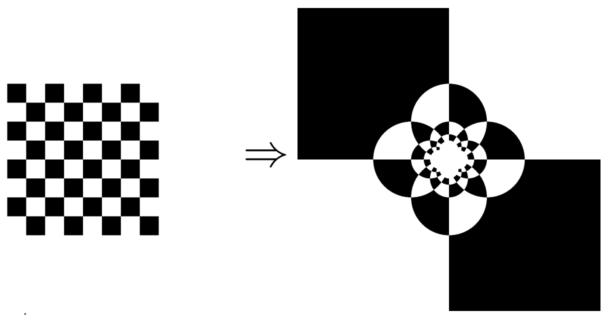
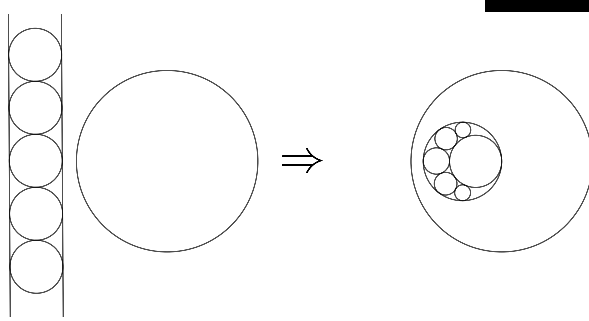
INVERSIVE GEOMETRY

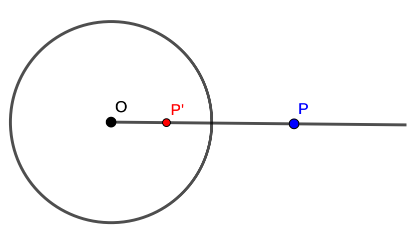
MATHCAMP 2021 QUALIFYING QUIZ

An *inversion* is a kind of geometric transformation. It swaps the inside and outside of a circle. Some examples are shown below.

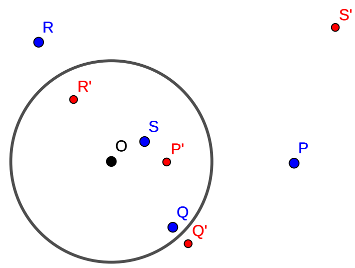




Inversion can be defined more precisely as follows. Suppose we are given a circle in the plane, centered at a point O, with radius r. For any other point P in the plane, we define the *inverse* of P to be the point P' on the ray OP satisfying $OP \cdot OP' = r^2$.



Note that the inverse of P' is P. The inverses of some more points are shown below.

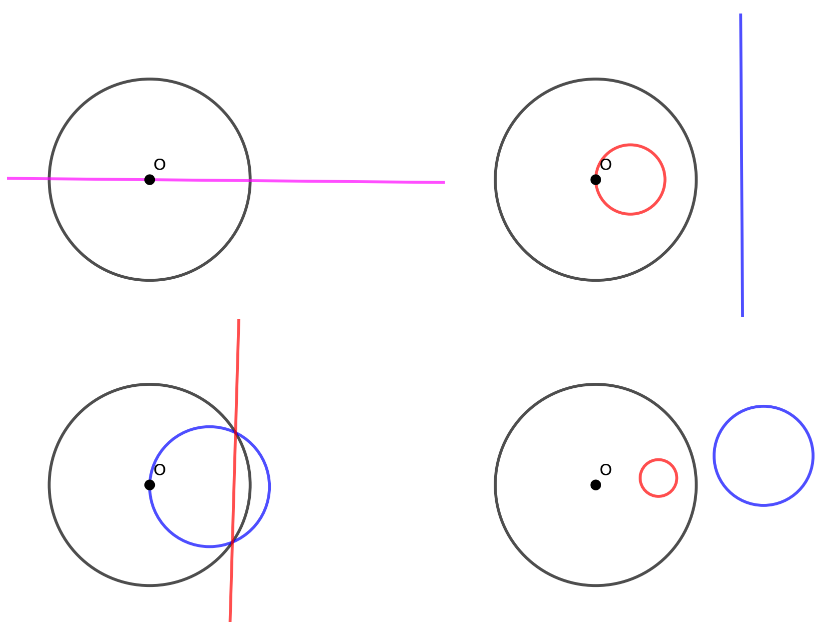


What about the inverse of the point O? If P is very close to O, then its inverse P' is very far from O. We introduce a "point at infinity" and say that O and the point at infinity are inverses of each other.

Some properties of inversion

From the above pictures, it is clear that, unlike translation or rotation, inversion does not preserve lines or distances. Nonetheless, inversion has some nice properties:
Theorem 1.

(1) *A line passing through O is its own inverse.*
(2) *The inverse of a line not passing through O is a circle passing through O.*
(3) *Similarly, the inverse of a circle passing through O is a line not passing through O.*
(4) *The inverse of a circle not passing through O is a circle not passing through O.*



If we define a "generalized circle" to be either a line or a circle, then the inverse of a generalized circle is always a generalized circle. We can think of a line as a generalized circle containing the point at infinity, and a circle as a generalized circle not containing the point at infinity.

