WEEK 5 CLASS PROPOSALS, MATHCAMP 2021

These class proposals don't have theme or class action information, but if you have questions, and you can't make a guess based on class format/blurb, feel free to ask the teacher or your AA!

Contents

Aaron's Classes

Compactness in combinatorics of coloring (🌶️🌶️🌶️, Aaron, 1–2 days)
We take a look at some of the things you can do if you throw the Compactness Theorem at combinatorics problems that involve coloring graphs and hypergraphs. We'll see how this can let us understand colorings of infinite graphs using finite graphs, and more surprisingly, understand colorings of finite graphs using infinite graphs.

Class format: Lecture on virtual whiteboard with probably a bit of optional homework

Prerequisites: Model Theory, some graph theory

Homework: Optional.

Frieze patterns and triangulations (🌶️🌶️, Aaron, 3–4 days)
In 1973, Conway and Coxeter published a paper on frieze patterns of natural numbers, classifying them and showing their relationship to polygons. However, this wasn't an ordinary paper — this was a problem set with solutions. Clearly we are meant to uncover the mysteries of frieze patterns for ourselves.

Class format: IBL: We will work in breakout rooms on a self-contained series of problems, inspired by Conway and Coxeter's original paper on the subject.

Prerequisites: A bit of linear algebra (facts I can review about determinants of mostly 2x2 matrices)

Homework: Optional.

Sperner's lemma and Brouwer's fixed point theorem (🌶️🌶️🌶️, Aaron, 2 days)
Take your favorite triangle, and paint its corners red, green and blue. Now subdivide it into a lot of tiny triangles, and paint all the corners of all of those with the three colors, so that each color on the edge between red and blue is red or blue, and each color on the edge between green and blue is green or blue, and each color on the edge between red and green is red or green.

Sperner's Lemma says there is at least one tiny triangle with a corner of each color.

Take your favorite continuous map from the disk to itself. Brouwer's Fixed Point Theorem says it has a fixed point.

In this IBL class, we will prove this very discrete lemma, and use it to prove this very continuous theorem.

Class format: IBL: We will solve problems in breakout rooms.

Prerequisites: Sperner's Lemma will require just a bit of graph theory, Brouwer's FPT will require the idea of Cauchy sequences.

Homework: Optional.

The Ra(n)do(m) graph (🌶🌶🌶, Aaron, 2 days)
Let's pick a random graph on countably many vertices... ok, here it is. A nice graph. Now let's pick another one... huh. That's weird, I got the same graph. Ok, let's try again... there must be some mistake, the random graph machine is broken, I keep getting the same graph!

Find out why there's only one Random Graph, conveniently known as The Random Graph or The Rado Graph, and find out how to build it, and how it contains the entire world of finite graphs within its murky depths.

Class format: Lecture, but with OPTIONAL homework problems connecting it to model theory.

Prerequisites: A bit of graph theory, a smidge of probability. (In particular, you do NOT need "Evolution of Random Graphs".) Model theory will be required for some OPTIONAL homework.

Homework: Optional.

Ultraproducts (🌶🌶🌶🌶, Aaron, 2–4 days)
If you took my model theory class, you'll remember that the Compactness Theorem is basically the fundamental theorem of logic. However, we didn't quite have time to prove it, so we just took the completeness theorem for granted instead. Let's fix that!

We'll prove the compactness theorem through problems. First we show that there exist these wacky objects called *nonprincipal ultrafilters*, and then we appeal to this magic ultrafilter to summon forth a model of any finitely-consistent theory.

This class will run for either 2 or 4 days. If it runs for 2 days, we prove the compactness theorem. If it runs for 4 days, we'll do that and then take the construction we just did, aim it at the real numbers, and build the hyperreals, which we can use to do nonstandard analysis!

Class format: IBL: Problem-solving in breakout rooms

Prerequisites: Model Theory (and calculus with epsilon-delta proofs for optional nonstandard analysis second half)

Homework: Optional.

Alan's Classes

Finite field Kakeya without the polynomial method (🌶🌶–🌶🌶🌶, Alan, 2 days)
Let $\mathbb{F}_q$ be a finite field. A *finite field Kakeya set* is a set $K \subset \mathbb{F}_q^n$ that contains a line in every direction. (More precisely, for every $v \in \mathbb{F}_q^n \setminus \{0\}$, there exists an $x \in \mathbb{F}_q^n$ such that $\{x + tv : t \in \mathbb{F}_q\} \subset K$.) In this class, we will prove some lower bounds on the cardinality of K.

(1) On the first day (2 chilis) we will use incidence geometry (more specifically, the fact that two lines intersect in at most one point) to prove that every Kakeya set in $\mathbb{F}_q^n$ has dimension at least $(n+2)/2$, or more precisely:
   - Every Kakeya set in $\mathbb{F}_q^n$ has cardinality at least $c_n q^{(n+2)/2}$, where the constant c_n depends only on n.

(2) On the second day (3 chilis), we will use additive combinatorics to prove that every Kakeya set in $\mathbb{F}_q^n$ has dimension at least $(6n+5)/11$. (For the precise statement, replace $(n+2)/2$ in the exponent above with $(6n+5)/11$.)

In week 1, Charlotte used the polynomial method to prove a result that is much stronger than the two above. Our proofs do not use the polynomial method. While our proofs give weaker results, they also work in the Euclidean case and show some of the ways in which the Kakeya conjecture is connected to many other areas of mathematics.

Class format: Lecture

Prerequisites: Know about finite fields, know the Cauchy–Schwarz inequality

Homework: Optional.

Fractal projections and a number theory question (🌶️🌶️, Alan, 2 days)
Let $K_0 \subset \mathbb{R}^2$ be the unit square. Divide K_0 into 16 squares of equal size, and let $K_1 \subset K_0$ be the union of the four corner squares. Repeat the same procedure on each of the four squares of K_1 to get K_2 (a union of sixteen squares), and so on. We define the four corner Cantor set to be the limit set $K = \bigcap_{n=0}^{\infty} K_n$.

In this class, we will discuss some interesting properties of the projections of the four corner Cantor set, including connections to the following number theory fact: If m and n are odd integers, then m/n can be written as the ratio of two numbers of the form $\sum_{j=0}^{\ell} \epsilon_j 4^j$, where $\epsilon_j \in \{-1, 0, 1\}$. (Incidentally, this number theory fact is proved in a paper called "An awful problem about integers in base four.")

Class format: Lecture

Prerequisites: None

Homework: Optional.

Kakeya sets via Baire category (🌶️🌶️🌶️, Alan, 2 days)
A set $K \subset \mathbb{R}^2$ is called a Kakeya set (or a Besicovitch set) if it has area zero and contains a unit line segment in every direction. We will use the Baire category theorem to prove that not only do Kakeya sets exist, they are actually "generic" in the sense of Baire category.

Class format: Lecture

Prerequisites: Baire category theorem (e.g., Charlotte's Week 4 class)

Homework: Optional.

Maximum cut (🌶️🌶️🌶️, Alan, 2 days)
The max-cut problem asks: Given a graph $G = (V, E)$ find a subset S such that the maximizes the number of edges between S and S^c. There is a simple randomized 0.5-approximation algorithm. We will study Goemans–Williamson approximation algorithm, which has the better approximation ratio of

$$\frac{2}{\pi} \min_{0 \le \theta \le \pi} \frac{\theta}{1 - \cos\theta} \approx 0.878.$$

This approximation works by considering the semidefinite programming relaxation of max-cut. (We will see what a semidefinite program is, but we will not have time to prove that semidefinite programs can be solved efficiently.)

Class format: Lecture

Prerequisites: Know what a graph it, know the definition of a positive definite matrix.

Homework: Optional.

