CLASS DESCRIPTIONS—WEEK 1, MATHCAMP 2022

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9:10 Classes

Computability theory (🌶🌶🌶🌶, Steve, TWΘFS)
There is a sense in which, for any reasonable interpretation of the term "solvable," there are some problems which are not solvable. However, this leaves two major follow-up questions open: are there any *natural* unsolvable problems, and can we meaningfully compare the difficulty of unsolvable problems (so that some problems are "more unsolvable" than others)?

In this class we'll study one particular perspective on the idea of solvability—**computability theory**. Roughly speaking, a set of natural numbers A is computable iff there is a computer program which successfully determines membership in A (regardless of "resource" issues, like runtime). We will start off by answering both questions above in the affirmative, giving natural examples of unsolvable problems and a robust notion of "computable relative to." We will spend the rest of the class attacking Post's problem, which roughly asks whether the natural examples of unsolvable problems we see are optimal. If we have remaining time, we will look at current problems in computability theory.

Homework: Required

Class format: Lecture

Prerequisites: Mathematical maturity, and ideally the uncountability of the reals

Introduction to graph theory (🌶, Narmada, TWΘFS)
A graph is an object with a bunch of things (called "vertices"), some of which have connections between them (called "edges"). You could argue that just about anything is a graph. As a corollary, you might deduce that graph theory is the most important subject in all of mathematics.

Exaggerated deductions aside, there are many problems that naturally lend themselves to being modeled by graphs. For example, can Euler traverse the islands of Königsberg without crossing any bridge twice? Can Hall successfully arrange marriages for his fussy friends? Can you color a map of the world so that neighboring countries have different colors? In this class, we'll learn how to formulate these (and other problems) in the language of graph theory to provide elegant solutions.

Homework: Required

Class format: Mostly split into small groups to work on problems

Prerequisites: None

Required for: Extremal graph theory (W2); The Ra(n)do(m) Graph (W2); Szemerédi's {theorem, regularity lemma} (W3); Problem solving: graph theory (W3); Baire necessities for Banach–Tarski (W4)

Introduction to number theory (🌶🌶🌶, Mark, TWΘFS)
How do you find $\gcd(a, b)$ for two large integers a and b without having to factor them? Which integers are the sum of two (or the sum of three, or the sum of four) perfect squares? What postages can you get (and not get) if you have only 8 cent and 17 cent stamps available? Besides the famous $3, 4, 5$ triangle (and scaled versions of it), what right triangles are there for which all the side lengths are integers? How does the RSA algorithm (used for such things as sending confidential information, such as your credit card number, over the internet) work? (If you know the answers to all these questions, please don't take this class; you'll be bored, and you might make others feel bad.)

Besides the answers to such questions, number theory offers insight into many beautiful and subtle properties of our old friends, the integers. For thousands of years professional and amateur mathematicians have been fascinated by the subject (by the way, some of the amateurs, such as the 17th century lawyer Fermat and the theoretical physicist Dyson who passed away in 2020, are not to be underestimated!) and chances are that you, too, will enjoy it quite a bit. Although we'll start from scratch, in order to touch on as many as we can of the topics mentioned above (and maybe a few others) the class will go at a good pace—thus the three chilis.

Homework: Recommended

Class format: Lecture

Prerequisites: None beyond modular arithmetic (which I can catch you up on if needed)

Required for: 2-adic computer science (W3); The distribution of prime numbers (W4); The abc's of polynomialand (W4)

Machine geometry (🌶🌶🌶, Misha, TWΘFS)
This class is inspired by, but not strictly about, computer algorithms for solving geometry problems. We avoid ugly coordinates, but also remain skeptical of proofs that rely on cleverly spotting the right similar triangle or cyclic quadrilateral.

We will begin with the area method, and use it to prove theorems in affine geometry: geometry where we can't measure distances or angles. We like distances and angles, though. To be able to handle those, we will define a notion of coarea, which relates to area as cosine relates to sine.

This is not a class on olympiad geometry, but we will often apply our method to examples from various math competitions.

Homework: Recommended

Class format: Lecture, which we will occasionally interrupt to solve problems together

Prerequisites: None

The answer is χ (🌶🌶, Assaf, TWΘFS)
In this class, we will prove:

$$\sum \text{ind} = \chi(S).$$

Though it's up to you to figure out what each of these symbols mean. Without spoiling too much, we will talk about triangulations of surfaces, height-functions, and why you can't comb a hairy sphere, but you can comb a hairy torus. Join this class if you are interested in strange links between surfaces and "additional structures" that surfaces may carry, and prepare to answer everything with: "The answer is χ!"

Homework: Optional

Class format: Moore method—I will only guide your exploration

Prerequisites: None

10:10 Classes

Cluster algebras from surfaces (🌶🌶🌶🌶, Kayla, TWΘFS)
Snake graphs, Farey trees, frieze patterns, cluster algebras, oh my! In this class, we will define the idea of a cluster structure. As an example, how many ways can you triangulate an n-gon into triangles? Come learn how the answer to this question leads to a rich algebraic and combinatorial structure called a cluster structure! A cluster structure takes a piece of initial data—e.g. a triangulated polygon, a directed graph, or a topological surface—a set of variables, and a mutation rule that prescribes a way to transform this data to create something new. The set of all things you can generate via this mutation rule itself has even more structure! Enter cluster algebras. This abstract phenomenon of initial seeds and mutation leads to a beautiful intersection of algebra, combinatorics, topology and geometry.