Experimenting with inversion

If you'd like, you can experiment with inversion using the interactive geometry program GeoGebra, https://www.geogebra.org/geometry. In GeoGebra, the "Reflect about Circle" tool performs an inversion. You may need to click on "MORE" twice to see this tool.

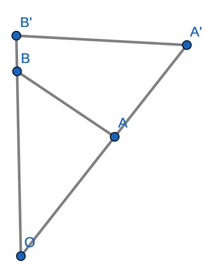
Proof of Theorem 1 and applications of inversion

The properties of inversion mentioned in Theorem 1 are all that are needed to solve Problem 4 on the Qualifying Quiz. But in case you are curious, we will give a proof of Theorem 1 and an application of inversion (Ptolemy's theorem).

Proof of Theorem 1. Item (1) is clear from the definition of inversion.

Let A, B, C, D be four points in the plane, and let A', B', C', D' be their inverses.

First, we claim that triangles OAB and $OB'A'$ are similar.



Indeed, angle AOB is congruent to angle $B'OA'$ and $OA \cdot OA' = OB \cdot OB' = r^2$, which implies

$$\frac{OB'}{OA} = \frac{OA'}{OB} = \frac{r^2}{AD \cdot BD}\,.$$

Therefore, angle $OA'B'$ is congruent to angle OBA. It follows that

$$m\angle C'A'B' = m\angle OA'B' - m\angle OA'C' = m\angle OBA - m\angle OCA\,.$$

If A, B, C are collinear in that order, then $m\angle OBA - m\angle OCA = m\angle BOC = m\angle C'OB'$. So angles $C'A'B'$ and $C'OB'$ are congruent, implying that O, A', B', C' lie on a circle. This proves item (2).

If A, B, C, O lie on a circle in that order, then $m\angle OBA = m\angle OCA$, implying $m\angle C'A'B' = 0$, i.e. A', B', C' are collinear. This proves item (3).

Finally, we have

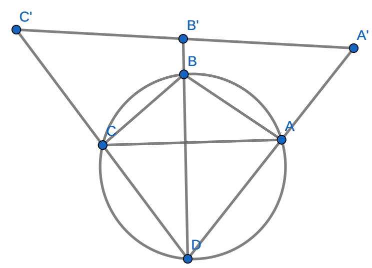
$$\begin{aligned} m\angle C'A'B' - m\angle C'D'B' &= (m\angle OBA - m\angle OCA) - (m\angle OBD - m\angle OCD) \\ &= (m\angle OBA - m\angle OBD) - (m\angle OCA - m\angle OCD) \\ &= m\angle DBA - m\angle DCA\,. \end{aligned}$$

In particular, if A, B, C, D lie on a circle in that order, then $m\angle DBA = m\angle DCA$, $m\angle C'A'B' = m\angle C'D'B'$, and A', B', C', D' also lie on a circle. This proves item (4). $\square$

The most famous application of inversion is probably Ptolemy's theorem. We present this theorem and its proof below. The proof can also be found in the video [1]. References [2, Ch. 5] and [3] describe quite a few other applications of inversion.

Theorem 2 (Ptolemy's theorem)**.** *Let A, B, C, D be four points lying on a circle (in that order). Then $AB \cdot CD + AD \cdot BC = AC \cdot BD$.*

Proof. Invert the figure about an arbitrary circle containing D. Let r be the radius of this circle. Let A', B', C' be the inverses of A, B, C, respectively.



By the same reasoning as in the proof of Theorem 1, triangles DAB and $DB'A'$ are similar. Hence

$$\frac{B'A'}{AB} = \frac{B'D}{AD} = \frac{A'D}{BD} = \frac{r^2}{AD \cdot BD}\,.$$

Therefore,

$$B'A' = \frac{r^2 \cdot AB}{AD \cdot BD}\,.$$

By similar reasoning,

$$C'B' = \frac{r^2 \cdot BC}{BD \cdot CD}$$

$$C'A' = \frac{r^2 \cdot AC}{AD \cdot CD}\,.$$

Since A', B', C' lie on a straight line (in that order),

$$B'A' + C'B' = C'A'\,.$$

Applying the above identities gives

$$\frac{r^2 \cdot AB}{AD \cdot BD} + \frac{r^2 \cdot BC}{BD \cdot CD} = \frac{r^2 \cdot AC}{AD \cdot CD}\,.$$

After multiplying by $\frac{AD \cdot BD \cdot CD}{r^2}$, we obtain

$$AB \cdot CD + AD \cdot BC = AC \cdot BD\,,$$

which is what we wanted to prove. □

References

[1] Numberphile. "A Miraculous Proof (Ptolemy's Theorem) - Numberphile." https://www.youtube.com/watch?v=bJOuzqu3MUQ

[2] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*. Random House, New York, 1967.

[3] T. Davis, "Inversion in a Circle." http://geometer.org/mathcircles/inversion.pdf