bit of linear algebra might show up.

Trail Mix Day 5: Integration by Parts and the Wallis Product.

Integration by parts is one of the only two truly general techniques for finding antiderivatives that are known (the other is integration by substitution). In this class you'll see (or review) this method, and encounter two of its applications: How to extend the factorial function, so that $(1/2)!$ ends up making sense (although the standard notation used for it is a bit different), and how to derive the famous product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5}\frac{6}{5}\frac{6}{7} \cdots,$$

which was first stated by John Wallis in 1655.

Chilies: 🌶️🌶️🌶️

Prerequisites: Basic single-variable calculus.

Class format: Mostly interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking some questions rather than just talking "at" you. I *may* set up breakout rooms for you to experiment for a little while at the beginning of the Prüfer Correspondence class, but probably not for the other classes.

Chilies	Class Actions	Themes (click for info)
🌶️ → 🌶️🌶️🌶️	No HW	Smörgåsbord

10:10 Classes

Evolution of random graphs (Misha, M$\boxed{\mathrm{TW\Theta F}}$)
Imagine the following picture: at midnight, we scatter ten thousand vertices in an empty night sky. Each of the $\binom{10\,000}{2}$ possible edges between these vertices chooses a random time between midnight and 1 AM. At its chosen time, the edge lights up.

At midnight, the sky is dark; at 1 AM, it is lit up by the complete graph $K_{10\,000}$. What does the sky graph look like between those times?

The sky process is random, but when the number of vertices is sufficiently large, we can make confident predictions about when the graph will acquire certain properties. When will the first cycle appear? When will the graph become connected? When will we see the first Hamiltonian cycle?

In this class, we will answer these questions and more—and learn some techniques of probabilistic combinatorics as we go.

Class format: Interactive lecture; I will present slides which I will annotate as we go.

Prerequisites: Introduction to graph theory, or comfort with basic graph theory concepts such as trees, subgraphs, vertex degrees, distance.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	HW Recommended	Discrete analysis Graph Theory

Knot theory (Emily, M$\boxed{\mathrm{TW\Theta F}}$)
In this class, we will explore the fascinating and dynamic field of knot theory. While learning what is a knot and what is not a knot, we will encounter various types of knots and not shy away from manipulating knots and knotting knots together. Whether we are knot sure or not sure if two knots are the same, there are a variety of invariants we can use to distinguish between knots. Why not come and have some knot fun!

Class format: Interactive lecture

Prerequisites: None

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	HW Recommended	Squishy shapes

Nonunique factorization in the Chicken McNugget monoid (Gabrielle, M$\boxed{\mathrm{TW\Theta F}}$)
Let's ponder one of life's greatest mysteries: The Chicken McNugget. Questions about Chicken McNuggets date all the way back to Ancient Grease. What are they made of? Are they even food? (I do not have answers for you there.) How many Chicken McNuggets is it possible to buy?

When Chicken McNuggets were first introduced, they were sold in packs of 6, 9, and 20. Under those assumptions, any number n that can be written as $6\cdot x_1+9\cdot x_2+20\cdot x_3 = n$ can be purchased. But such a representation does not have to be unique. There are two ways to buy 18 Chicken McNuggets, by buying 3 6-packs or 2 9-packs, corresponding to the two ways of writing

$$18 = 6 \cdot 3 + 9 \cdot 0 + 20 \cdot 0$$

or

$$18 = 6 \cdot 0 + 9 \cdot 2 + 20 \cdot 0.$$

We can think of these two expressions as two distinct factorizations of the number 18. This brings us to the big question that we will answer, to some extent, in this class: How nonuniquely can we factor quantities of Chicken McNuggets (and what should this mean)?

This class is not sponsored by McDonald's.

Class format: IBL, minimal lecture.

Prerequisites: None

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	📖 (Optional: 💬 👥) HW Optional	Algebraic structures

Nowhere differentiable but continuous functions are everywhere! (Charlotte, M$\boxed{\text{TW}\Theta\text{F}}$)
Analysis is rather well-known for, and well-loved by many because of, its pathological examples. An example that definitely deserves this qualification is Weierstrass's "Monster of Analysis": a function that is continuous everywhere, but not differentiable anywhere. Instead of constructing such a function, and therefore proving that such a function exists, we will show that there are *many, many* such functions, without even constructing a single one! Somehow, this will be easier.