Alan and Charlotte's Classes

Completeness of the real numbers (🌶️🌶️, Alan and Charlotte, 1 day)
Our Week 2 Introduction to Analysis class was *full of holes* (:partyparrot:) because there were many topics we did not have time to cover (:actualsadparrot:). This class is an attempt to *fill in some holes* (:partyparrot:) by discussing the completeness of the real numbers. Recall that the rational numbers are full of holes! The way to fill in the holes is by "completing" them, thus obtaining the real numbers. This can be done in various ways, including via Cauchy sequences, monotone sequences, least upper bounds, and more. In this class, we will discuss these various ways to think about completeness of the reals.

Class format: The class is mostly lecture-based. We'll spend some time in breakout rooms discussing problems.

Prerequisites: The Week 2 analysis class, or the Week 4 nowhere differentiable but continuous functions class; more specifically, you should know the epsilon-delta definitions of convergent sequences and Cauchy sequences.

Homework: Optional.

Uniform convergence (🌶️🌶️, Alan and Charlotte, 1 day)
In our Week 2 Introduction to Analysis class, we did not have enough time to cover all the topics *uniformly* (:partyparrot:), which may have led to *discontinuities* (:partyparrot:) in your knowledge of various analysis topics (:actualsadparrot:). This class is an attempt to address that by discussing uniform convergence. Recall that the uniform limit of a sequence of continuous functions is continuous. In this class, we will discuss other properties of uniformly convergent sequences of functions, including how uniform convergence interacts with integration and differentiation.

Class format: The class is mostly lecture-based. We'll spend some time in breakout rooms discussing problems.

Prerequisites: The Week 2 analysis class, or the Week 4 nowhere differentiable but continuous functions class; more specifically, you should be comfortable with epsilon-delta type proofs, sequences of functions, and uniform convergence.

Homework: Optional.

Assaf's Classes

Geometric group theory (🌶️🌶️–🌶️🌶️🌶️, Assaf, 5 days)
Geometric group theory is, broadly speaking, the study of groups by their actions on metric spaces[1]. One such metric space is the Cayley graph, which is a well-defined geometric object up to quasi-isometry, which is like an isometry, but allows for finite amounts of scaling and fudge factors.

In this class, we will explore this further, and prove the fundamental theorem of geometric group theory: The Svarc-Milnor Lemma, and other related concepts.

Class format: Lecture

Prerequisites: Group theory, a bit of topology, knowing what a metric space is

Homework: Recommended.

[1] the "geometric" in geometric group theory referring to metric spaces

Geometry and quaternions (🌶🌶🌶🌶, Assaf, 5 days)
The quaternion algebra is a four-dimensional algebra generated by i, j, k, and having the relation:

$$i^2 = j^2 = k^2 = ijk = -1.$$

But in this class, we will not be talking about the algebra, but rather how to understand the algebra in terms of geometry. Specifically, we'll relate these to rotations of $\mathbb{R}^3$, the cross product, and the Hopf-Fibration. All this will lead us to the real treat – Octonions.

Class format: Lecture with IBL portions

Prerequisites: Linear algebra, some group theory may be helpful

Homework: None.

The Hopf–Poincaré index formula (🌶🌶🌶, Assaf, 4–5 days)
The Hopf–Poincaré index formula is an incredible piece of math that says that if V is a continuous vector field on a surface S with isolated zero set $Z(V)$, then

$$\sum_{p \in Z(V)} ind_p(V) = \chi(S)$$

I love this formula. It's so cool because it involved topology, vector fields, and relates them in such a concise way. Also, its proof is super awesome.

In this class, we will prove this theorem. Along the way, we will talk about Euler characteristics, prove the (combinatorial) Gauss–Bonnet theorem, and the hairy ball theorem, and explain why soccer balls must have pentagonal patches.

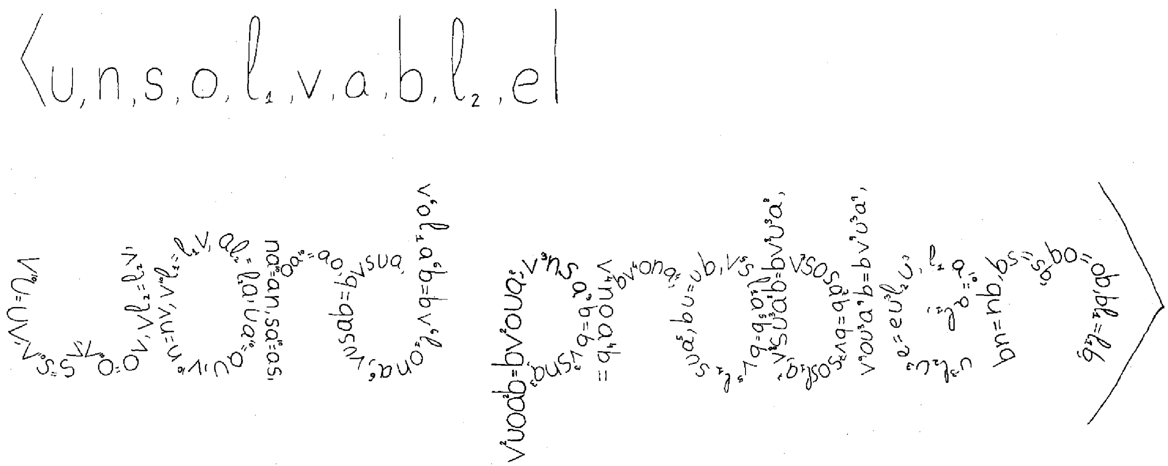
Class format: IBL/worksheets

Prerequisites: Should know what a surface is, and an intuitive definition of a continuous function

Homework: Recommended.

The word problem for groups (🌶🌶–🌶🌶🌶, Assaf, 3–4 days)
A group can be thought of as a collection of reversible operations, with some rules about how they relate to each other. Such a way of thinking is called a *group presentation*, and as an example, we have that the cyclic group of order 3 can be written as $\langle x | x^3 = e \rangle$, where x is a generator, and $x^3 = e$ is a relation. Given a group presentation with a finite set of generators and relations, does there always exist an algorithm to tell if a word in the generators is the identity? This is called the Word Problem, and it was posed by Max Dehn in 1911. The surprising answer, shown by Pyotr Novikov in 1955, is no - there does not exist such an algorithm. In this class, we will prove this result by studying how we can embed "uncodable" sets in groups. This will give us a candidate for a "bad" group for which a solution to the word problem would contradict the "uncodablity" of the set.

Additionally, this class comes with a really sweet t-shirt design that we should totally reprint this year, and which contains a group presentation that has unsolvable word problem (we will not prove that this specific presentation does not have solvable word problem in this class):



Class format: Lectures with some homework

Prerequisites: group theory

Homework: Recommended.

Traffic and the price of anarchy (🌶, Assaf, 1 day)
If we all took the bus, the roads would be empty, and transit would be fast. And yet, for some reason, it always seems like there is gridlock. Why can't we all just co-operate? Why do drivers always have to cut me off when I'm biking to campus? Why is it that closing roads produces better traffic in the rest of the city? How is all of this related to COVID-19 and vaccines?

Everyone wants to be rational, but sometimes our irrational human nature comes out and bites the collective. In this class, we'll explore scenarios where this effect happens. We'll look at Braess' Paradox, the Bus Motivation Problem, and spend some time discussing the formalism of congestion games.

Class format: Lecture

Prerequisites: know what a bus is

Homework: Recommended.

Ben's Classes

A curve with area (🌶🌶, Ben, 2 days)
If you take a pencil, crayon, or marker, it's pretty easy to fill in a square with its drawing line—it's so easy because the line has some thickness, so you can fill in a square with a finite amount of work. Clearly, as we make the line thinner and thinner, it will get harder and harder to fill in the square. Intuitively, this *might* make it seem like it will be infinitely hard to fill in a square using a line of zero thickness[2], and that "infinitely hard" might mean "impossible."

Infinitely hard or not—you decide! Because there are infinitely thin curves that still fill in a square—curves that have area, space-filling curves. In this class, we'll see one elegant construction of a space-filling curve, due to Hilbert.