Homework: Optional

Class format: Lecture

Prerequisites: Some exposure to algebra would be helpful (more specifically, understanding defining a generation of an algebraic structure by some set e.g. seeing a basis from linear algebra would be good, group by a presentation, etc.)

Complexity theory (🌶🌶🌶, Linus, TWΘFS)
P is—roughly—the class of problems that an algorithm can efficiently solve. For example, deciding whether a graph is planar. On the other hand, NP is—roughly—the class of problems where one can efficiently *check* a purported answer. For example, Sudoku or integer factorization. I'll leave it as an exercise whether P = NP.

In this class, we'll more formally introduce P and NP, alongside a host of other complexity classes such as coNP, BPP, and P/poly. We'll prove that some of them are equal, some of them aren't, and answer vital questions such as "Is deciding which Mathcamp classes to take NP-complete?"

Homework: Recommended

Class format: Lecture

Prerequisites: None

Generating functions, Catalan numbers, and partitions (🌶🌶, Mark, TWΘFS)
Generating functions provide a powerful technique, used by Euler and many later mathematicians, to analyze sequences of numbers; often, they also provide the pleasure of working with infinite series without having to worry about convergence.

The sequence of Catalan numbers, which starts off $1, 2, 5, 14, 42, \ldots,$ comes up in the solution of many counting problems, involving, among other things, voting, lattice paths, and polygon dissection. We'll use a generating function to come up with an explicit formula for the Catalan numbers.

A *partition* of a positive integer n is a way to write n as a sum of one or more positive integers, say in nonincreasing order; for example, the seven partitions of 5 are

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ \text{and } 1+1+1+1+1.$$

The number of such partitions is given by the partition function $p(n)$; for example, $p(5) = 7$. Although an "explicit" formula for $p(n)$ is known and we may even look at it (in horror?), it's quite complicated. In our class, time permitting, we'll combine generating functions and a famous combinatorial argument due to Franklin to find a beautiful recurrence relation for the (rapidly growing) partition function. This formula was used by MacMahon to make a table of values for $p(n)$ through $p(200) = 3972999029388$, back when "computer" still meant "human being who does computations".

Homework: Recommended

Class format: Lecture

Prerequisites: Summation notation; geometric series. Some experience with more general power series may help, but is not really needed. A bit of calculus may come in handy, but you should be able to get by without.

Introduction to group theory (🌶🌶, Susan, TWΘFS)
Let's list some operations on sets! Let's see... there's addition on the integers, multiplication on the rationals, and taking the average of two real numbers. Those are three good ones. Now let's think of some more exotic operations. How about multiplying square matrices over the complex numbers, or composing continuous functions from the reals to the reals?

Broadly, abstract algebra is the study of operations of sets. A binary operation that satisfies a collection of particularly nice properties (associativity, identity, and inverses) is called a *group operation*, and the set that it acts on is called a *group*. Groups pop up all over mathematics. They can be used to explore symmetries of geometric objects, prove the existence of an unsolvable quintic polynomial, and solve tricky counting problems in combinatorics. But they are also a beautiful class of mathematical objects in their own right.

In this class, we'll be getting to know our good friend the group. Starting with a few simple axioms, we'll build a fundamental toolkit of theorems that give us an instinct for how these objects behave. We'll learn what it means for two groups to be secretly "the same." We'll learn about subgroups and what Lagrange's theorem has to say about their sizes. We'll go over the construction for building quotient groups, and if we have time we'll talk about the group isomorphism theorems.

Homework: Recommended

Class format: This will be a lecture-based class with substantial problem sets to work on between classes. Though homework is recommended rather than required, campers will get much more out of the class if they engage substantially with the problem sets. Come join my problem solving parties during TAU!

Prerequisites: None

Required for: Ring theory (W2); Bonus group theory part 2 (W2); Grammatical group generation (W2); Representation theory (week 1) (W3); The 17 wallpaper patterns (W3); Commutative algebra and algebraic geometry (W3); In-fun-ite groups (W3); Algebraic solutions to Painlevé VI (W4); Algebraic topology: homology (W4); Representation theory (week 2) (W4); Baire necessities for Banach–Tarski (W4); Introduction to Galois theory (W4)

The geometry of music (🌶, Emily, TWΘFS)
We all know what music sounds like, but what does it look like? In this class, we will study just that! We will learn how to visualize scales, chords, and rhythms as manipulations of polygons, and further use geometric ideas to construct our own rhythmic patterns. In addition, we will build a lattice-like structure of chords called the Tonnetz from which we can visually study many things, such as chord progressions and consonance versus dissonance.