We will introduce a way of measuring the size of a set in terms of its density (known as Baire category, but it's not what it probably sounds like to you). In the sense of Baire category, we will show that the set of continuous but nowhere differentiable functions is "large." Consequently, we'll see that the aforementioned Monsters of Analysis, while seemingly quite pathological, may in fact be a rather generic example of a continuous function. [1]

An unintended BONUS for taking this class: You will learn what the heck the "bairespace" emoji in Slack is referring to.

Class format: Most of the class will be an interactive lecture. We will sprinkle in some time for activities and problem solving.

Prerequisites: You should have some background in analysis. In particular, you should feel comfortable with ϵ-δ proofs, know the definition of a sequence in terms of ϵs, and have worked with sequences of functions before. It would be useful, but not strictly necessary, to be familiar with metric spaces, uniform convergence, and Cauchy sequences. I will have a quick intro to all of these concepts.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️🌶️	🤔 👥 (Optional: ✏️ 💬) HW Recommended	Analysis

The fundamental theorem of algebra and its many proofs (Jorge, M$\boxed{\text{TW}\Theta\text{F}}$)
The Fundamental Theorem of Algebra states that any complex polynomial must have a complex root. Although this problem would seem to lie at the intersection of the theory of numbers and the theory of equations, it is absolutely baffling how many seemingly-unrelated areas of mathematics can be interweaved to come up with different beautiful proofs of the result!

Indeed, in this class we will think of the FTA in terms of advanced calculus, complex analysis, field extensions and topology. We will give an independent proof inside of each domain—a testament to

[1]!!

the richness of the complex numbers as a mathematical structure. Perhaps more importantly, this gives us an excuse to learn advanced mathematics with an immediate theoretical application.

As we go through each of the proofs, I will encourage us to discuss what is it about them that makes them fail on a different field, such as $\mathbb{Q}$ or $\mathbb{R}$. This kind of reasoning will help you truly understand what $\mathbb{C}$ and $\mathbb{R}$ are, and what makes them unique.

Class format: Lecture with QA.

Prerequisites: Complex numbers arithmetic, in particular Euler's formula and De Moivre's formula. Calculus: continuity, integration, and the extreme value theorem. Abstract algebra: the definitions of groups, rings and fields. Linear algebra: the definition of a vector space.

Chilies	Class Actions	Themes (click for info)
🌶🌶🌶🌶	HW Optional	Smörgåsbord Analysis

12:10 Classes

Finite Fourier analysis (Mike Orrison, $\boxed{\text{MTW}\Theta\text{F}}$)
In this course, we will focus on how and why you might want to rewrite a complex-valued function defined on a finite abelian group as a linear combination of simple functions called characters. As we will see, doing so quickly leads to discussions of far-reaching algorithms and ideas in mathematics such as discrete Fourier transforms (DFTs), fast Fourier transforms (FFTs), and the Uncertainty Principle.

Class format: I was thinking I'd keep it pretty simple by just sharing my slides on Zoom, and annotating them along the way when helpful. I'm not planning on using any breakout rooms, etc.

Prerequisites: Complex Numbers (basic arithmetic and geometric interpretations), Linear Algebra (bases, invertible matrices, eigenvalues, complex inner products, orthogonality), and Group Theory (examples of finite abelian groups and their subgroups, cosets, homomorphisms).

Chilies	Class Actions	Themes (click for info)
🌶🌶🌶	HW Recommended	Discrete analysis

Learning online learning online (*Eric Neyman*, $\boxed{\text{MTW}\Theta\text{F}}$)
You should read the title of this class as Learning (Online Learning) Online. Online learning – and more generally online algorithms – is a topic within the study of algorithms. The word *online* here does not mean "over the internet"; instead it means *sequential decision-making*. For example, let's say that once the pandemic is over, you want to celebrate by eating out at the restaurant down the block once a week for a year. (It just opened up so you haven't been there, but the menu looks good!) You order a dish, and really like it. The following week, you need to make a decision: get the same dish again, or try something new? Getting the same dish again means you'll probably like it a lot, but if you get something new then on the off chance that you like it *even more*, that'll be good to know for the future. How should you balance trying new things with getting things you already know you like? If you find this question intriguing and want to learn more, come to my class!

Class format: I'll be using my iPad as a whiteboard (so I'll be screen sharing my iPad and writing on it). Probably around 20% of the time will be spent in breakout rooms. Of the other 80%, probably 25% will discussion and 75% will be interactive lecture.

Prerequisites: basic facts about expected values, like linearity of expectation. Basic calculus will at times be helpful, but you should be able to follow most of the class without it.