Along with being mathematically interesting, the process of constructing the curve gives us a number of fun fractally shapes! These are great to draw and embroider because (a) they're made of a bunch

[2] I'm pretty sure reading that sentence doesn't break Rule 4. Pretty sure.

of straight lines and thus are not *too* complicated and (b) it's just repetitive enough that I find it relaxing and requires just enough thought that I find it fun.

Class format: Largely interactive lecture

Prerequisites: Introductory analysis (Uniform convergence and completeness)

Homework: Optional.

A study in smallness (🌶️🌶️🌶️–🌶️🌶️🌶️🌶️, Ben, 3–5 days)
If you've read my other blurbs here, you'll know that there's something called "compactness" which is a kind of smallness condition on a topological space. In the other blurb, I'm very focused on a few specific examples—what kinds of things in Euclidean space are compact, what kinds of things in this function space are compact?

What if we want to ask questions that are more general? *Vastly* more general? In this course, we'll look at more general definitions of compactness, in more general spaces, and ask questions about how to combine these "small" spaces to make other, bigger, small spaces. For example, the union of finitely many compact spaces is compact; the union of infinitely many is usually not. However, the product of arbitrarily many compact sets is, in fact, compact! This is Tychonoff's Theorem, one of the most important theorems of point-set topology.

But wait, there's more! All of that is the plan for the three-day version of this class. If we have five days, we will then move on, and investigate how to make spaces smaller[3]. The answer is quite simple—you make them bigger[4]! In particular, we will ask a simple question: if we have a space that *isn't* compact, what is the biggest compact space we can build out of it in a remotely sensible way?

Or, to rephrase that in a vastly less helpful way: if we have something big, and we want to make it smaller by making it bigger, what's the biggest way we can make it small?

Class format: Mostly lecture

Prerequisites: Enough topology to have heard of "open sets," "closed sets," and "limits." Having encountered "compactness" in some context will help, but we'll cover that—albeit quickly—in this class.

Homework: Recommended.

Ben's favorite game theory result (🌶️–🌶️🌶️, Ben, 2–3 days)
In my Week 2 class "The Pirate Game," I spent several minutes bad-mouthing the Prisoners' Dilemma, saying that I just found it utterly boring and completely, profoundly devoid of anything resembling interest.

That might have been a bit harsh, and I'm here now to also admit that it was a bit unfair. I do know of one theorem that makes the Prisoners' Dilemma a little more interesting—well, a LOT more interesting, truth be told. This is one of the so-called "Folk Theorems" of game theory.

Usually, when we analyze sequential games, we start at the "end" of the game and work backwards. But what if we don't know when the game is going to end? In practical "games" that occur in "real life," this is often the case because most of us cannot predict the future.[5] It turns out that analyzing such games is much more difficult, and therefore much more interesting.

In this class, we'll talk more about credible threats, about how to even start thinking about "infinite games" (i.e. games where there's no fixed end point), and a kind of strategy called a "grim trigger." Time permitting, we'll also talk about why all the game theorists HATE Ben's Favorite Game Theory

[3] i.e. compact

[4] i.e. you add more points

[5] If you can predict the future, please contact me exactly one year ago to let me know.

Result. If even more time permits, we might even talk about mercy, forgiveness, and grace, but I really can't make any promises on that front.

Class format: Mostly interactive lecture

Prerequisites: Familiarity with geometric series

Homework: Optional.

Compactness in function spaces (🌶🌶–🌶🌶🌶, Ben, 3 days)
In the calculus of variations, we often want to find minimizers of various functions-of-functions. One common approach is the so-called "direct method" which is a four-step process:

(1) Prove that your function $J(f)$ is always at least some value M, and find a sequence of functions f_n so that $J(f_n)$ converges to M,
(2) Argue that J is continuous,
(3) Prove that some subsequence of your f_n converges to some f and, most importantly,
(4) Win by observing that f is a minimizer.

This is all very well as far as it goes, but step (3) leaves a few details unspecified, such as "how do we do that?" One way to do that is to first make up a new condition that means "Sequences like this are guaranteed to have convergent subsequences" and give it a fancy name, like "compactness." Then, if we can show that some (nicer-looking) conditions are equivalent to compactness, we're in business! All we need to do to do step (3) is check those equivalent conditions, and if they hold then we win and if they don't… well, anyways.

In this course, we'll discuss compactness in general metric spaces, and then investigate conditions that are equivalent to compactness in Euclidean space and in the function space of continuous functions. In our studies of the latter, we will see one of the fundamental results of mathematical analysis, the Arzelà–Ascoli Theorem.

Class format: Mostly interactive lecture

Prerequisites: Introductory analysis, for the definition of convergence and uniform convergence

Homework: Optional.

Draw every curve at once (🌶🌶🌶, Ben, 2–3 days)
Isn't it so tiresome to have to draw different things at different times? When I want to write an "A," I have to use an entirely different process from when I write a "B," and wouldn't it be a lot more convenient if I could just draw everything at once? Every single curve imaginable, all packed into one shape?

Not only would this be marvelously convenient[6], it is also possible[7] and easy[8]! You, too, can draw a universal curve[9] and in this class, we'll see how!

Class format: Interactive Lecture

Prerequisites: Knowing what metric spaces are

Homework: Optional.

Sylow theorems and simple groups (🌶🌶–🌶🌶🌶, Ben, 2 days)
In 2004, the classification of finite simple groups was finally wrapped up, describing the kinds of groups that are in some sense "prime-like" in that other groups can be broken down into them. We can't

[6] For sufficiently inconvenient values of convenient.

[7] For sufficiently impossible values of possible.

[8] For sufficiently difficult values of easy.

[9] For sufficiently… weird values of curve, of course

really prove this result because it's (a) not something I know the proof of, (b) spread out over a few thousand pages, and (c) neither this class, nor indeed this Mathcamp, will run for long enough to handle point (b) even if point (a) weren't a factor.

But we can find some conditions that tell us that a group *can't* be simple, based on nothing more than the cardinality of the group. We can use group actions to investigate what kinds of subgroups a group has to have, investigate relations between them, and prove such results as "Every group of order 24 has a normal subgroup" or "There is only one group of order 15 (up to isomorphism)."

Class format: Mostly lecture

Prerequisites: Knowing the orbit-stabilizer theorem in group actions

Homework: Optional.

Charlotte's Classes

An essential and singular complex analysis class (🌶🌶, Charlotte, 1 day)
From Mark's complex analysis class, you may recall the statement of Picard's theorem – that an analytic function takes on every possible complex value, with at most one exception, infinitely many times *in any neighbourhood* of an essential singularity. In this class, we won't prove this insane result, but instead prove a similar statement that is arbitrarily close in craziness.

Class format: About half interactive lecture, half problem solving in breakout rooms

Prerequisites: Mark's complex analysis class; specifically, you should be familiar with Laurent series.

Homework: None.

Arithmetic progressions and primes and parrots (🌶🌶🌶, Charlotte, 2–3 days)[10]
The Green–Tao Theorem is a celebrated result in mathematics: the primes contain arbitrarily large arithmetic progressions. That is, for any $k \in \mathbb{N}$, the set of primes contains some sequence of points in the form a, $a + d$, $a + 2d, \ldots, a + kd$.

In this class we'll look at something much easier to prove, that the primes get arbitrarily close to arbitrarily long arithmetic progressions. Proving this should help shed some light on why the Green–Tao Theorem is true – and we'll see that it has a lot to do with the "size" of the set of prime numbers.

Consequently, we'll spend some time looking at a variety of ways to define the "size" of a subset of the natural numbers, and consider whether or not sets that are large or small in these notions of size should or could contain arbitrarily large arithmetic progressions.

Class format: Mostly interactive lecture with some time for problem solving

Prerequisites: You should be comfortable with limits and infinite series. It would be helpful if you've already seen sups, infs, and limsups before, but I'll introduce these quickly.

Homework: Optional.

Fourier approach to cap sets (🌶🌶🌶, Charlotte, 2 days)
A "cap set" is a subset of $\mathbb{Z}_3^n$ that contains no 3-term arithmetic progression. A natural question that arises from this definition is, how large can a cap set be? Fun fact: this question is actually identical to the question, in the game "Set," what is the maximum number of cards that does not contain a set?

This question was fully settled using the polynomial method; we'll discuss this approach, but we'll mainly talk about the Fourier analytic approach that made decent progress in resolving the cap set

[10] Not to be confused with Viv's class, "Arithmetic progressions and primes and parrots." And actually, there will be no parrots in this class (sad).

conjecture before its resolution via polynomials. We'll do this because (1) Fourier analysis is fun! (2) the proof strategy is much more similar to the analogous problem for subsets of the natural numbers.

Class format: interactive lecture

Prerequisites: Mike's Finite Fourier Analysis class, and some probability—enough to know about linearity of expectation.

Homework: Optional.

The joints problem, solved via polynomials (🌶🌶🌶, Charlotte, 1 day)
The Joints Problem asks the following question: given a collection of ℓ lines in $\mathbb{R}^3$, how many "joints" can they form? A "joint" is a point where at least three non-coplanar lines from the collection intersect. You may remember from my Week 1 class that the Finite Field Kakeya Conjecture was tackled with polynomials, and the method used there extends almost with ease to the Joints Problem. Using polynomials, we'll show that a collection of ℓ lines can generate at most $10\ell^{3/2}$ joints.

Class format: Interactive lecture

Prerequisites: Multivariable calculus (in particular, we'll use the gradient), and my Week 1 Kakeya class (or, if you are comfortable with polynomials and the rank nullity theorem, then you could probably get up to speed quickly).

Homework: None.

Emily's Classes

Conjugation in the symmetric group (🌶🌶🌶, Emily, 2 days)
Two elements x, y of a group G are said to be conjugate if there exists a $g \in G$ such that $gxg^{-1} = y$. If we consider the set of all elements that are conjugate to x in G, this is called the conjugacy class of x. Now if we let G be the symmetric group S_n, the conjugacy classes partition S_n is a rather nice way - by cycle type of the permutations. In this class, we will prove this beautiful fact, as well as find a formula for the size of each conjugacy class.

But we won't stop there! What happens if we restrict the g's used above to be elements of a subgroup of G, say A_n? Will the conjugacy classes in A_n also be completely determined by cycle type? Come to this class to find out!

Class format: Interactive lecture

Prerequisites: Group theory—should be comfortable with the symmetric group and cycle notation.

Homework: None.

Symmetries of a (hyper)cube (🌶🌶, Emily, 1 day)
The dihedral group D_4 can be described as the symmetries of a square, which has four rotations and four reflections. We can bump this up to the next dimension and construct a group that is the symmetries of a cube. And bump it up again to a 4th dimension cube. And again and again for every *hypercube*, or n-dimension cube. We will build up the elements of this group when $n = 3$, find the size of this group for any n, and look at a nice way of representing its elements as permutations (but not the kind you are used to!)

Class format: Interactive lecture

Prerequisites: Group theory—know what the symmetric and dihedral groups are

Homework: Optional.

The math I actually do (🌶🌶🌶–🌶🌶🌶🌶, Emily, 1–2 days)
Ever wonder what kind of math the Mathcamp staff does in their day-to-day life? If so, come learn about one object in particular that I work with on a regular basis: p-subgroup complexes. These are simplical complexes built from posets built from subgroups of a group, so they live at the intersection of topology, combinatorics, and group theory. We will explore the rich history of these complexes, discuss some influential results from the last few decades, and see what people are doing with them today.

Class format: Lecture

Prerequisites: Group theory for sure, having seen homotopies before would be beneficial. Know what a poset is. I will mention group actions, so it would be swell if you've taken my week 3 class, though not required.

Homework: None.

Eric's Classes

Counting curves with linear algebra (🌶–🌶🌶, Eric, 1–2 days)
In this class we will prove many cool facts about plane curves with a single tool: dimensions of vector spaces! The basic form of our main theorem is the statement that "five generic points in the plane determine a conic." This word "generic" will be explored in some depth in this class, which you can think of as a very low-key introduction to some big ideas in algebraic geometry.

Class format: Small amount of lecturing and otherwise worksheets.

Prerequisites: You should be comfortable with the formula

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

for vector subspaces U and W.

Homework: Optional.

Finite fields: the power of Frobenius (🌶🌶🌶, Eric, 1–2 days)
Galois theory as a whole is not the easiest of subjects, but it turns out that Galois theory for finite fields boils down to understanding a single map: the Frobenius homomorphism $x \mapsto x^p$! In this class we'll pick up where Viv's week 3 class left off and dive into the algebraic side of finite fields. We'll see how the Frobenius homomorphism and its powers are a tool for distinguishing between different finite fields, we'll see how they fit together, and we'll understand (in the 2 day version) what the algebraic closure of $\mathbb{F}_p$ is and the structure of its subfields.

Class format: Mixture of lecture and in-class worksheets.

Prerequisites: Viv's week 3 finite fields class.

Homework: Optional.

How to ask questions (🌶, Eric, 1 day)
In this class you will learn about asking questions and also ask questions, though possibly not in that order. You will have the opportunity to learn practical wisdom on how to ask questions in a mathematical context and how to be intentional about your question asking.

Your homework will be to ask questions, in this class and others.

Class format: Inquiry based learning during lecture about learning through inquiry during lectures.

Prerequisites: None!

Homework: Required.

Ringing bells, but secretly just enumerating symmetric groups (🌶️, Eric, 1–2 days)
This class is a very quick introduction to some of the mathematics behind change ringing! We'll talk about a couple of different algorithms for enumerating symmetric groups, while building up the "axioms" of change ringing and discussing the historical and physical motivations behind them. There will be a lot of videos and pictures in this class, we'll detour to the Long Now Project's set of giant bells installed in a mountain and explain why they're boring, and you'll be able to try change ringing for yourself during TAU!

Class format: Lecture feat. many demonstrations

Prerequisites: Comfort with cycle notation for permutations

Homework: Optional.

Gabrielle's Classes

The class group (🌶️🌶️🌶️🌶️, Gabrielle, 4–5 days)
In an ideal world, all rings would have unique factorization and Fermat's Last Theorem would have been proven hundreds of years earlier. Unfortunately, that isn't the case. But maybe we can ring some lemonade out of those lemons by looking at ideals. In this class, we'll very quickly explore the notions of a number field and a ring of integers, before defining the ideal class group, showing that it's a group, and showing what in the world it has to do with unique factorization. (And yes, this does have something to do with quadratic forms!)

Class format: Lecture

Prerequisites: You should feel comfortable with the notions of rings, ideals, prime ideals, and maximal ideals. You should also feel comfortable with the definition of a group.

Homework: Recommended.

Vitali's curse (🌶️🌶️🌶️–🌶️🌶️🌶️🌶️, Gabrielle, 3–4 days)
I failed my real analysis qualifying exam in fall 2019 because I did not read the problem and answered a far easier question than what I had been asked. (Pro tip: Don't do that.) And because I didn't have the Vitali Covering Lemma in my heart. As punishment, I have been doomed to roam the earth, telling people about the Vitali Covering Lemma. We'll learn a little bit of measure theory, answer the question I thought I was supposed to answer, prove the Vitali Covering Lemma, and finally vindicate Past Gabrielle by solving the problem I was supposed to solve. Time permitting, we'll see less redemption-based applications.

Class format: Lecture

Prerequisites: Failure. Also helpful to have comfort with epsilon-delta definitions of continuity and differentiability.

Homework: Optional.