Chilies	Class Actions	Themes (click for info)
🌶️🌶️	💬 🤔 👥 HW Recommended	CS & algorithms Math in real life

Non-Euclidean geometries (Samantha, $\boxed{\text{MTW}\Theta\text{F}}$)
The geometry you've seen in school was likely Euclidean geometry, and it's based on five different axioms, or assumptions. One of those axioms is the Parallel Postulate, which says that if you have a line and a point that's not on the line, you can draw exactly one line through that point which is parallel to the first line.

What happens if you remove this axiom? You'll get different types of geometry called spherical and hyperbolic geometry. In this class, we'll explore basic geometry concepts in these geometries and see what's the same and what's different from Euclidean geometry. Note: Very few proofs (if any) will be given. Instead, you'll be exploring the concepts using online applets and testing out lots of examples!

Class format: First day will be a lecture and the remaining days will be IBL, meaning I'll give you lots of questions to explore in groups, and you'll learn by working through the problems (instead of being taught directly by me).

Prerequisites: Euclidean geometry

Chilies	Class Actions	Themes (click for info)
🌶️	📖 💬 🤔 👥 No HW	Rigid shapes

Noncommutative ring theory (2 of 2) (Susan, $\boxed{\text{MTW}\Theta\text{F}}$)
A ring R is called "simple" if its only ideals are $\{0\}$ and R itself. A commutative ring is simple if and only if it is a field. A good first guess would be that a noncommutative ring is simple if and only if it is a division ring. This would also be a wrong first guess. Unfortunately, in noncommutative rings simplicity gets complicated.

In this class we'll prove the Wedderburn–Artin's structure theorem for semi simple rings, and see how simplicity and division rings are related. We'll start this week with a quick jaunt through module theory. We'll define modules, prove Zassenhaus's lemma and the Schreier refinement theorem, and use these results to prove the Jordan–Hölder's theorem, which essentially says that every sufficiently nice module has a unique decomposition into its simplest components.

Class format: Interactive lecture

Prerequisites: Noncommutative ring theory (1 of 2)

Chilies	Class Actions	Themes (click for info)
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PDEs part 1: Laplace's equation (Assaf, $\boxed{\text{MTW}\Theta\text{F}}$)
Laplace's equation is the deceptively simple

$$\Delta f = g,$$

perhaps with some initial/boundary conditions, where $\Delta f = \sum_i \frac{d^2}{dx_i^2} f$ is the Laplacian. $g = 0$, we say that f is *harmonic*, and it turns out that there are lots and lots of strange, yet obvious-in-hindsight properties of harmonic functions. For example, nonconstant harmonic functions on $\mathbb{R}^n$ never attain an extreme value, and we can use this to provide a proof of the fundamental theorem of algebra using only partial differential equations! As another example, the value of an harmonic function at a point is equal to the average value of it on a circle around that point.

If any of this seems a bit like complex analysis, you're mistaken! We will prove all of these facts using just the fact that $\Delta f = 0$!

In this class, we'll go through basic properties of Laplace's equation with various boundary and initial conditions. We'll prove the maximum principle, mean-value property, and link them to Brownian motion. We'll talk about energy methods, and possibly move on to other, more advanced topics in elliptic PDEs.

Get ready to integrate by parts a lot!

Class format: Lecture with small work periods

Prerequisites: Multivariable calculus -or- the Derivative as a linear transformation. Integration by parts is highly recommended. Complex analysis mildly recommended.

Chilies	Class Actions	Themes (click for info)
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Class action descriptions

Different classes will have you engage with the content in different ways. The following actions are meant to help you get a sense for what you might expect to do in each class.

If an action is selected, this means that the teacher is intentionally building this activity into your learning experience. An action may be listed as optional if the teacher is planning to offer space for this activity, but will not expect everyone to participate in it.

- **Active Listening:** You should be ready to do this for every class. May include listening to someone present information, reading things on slides, taking notes, and having opportunities to ask or respond to questions.
- 📖 **Reading:** Reading and interpreting a document for yourself (problem statements, paragraphs, etc.). This does not include anything that will be explained out loud at the same time, such as lecture notes or slides.
- 💬 **Speaking:** Sharing things out loud with others in the class. Ideas, problem solutions, etc.
- ✏️ **Writing:** Drawing or typing your thoughts during the class. Proofs, responses to questions, shared whiteboard, etc.
- 🤔 **Problem solving:** Thinking about how to solve problems that have been given to you.
- 👥 **Collaborating:** Working with others in a small group to accomplish a task.
- 🚣 **Other activitying:** Anything not included in the previous categories: playing a game, using a piece of software (besides Zoom), building a craft, etc.