Yet another proof of Euler's pentagonal number theorem (🌶️🌶️🌶️, Gabrielle, 1–2 days)
If you took A Combinatorial Proof of the Jacobi Triple Product Identity in Week 1, you saw two different ways to prove Euler's Pentagonal Number Theorem: One was purely combinatorial, and the other was a simple substitution into the Jacobi Triple Product Identity. I noted on homework that there is yet another proof of Euler's Pentagonal Number Theorem, using the concept of the rank of a partition, but I didn't hint at it very well. So, you can learn about rank, a few of Ramanujan's conjectures, and see (yet another) proof of Euler's Pentagonal Number Theorem.

Class format: Lecture

Prerequisites: Partitions, Young Diagrams, and generating functions. If you don't know what they are, talk to me!

Homework: Optional.

J-Lo's Classes

Absolute value ~~Pokemon~~ GO (🌶️–🌶️🌶️, J-Lo, 1–3 days)
An absolute value is a method for determining the "size" (or "worth," if you took Eric's week 3 class) of a number. There's the standard absolute value, an absolute value that measures "how even" a rational number is, an absolute value that just detects whether a number is zero or not, and many, many more. How many absolute values do the rational numbers have? Gotta Catch 'Em All!

Class format: Lecture, possibly with some breakout rooms if the class is longer

Prerequisites: None

Homework: Optional.

A combinatorial proof of "the" quintic formula (🌶️🌶️🌶️–🌶️🌶️🌶️🌶️, J-Lo, 2–4 days)
If we are only allowed to use $+$, $-$, $\times$, $\div$, and $\sqrt[n]{\ }$ for any n, it is impossible for us to write down the roots of a general quintic equation. However, if we allow ourselves to use other functions, then solutions do exist! One of these solutions uses the extra function

$$F(x) = \sum_{k=0}^{\infty} \binom{5k}{k} \frac{x^{4k+1}}{4k+1}.$$

We will find a combinatorial proof of this version of the quintic formula using a generalization of the Catalan numbers.

Class format: Lecture

Prerequisites: None

Homework: Recommended.

Axiomatic music theory (🌶️–🌶️🌶️, J-Lo, 2–5 days)
What makes some combinations of notes more pleasant to listen to than others? Why does the chromatic scale have 12 notes? At first glance, music might seem like a collection of completely arbitrary facts that just coincidentally combine in ways that happen to sound nice.

But it turns out that many of these complicated musical constructions can be derived as corollaries from just a couple of basic conditions that we might want our music to satisfy! You will reach these conclusions for yourselves by working through a set of problems in groups. The rich, complex intricacy? That's actually number theory under cover!

Class format: Almost entirely problem-solving in breakout rooms (a bit of lecture at the start and end of each day)

Prerequisites: None, but the class will be much easier to follow if you have some musical background (being able to play an instrument or read sheet music should be sufficient)

Homework: Recommended.

Minus Choice, Still Paradoxes (🌶️🌶️🌶️, J-Lo, 2 days)
The Banach-Tarski paradox says that a ball can be broken into finitely many pieces, and using only translations and rotations, can be rearranged into two balls, each the same volume as the original.

The "Axiom of Choice" is a crucial part of this construction, and historically some people have used Banach-Tarski as an argument against this axiom.

But it turns out that paradoxical decompositions (breaking something into finitely many pieces and rearranging them into two copies of the original) exist even without the Axiom of Choice! This class will discuss some of these constructions, which are all based in group theory, and see what specific role Choice plays in the Banach-Tarski case.

Class format: Lecture

Prerequisites: Group theory

Homework: None.

The Borsuk problem (🌶🌶, J-Lo, 1–2 days)
The *diameter* of an arbitrary shape is the largest possible distance between two points in the shape. In 1932, Karl Borsuk asked the following question:

Given a bounded subset of $\mathbb{R}^n$*, can it be split into* $n+1$ *pieces, each of which has smaller diameter than the original set?*

For example, if you try to split an equilateral triangle into two pieces, one of them will always contain two of the vertices, and so its diameter will not be any smaller than that of the original triangle . However, any figure in the plane can be decomposed into 3 pieces of smaller diameter.

Over the course of the 20th century, evidence began to grow that $n+1$ pieces would always be enough. That is, until 1993, when the first counterexample was found... in 1325 dimensions. Currently the smallest known counterexample is a 64-dimensional set which requires more than 65 pieces.

Oddly enough, the counterexamples come from graph theory. This class will explain what on earth graph theory has to say about this inherently geometric problem.

Class format: Lecture

Prerequisites: Introductions to Graph Theory and linear algebra

Homework: None.

The missing dimensions (🌶🌶🌶, J-Lo, 2–4 days)
One powerful feature of the complex numbers (a 2-dimensional number system) is that they can be used to describe rotations in the plane. William Rowan Hamilton spent years trying to find a number system that could be used to describe rotations in 3–d space, before he finally had a eureka moment and carved his discovery into Brougham Bridge: the quaternions, a *four*-dimensional number system.

Why did he need four dimensions to describe three-dimensional rotations? And why was it so hard for him to find a three-dimensional number system? More generally, for which natural number systems n can we build an n-dimensional number system?

Class format: Lecture. In a longer version of the class, there would also be some breakout room sessions centered around getting used to how the quaternions work.

Prerequisites: Comfort with complex number arithmetic (including Euler's formula) and matrix multiplication

Homework: Recommended.

Jorge's Classes

Hash functions and cryptocurrencies (🌶🌶, Jorge, 2 days)
Bitcoin was the first cryptocurrency to gain traction around the world, and it has certainly made the news with its all-time-highs, crashes, and clones (Dogecoin, anyone?). It has also been instrumental in the adoption of other cryptocurrency projects. Amidst this enormous amount of clout, it is very

easy to forget the interesting cryptographic and probabilistic bases that make the existence of Bitcoin possible.

In this class, we set out to explore these bases. Indeed, Bitcoin's robust security infrastructure depends on using a cryptographic hash function—a function that is easy to compute but virtually impossible to invert, among other properties (this is a concept related to that of digital signatures). The cryptographic hash function used in Bitcoin's implementation is known as SHA-256. We will study hash functions in general and present the current state of knowledge around them, giving special attention to SHA-256.

We will conclude the class with an overview of how SHA-256 is used in the context of Bitcoin. By the time we are done, students will have all the mathematical bases to understand how Bitcoin works!

Class format: Lecture

Prerequisites: Basic knowledge of discrete probability.

Homework: None.

Supervised machine learning: the essentials (🌶🌶–🌶🌶🌶, Jorge, 2–3 days)
Machine learning (ML) is a type of artificial intelligence (AI) that allows software applications to become more accurate at predicting outcomes without being explicitly programmed to do so. One subfield of ML is known as *supervised learning*, in which we use historical data (e.g. past measurements) as inputs in order to obtain these predictions.

In order for ML to pull off the futuristic cool stuff one day (such as self-driving cars), we'll need to start small. An example: I have a polynomial curve of unknown degree (but let's say the degree is less than 10), and I give you the coordinates of 10 points on the curve. We want to find a polynomial formula that accurately describes these points and any future points on my curve that I give you. How do we go about this?

The tricky part about it is that solving the problem exactly (via the so-called Lagrange interpolation formula) won't necessarily give you a better solution. In fact, it is likely to give you something terrible that won't be even close to how the actual curve looks like, due to a phenomenon called *overfitting*.

In order to solve the problem correctly, the class will go over the essentials of supervised learning: doing a linear regression in order to find suitable formulas and using *regularization* to avoid overfitting. What do all these buzzwords mean? Come to class and find out!

Class format: Lecture

Prerequisites: Linear algebra (vector spaces, bases, linear combinations, matrix equations) and some probability (knowing about Gaussian distributions and conditional probability)

Homework: None.

The quantum factorization algorithm (🌶🌶🌶–🌶🌶🌶🌶, Jorge, 2–3 days)
What makes quantum computing special, apart from just providing us with an overly complicated way of transmitting information? I'm so glad you asked...

Just before the turn of the century, Peter Shor came up with an algorithm that can use quantum computers to factor numbers in $O(b^3)$ time, where b is the number of bits of the number being factored. In comparison, not even the state-of-the-art algorithms on classical computers are able to factor arbitrary numbers in polynomial time. Because of this, and given that the most common cryptographic algorithm (RSA) depends on the intractability of factoring large pseudo-primes by brute force, there is a large interest in quantum computers from a security standpoint.

Shor's paper is very interesting in its own right, and its study can be split into two. First, we study the "classical" part of the algorithm, which shows how we could factor numbers quickly if we were able to find the order of numbers with respect to the right modulus. The quantum part of the algorithm is precisely how to find these orders, and introduces the very interesting quantum Fourier transform.

Class format: Lecture

Prerequisites: "Intro to quantum computing" week 1 class. Linear algebra: bases, dimension, and the matrix representation of an operator. Also, some familiarity with elementary number theory: divisors, primes, modular arithmetic.

Homework: None.

Linus's Classes

P =NP , if you interpret each space as the word "SPACE" (🌶🌶, Linus, 1 day)
We will prove P =NP . Unfortunately, the Clay Millenium Institute will not give us a million dollars.

Class format: Interactive lecture

Prerequisites: Be comfortable working with algorithms, on the level of "This algorithm takes approximately n^2 time"

Homework: Optional.

There are less than 10^{39} Sudoku puzzles (🌶🌶🌶🌶, Linus, 1 day)
We'll introduce "entropy," which measures how much information you learn when you reveal the value of a random variable. We'll use it to upper bound $\binom{N}{k}$ as well as the number of 9-by-9 Sudoku puzzles.

Class format: Lecture

Prerequisites: linearity of expectation

Homework: None.

Mark's Classes

Counting, involutions, and a theorem of Fermat (🌶🌶, Mark, 1 day)
Involutions are mathematical objects, especially functions, that are their own inverses. Involutions show up with some regularity in combinatorial proofs; in this class we'll see how to use counting and an involution, but no "number theory" in the usual sense, to prove a famous theorem of Fermat about primes as sums of squares. (Actually, although Fermat stated the theorem, it's uncertain whether he had a proof.) If you haven't seen why every prime $p \equiv 1 \pmod 4$ is the sum of two squares, or if you would like to see a relatively recent (Heath-Brown 1984, Zagier 1990), highly non-standard proof of this fact, consider voting for this class.

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: None

Homework: None.

Elliptic functions (🌶🌶🌶–🌶🌶🌶🌶, Mark, 3–5 days)
Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and

elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k)\,,$$

where $\sigma_i(k)$ is the sum of the i-th powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120\left(1(1^3+2^3+4^3) + (1^3+2^3)(1^3+3^3) + (1^3+3^3)(1^3+2^3) + (1^3+2^3+4^3)1\right)\,,$$

which you are welcome to check if you run out of things to do.)

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: Functions of a complex variable, in particular Liouville's Theorem. (If you took only the first week of the FCV class, I could easily get you caught up on that theorem.)

Homework: Optional.

Galois theory crash course (🌶️🌶️🌶️🌶️, Mark, 4–5 days)
In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully. Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later). If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we might conceivably be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class.) Even if we don't get that far, the so-called Galois correspondence (which we should be able to get to, and probably prove) is well worth seeing!

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: Group theory; linear algebra; some familiarity with fields and with polynomial rings.

Homework: Optional.

Multiplicative functions (🌶️🌶️–🌶️🌶️🌶️, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m,n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider voting for this class.

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)

Homework: Optional.

Quadratic reciprocity (, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is q a square modulo p?"
(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: Some basic number theory (if you know Fermat's Little Theorem, you should be OK)

Homework: Optional.

The Cayley–Hamilton theorem (–, Mark, 1 day)
Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: Linear algebra, including a solid grasp of determinants

Homework: None.

Wedderburn's theorem (, Mark, 1 day)
You may well have seen the quaternions, which form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$ with basis $1, i, j, k$ and multiplication rules $i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$. Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof

of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Class format: Interactive lecture (over Zoom). I'll be using a document camera like a "blackboard" (and scanning the notes afterward), looking out at your faces even when you can't see mine (when I'm not actually writing, you will see mine), and asking questions to help us go through the material together.

Prerequisites: Some group theory; knowing what the words "ring" and "field" mean. Familiarity with complex roots of unity would help.

Homework: None.

Mia's Classes

A Property of a^n (🌶️, Mia, 1 day)
Suppose I asked you to give me a power of two starting with '6'; you'd probably quickly reply '64'. And if I asked you to give me a power of two starting with '41', well, that get's a little trickier, but perhaps you know that $2^{22} = 4194304$. However, if I asked you to find me a power of two starting with '616263646566', that's a whole other story. Rather remarkably though, such a power exists. In fact, for any sequence of digits S (which is not a power of 10), there exists a power of two that starts with those digits! The delightful proof comes from Ingenuity in Mathematics and will be the starting point for the class.

Class format: Interactive lecture

Prerequisites: None

Homework: None.

Forbidden minors (🌶️🌶️🌶️, Mia, 3–4 days)
Given a graph G with n vertices, how many edges does G need to guarantee that H is a minor? A topological minor? The first fact we will prove is that every graph of average degree at least $2^{(r-2)}$ has a K_r minor. But in fact we can do much better; a recent theorem by Kostochka says that an average degree of at least $cr\sqrt{\log r}$ is sufficient. Unfortunately, topological minors are a little trickier to guarantee and we'll only be able prove that every graph of average degree at least cr^2 has a K_r topological minor. And lastly, if there is time, we'll look at a beautiful proof by Thomassen which says that, rather counterintuitively, we can force a K_r minor simply by raising the girth.

Class format: Interactive Lecture

Prerequisites: Graph theory

Homework: Optional.

Getting ourselves oriented (🌶️🌶️, Mia, 1 day)
In Graph Colorings, we very briefly saw how applying an orientation to a graph could be a useful way of extracting information from it. In particular, we used a theorem that said that for an oriented graph D, it is always the case that $\chi(L(D)) \geq \log_2(\chi(D))$, but we never proved this ABSOLUTELY DELIGHTFUL theorem. In this class, we'll go back, prove this theorem, and explore several other proofs that start by cleverly applying an orientation to our graph.

Note: Graph Colorings is not a prerequisite.

Class format: Interactive lecture with 15 minutes of breakout rooms

Prerequisites: Graph theory

Homework: None.

Perfection (🌶🌶🌶–🌶🌶🌶🌶, Mia, 2 days)
Commutative algebraists have their excellent rings and algebraic geometers have their wonderful compactifications, but no one achieves perfection like graph theorists. In this class, we will prove none other than the Perfect Graph Theorem which, in addition to having an excellent[11] name, has an exceedingly clever proof.

So, what is perfection? In Graph Colorings, we proved that $\omega(G) \le \chi(G)$ and then asked, can we push those two invariants arbitrarily far apart. An alternative question one might ask is, what graphs achieve equality? Or even better, which graphs achieve equality and have that their subgraphs achieve equality too? The answer, perfect graphs! And what's more, the Perfect Graph Theorem gives us an elegant characterization of these graphs.

Note: Graph Colorings is not a prerequisite.

Class format: Interactive Lecture

Prerequisites: Graph Theory

Homework: Optional.

Misha's Classes

Problem solving: brute force counting (🌶🌶🌶, Misha, 1 day)
In other classes, you may have learned about beautiful formulas for the Catalan sequence or the Stirling numbers that solve tricky combinatorics problems for you.

This class is about what happens when those tools fail you, and you have nothing left but your fists. We will use those fists to solve combinatorics problems that are awful and have no nice approach, but can be made a tiny bit less terrible with planning and care.

Class format: Extra-interactive lecture, with a slide for every problem that I'll annotate as we solve it.

Prerequisites: None.

Homework: Optional.

Problem solving: convexity (🌶🌶, Misha, 1 day)
You see lots and lots of olympiad problems about inequalities. These are often solved by applying obscure theorems due to Scottish mathematicians.

In the real world, mathematicians study inequalities too, but their approach is different. Inequalities that arise from convex functions are much more important.

So in this class, we will take the best of both worlds and use convexity to solve olympiad math problems.

Class format: Extra-interactive lecture, with a slide for every problem that I'll annotate as we solve it.

Prerequisites: None.

Homework: Optional.

Problem solving: floor function (🌶🌶, Misha, 1 day)
If you're tired of doing math and want to lie down on the $\lfloor \cdot \rfloor$ and relax, this class will not help you do that.

Instead, in this extremely tenuously themed problem solving class, we will solve problems that involve rounding down.

[11] Excellent under the English definition, not the algebraic one.

Class format: Extra-interactive lecture, with a slide for every problem that I'll annotate as we solve it.

Prerequisites: None.

Homework: Optional.

Problem solving: lecture theory (🌶, Misha, 1 day)
This class will teach you about the dark side of problem-solving: how to make educated guesses, how to use problem statements to your advantage, and how to exploit the one piece of extra information contest writers can't help giving you: that the problem has an answer.

(Due to its nature, this class is primarily focused on US contests like the AMC, AIME, and ARML, where you don't have to prove that your answers are correct.)

Class format: Extra-interactive lecture, with a slide for every problem that I'll annotate as we solve it.

Prerequisites: None.

Homework: Recommended.

Problem solving: linear algebra (🌶🌶🌶, Misha, 1–2 days)
Most high school math contests (the IMO included) do not use any topic considered to be too advanced for high school, such as linear algebra. This is a shame, because there have been many beautiful problems about linear algebra in undergraduate contests such as the Putnam Math Competition.

In this class, we will look at linear algebra from a new perspective and use it to solve olympiad problems.

Class format: Extra-interactive lecture, with a slide for every problem that I'll annotate as we solve it.

Prerequisites: Linear algebra: in particular, comfort with determinants and eigenvalues.

Homework: Optional.

Problem solving: tetrahedra (🌶🌶🌶🌶, Misha, 1 day)
In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

Class format: Extra-interactive lecture, with a slide for every problem that I'll annotate as we solve it.

Prerequisites: None.

Homework: None.

The liar's guessing game (🌶🌶🌶, Misha, 1 day)
This is not a problem-solving class. This will just be a standard lecture class about ideas that appear in the following olympiad problem:

> **IMO 2012, Problem 3.** The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.
>
> At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some

previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

(1) If $n \geq 2^k$, then B can guarantee a win.
(2) For all sufficiently large k, there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win.

Class format: I will lecture and annotate slides.

Prerequisites: None.

Homework: None.

Samantha's Classes

Quiver representations (🌶🌶–🌶🌶🌶, Samantha, 2–3 days)
A quiver is a directed, acyclic graph. A representation M of a quiver Q is an assignment of a vector space M_v to each vertex v in the quiver and a linear map $\phi_{v,w} : M_v \to M_w$ to each edge $v \to w$ in Q. We will consider a few examples of invariants of quiver representations, and will state Gabriel's Theorem (a foundational theorem that describes which quivers have "nice" representations in a certain sense).

Class format: Lecture

Prerequisites: Intro linear algebra

Homework: Optional.

Taxicab geometry (🌶, Samantha, 2 days)
Taxicab geometry is the geometry resulting from taking the Euclidean plane but defining the distance between points (a,b) and (c,d) to be $|a-c|+|b-d|$. It turns out that this geometry has very different properties from Euclidean (and non-Euclidean) geometry! For example, circles are now square-shaped! In this class, you'll explore some of the basic properties of taxicab geometry; this class will be structured very similarly to my week 4 non-Euclidean course, but with more proofs!

Class format: IBL

Prerequisites: None

Homework: None.

The mathematics of voting (🌶, Samantha, 1 day)
What is the fairest way to vote on something? Perhaps we merely have everyone write down their top choice and the winner is whatever choice received the most votes. Or maybe we allow each voter to rank their preferences, and then we count each preference in a weighted way and allow that to determine the winner. It turns out, neither of these methods is fair, according to economist Ken Arrow's definition of "fair." In this class, we'll consider a few examples of voting systems and their unfairness, then we'll prove Arrow's Impossibility Theorem.

Class format: Lecture

Prerequisites: None

Homework: None.

The Sylow theorems (🌶🌶🌶, Samantha, 3 days)
Lagrange's theorem tells us that the size of a subroup always divides the size of the group. We could ask a related question—suppose n divides the size of our group; does there exist a subgroup of size n? The Sylow Theorems consider the case $n = p^k$, where p is prime and k is the largest power of p that divides the size of the group. There are some really cool consequences to these Theorems!

Class format: Lecture

Prerequisites: Intro group theory

Homework: Recommended.

Susan's Classes

Asymptotic cones (🌶🌶🌶–🌶🌶🌶🌶, Susan, 2 days)
The Cayley graph of a finitely generated group depends on the set of generators, but if you zoom out, it always seems to end up approximately the same shape. (For instance, the integers are a stick!). In this class we'll explore in detail the construction of the asymptotic cone of a finitely-generated group and prove that it is independent of the choice of generators.

Class format: Lecture

Prerequisites: Familiarity with group presentations with generators and relations.

Homework: Optional.

Just one more ring thing! (🌶🌶🌶–🌶🌶🌶🌶, Susan, 1–2 days)
Like rings? Oh man, me too! Want just a little more ring theory this camp? Gosh, so do I!

The exact content of this class will depend on how far we get in noncommutative ring theory. Maybe we'll explore some more exotic polynomial-like ring structures. Maybe we'll talk about what happens when you try to add multiplicative inverses to a ring that doesn't have them. Or maybe we'll look at a cool example of a simple ring which isn't semisimple.

Let's cram in just one more ring thing before the end of camp!

Class format: lecture

Prerequisites: Intro Ring Theory

Homework: Optional.

Martin's axiom (🌶🌶🌶🌶, Susan, 2–4 days)
You probably know that there are different sizes of infinity. The real numbers are provably larger than the natural numbers. In fact, the power set of any set A is larger than A. In fact in fact, the power set of $\mathbb{N}$ turns out to be precisely the size of $\mathbb{R}$. This begs the question: are there sizes of infinity between $|\mathbb{N}|$ and $|\mathbb{R}|$?

In this class, we will not answer that question. We will blow right past that question, assume that we're in a universe with intermediate sizes of infinity, and ask ourselves: how do those intermediate sizes of infinity behave? Using an extra-set-theoretic axiom called Martin's Axiom, we can show that intermediate sizes of infinity behave in several crucial ways like the natural numbers.

If you've ever looked at an induction proof and thought, "Man, this is cool and all, but I really wish it involved more posets," this could be the class for you!

Class format: lecture

Prerequisites: none

Homework: Recommended.

Tim! Black's Classes

Mechanical computers (🌶–🌶🌶, Tim! Black, 3 days)
A few years ago, I came across a YouTube video about a game called Dr. Nim (https://youtu.be/9KABcmczPdg). It's a little computer that can play a simple game against a human with perfect strategy. It's a fairly simple game — one-pile nim — but the amazing part is that there are no electronics. Just using mechanical pieces and gravity, the "computer" chooses how many marbles to drop, and drops them all on its own. It's not the first example of a mechanical computer — people started dreaming up computers long before modern electronics. But is Dr. Nim really a computer at all? It can really only do one thing.

A new game came out in 2018 called Turing Tumble (https://www.turingtumble.com/). It's a marble contraption similar to Dr. Nim, but has components that can be plugged in and arranged on a big peg board. It can do a lot of things; for example, it can do addition and multiplication, and it can even simulate Dr. Nim! Does this count as a computer? It certainly can't output to an electronic display. What does it mean to be a computer?

In this class we'll:

- Look at early attempts at inventing computers (back when mechanical contraptions were the only option).
- Play with Turing Tumble (you can use an online simulator) and see what we can get it to do.
- Discuss what it means to be a computer, and define *Turing Completeness* as one answer to this question.
- See other examples of Turing Complete Systems, and try to prove that Turing Tumble is Turing Complete.

Class format: Lecture/Discussion

Prerequisites: None

Homework: Recommended.

Viv's Classes

Arithmetic progressions and primes and parrots (🌶🌶🌶, Viv, 3 days)[12]
Did you know that there are infinitely many primes?

This fact has been proven many times in many different ways, but in this class we're actually going to ask a different question.

Did you know that there are infinitely many primes congruent to $a \pmod q$, as long as a and q are relatively prime?

Dirichlet showed this in the 1900s using some of my favorite ideas in all of mathematics. The ideas that he used include group theory, complex analysis, and his class number formula (see week 2). The ideas that we will use... are a subset of those, but a pretty big subset.

Class format: Lecture

Prerequisites: Familiarity with complex numbers. Some knowledge of groups and basic complex analysis (such as the word "pole") is helpful.

Homework: Recommended.

How not to prove a group isn't sofic (🌶🌶🌶–🌶🌶🌶🌶, Viv, 2–3 days)
Cayley's Theorem tells us that finite groups are all subgroups of finite permutation groups. A sofic

[12] Not to be confused with Charlotte's class, "Arithmetic progressions and primes and parrots." And actually, there will be no parrots in this class (sad).

group is a possibly-infinite group that we sort of maybe want to have the same property roughly speaking. The group can't always be a subgroup of a finite permutation group, so instead we just require that all finite subsets of our sofic group act kind of like finite subsets of permutation groups. This ends up being a super-useful definition, but we're left with a burning question: do non-sofic groups exist? We don't actually know the answer to this one. We'll spend the class talking about a hopeful candidate for a non-sofic group and one great way not to prove that it isn't sofic.

Class format: Lecture

Prerequisites: None.

Homework: Optional.

Mathematical art history (🌶️, Viv, 2–4 days)
Lucia Pacioli, one of Leonardo da Vinci's contemporaries, is quoted as saying

"Without mathematics, there is no art."

Now, it's certainly easy for many of us to agree with that, but there are also many examples throughout history of times when the art world was obsessed, wittingly or unwittingly, with mathematical ideas. We'll talk about some of these times, including topics like perspective, the golden ratio, proto-Cubism, and fractals.

Class format: Lecture

Prerequisites: None

Homework: None.

Oddtown and eventown (🌶️–🌶️🌶️, Viv, 1–2 days)
The n inhabitants of an odd town called Oddtown have joined many clubs, which satisfy the following strict rules: each club must have an odd number of members, and any two clubs must have an even number of shared members. How many clubs can Oddtown have? The answer is no more than n, with a beautifully short proof that relies on linear independence of vectors over the field with two elements. We'll explore this proof as well as the situations in the neighboring towns.

Class format: IBL

Prerequisites: None.

Homework: Optional.

The distribution of primes (🌶️🌶️–🌶️🌶️🌶️, Viv, 3–5 days)
You may have heard that an integer less than X has a probability of approximately $\frac{1}{\ln X}$ of being prime. But how do primes interact with each other? Are these probabilities independent? They can't be completely independent, because for example either n or $n+1$ is even, so n and $n+1$ can't both be prime. But then how close are they to being independent?

We'll introduce a randomized model for prime numbers and explore how it works and how it doesn't work, and then discuss an absolutely beautiful set of conjectures for prime constellations, and see how it interacts with our randomized model.

Note: This class will be in many ways a fleshed out version of the class I taught as a visitor last year.

Class format: Lecture

Prerequisites: It'll help to have seen big-O notation and probability before.

Homework: Optional.

Topological Tverberg's theorem (🌶🌶🌶🌶, Viv, 5 days)
The convex hull of a set X of points is the smallest set C containing X and all lines between points in C. Given four points in the plane, I can always partition them into two sets whose convex hulls intersect. And if I live in any Euclidean space and I'm given enough points, I can do the same thing. And if, before starting my partitioning process, I draw my convex hulls however I like, rather than through such silly procedures as "following the definition", I can still do the same thing. And if I'd like to split my set into arbitrarily many subsets instead of two, I can still do the same thing... provided I'm partitioning into a prime power number of subsets.

Class format: Lecture

Prerequisites: Linear Algebra, Group Theory

Homework: Recommended.

Zoe's Classes

Obstructions to graph properties (🌶🌶–🌶🌶🌶🌶, Zoe, 1–3 days)
Descriptive set theory holds the answers to a lot of questions of "what is the most generic object that makes ______ impossible" for properties varying from compactness to two-colorability. Most methods will focus on recursive structure in Polish spaces and most theorems come in dichotomy, which aside from being a cool word, ends up tying in a lot of interesting ideas. This subject gives an interesting blend of combinatorics, set theory and topology. For anyone who knows measure theory, we will look at related ideas from a completely opposite perspective!

Class format: Lecture

Prerequisites: Graph theory and some topology

Homework: Recommended.

Party parrot workshop (🌶–🌶🌶, Zoe, 1–5 days)
So many dancing parrots. The party begins!

Mathematical content will be provided.

Class format: IBL

Prerequisites: none

Homework: Recommended.

Ultra-fantastic ultra filters (🌶–🌶🌶🌶, Zoe, 1–3 days)
We want to figure out what it means for a subset of the Reals to be big! We will discuss what are the ideal properties for a 'Big' set to have. The surprising aspect is that when we have a good list of these properties it is actually hard to show that there exists an object with these properties! These are surprisingly useful objects in set theory and combinatorics.

Class format: Lecture

Prerequisites: none

Homework: Recommended.

Atticus Cull's Classes

Computability theory and finite injury (🌶🌶🌶🌶, Atticus Cull, 1 day)
Computability theory is what you would get if you were to do complexity theory without caring about the efficiency of your algorithms. Instead of bounding ourselves by polynomials, we indulge in the full

extent of algorithms. Questions you might consider are, can you come up with a description for every subset of $\mathbb{N}$? Is being able to list the elements of a set the same as being able to tell what's in the set and what isn't? This class will explore fundamental limitations of computation, what it means for sets of naturals to compute each other, and a curious partial order on $\mathcal{P}(\mathbb{N})$, culminating in my hands down favorite proof method: finite injury. It's not as scary as the name suggests - we could be doing infinite injury!

Class format: Lecture

Prerequisites: None

Homework: Optional.

Quinn Perian's Classes

Is math real? (🌶️, Quinn Perian, 1 day)
In the philosophy of math, realism is commonly explained to be a position according to which mathematical objects exist in a sense independent of human thoughts or practices (though this precise definition is itself far from cut and dry). How exactly can we interpret what it means for a mathematical object to exist in this sense? What reasons are there in favor of a realist position? How about against a realist position? What are the alternatives to mathematical realism? In this class, we will try to provide some answers to these questions, and see how different philosophers over time would answer the question "Is math real?"

Class format: The class will be mainly interactive lecture, consisting of lecture along with several opportunities for campers to contribute their own opinion to the discussion of various philosophical positions (on mic or in chat).

Prerequisites: None.

Homework: None.

William Ding's Classes

Introduction to auction theory (🌶️, William Ding, 1 day)
There are many ways to sell stuff, and often the price of the stuff that's being sold is not fixed. In this class, we will analyze common auction formats and optimal bidding under a few convenient assumptions. We'll then loosen those assumptions to begin to understand auctions in the real world, specifically the government-run "spectrum auctions" of the '90s and '00s: some were record-breaking successes, while others, for reasons that we will explore, were so disappointing as to warrant antitrust investigations.

Class format: Interactive lecture

Prerequisites: None

Homework: None.