Homework: Recommended

Class format: Lecture

Prerequisites: No mathematical prereqs. Musical prereq: be able to read sheet music

11:10 Classes

Degree theory (🌶🌶🌶, Zoe, TWΘFS)
Zero is not an odd number. This fact has a surprising number of powerful consequences when degree theory is involved. When considering the functions $f(x) = x^n$ for $x \in \mathbb{C}$ we have an easy way to

talk about degree and we have some intuition as to the properties this function might have. However, what about when we have a less clearly defined function? Or what about if we are considering some arbitrary space? We can still come up with a notion of the degree of a map which has various useful associated properties. The best part is that this notion of degree gives us a lot of the most crucial information without having to consider how bad a space or function might behave in specific areas. For example, degree theory allows us to say that the Petersen graph is 3-colorable (and not 2-colorable) by the fact that zero is not an odd number!

Homework: Recommended

Class format: Lecture

Prerequisites: None

Introduction to linear algebra (🌶🌶, Misha, TWΘFS)

Linear algebra is something that shows up in basically every field of math! Ostensibly it's about solving systems of linear equations. By the end of the course you'll be a master of understanding how to solve things like

$$13x + 3y = 20$$
$$19x + 16y = 22.$$

But the real beauty of linear algebra is that the techniques we'll learn reach so far beyond just solving systems of equations. Expect some homework problems where you learn to use linear algebra to solve some very neat problems that look decidedly non-linear to start with! This course will set you up to use linear algebra in a variety of situations throughout the rest of camp.

Homework: Recommended

Class format: Lecture

Prerequisites: None

Required for: Quantum computation (W2); Teichmüller theory of the torus (W2); Schubert calculus (W3); Representation theory (week 1) (W3); The 17 wallpaper patterns (W3); Machine learning (NOT neural networks) (W3); Algebraic solutions to Painlevé VI (W4); Problem solving: cheating in geometry (W4); High-dimensional potatoes (W4); Algebraic topology: homology (W4); Finite fields (W4); Representation theory (week 2) (W4)

Introduction to real analysis: epsilons and deltas (🌶🌶, Charlotte, TWΘFS)

You've probably heard of sequences, limits, and derivatives from calculus before. In this class you'll get a new introduction to all of these concepts from a much more rigorous (and in my opinion, satisfying and fun!) perspective. We'll learn various types of "epsilon-delta" definitions and get lots of practice with "epsilon-delta" proofs, which are ubiquitous in math.

Approaching these topics rigorously will help us discover some counterintuitive examples, like a function that is continuous at every irrational point, but discontinuous at every rational. Conversely, it'll also help explain some strange phenomena: for example, some infinite series can be rearranged to seemingly sum to different values.

Homework: Required

Class format: A split of lecture and group work, likely 60:40

Prerequisites: Some basic calculus—in particular, have an intuitive idea of what limits are, and know about derivatives & integrals.

Required for: Cantor before set theory (W4)

Overly convoluted plans (🌶🌶🌶, Ben, TWΘFS)
Some integrals are practical[1] to solve in the sense that you can use some combination of u-substitution, memorized integrals, and sensible clever tricks to work out an exact answer. However, some integrals, while easy to write down, are not quite as practical to solve, such as

$$\int_0^\infty \frac{\sin(x)}{x}dx, \quad \int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}dx, \ \ldots, \ \int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/15)}{x/15}dx.$$

(OK, the last one there isn't quite so easy to write down either, since we're taking a product of eight terms of the form $\frac{\sin(blah)}{blah}$, for $blah = x, x/3, x/5, \ldots, x/13, x/15$.)

In this class, we'll learn one way to solve these exactly (... except for the last one, which is a lot harder). If you'd like to skip the hard work of taking this class, they're all $\frac{\pi}{2}$, except the last one, which is *very slightly less* than $\frac{\pi}{2}$.

Why does this pattern of $\frac{\pi}{2}$s break down? What does French mathematician Joseph Fourier have to do with this? And how does it all relate to the convolution product—whatever that is? We'll discuss all this, and more!

Homework: Recommended

Class format: Lecture

Prerequisites: Calculus (e.g. you should know how to take derivatives, integrals, and improper integrals). Having seen ϵ-δ proofs in the past will help but should not be strictly necessary.

The mathematics of forbidden words (🌶🌶, Travis, TWΘFS)
Don't stumble under the yoke of society and tradition—take control of forbidden words with mathematics! Dynamical systems are a cool and super-applicable field of mathematics, but studying it usually requires calculus and differential equations and all sorts of advanced tools—unless it doesn't! In this class, we'll look at discrete dynamical systems, no calculus required, and answer questions about discrete dynamical systems including, but not limited to: What are they? How do they work? And how do *forbidden words* help us understand them?

Homework: Recommended

Class format: Lecture

Prerequisites: None

1:10 Classes

Crash course (🌶, Assaf, TWΘFS)
Math is useless unless it is properly communicated. Most of math communication happens through a toolbox of terminology and proof techniques that provide us with a backbone to understand and talk about mathematics. These proof techniques are often taken for granted in textbooks, math classes (even at Mathcamp!) and lectures. This class is designed to introduce fundamental proof techniques and writing skills in order to make the rest of the wonderful world of mathematics more accessible. This class will cover direct proofs from axioms, proofs using negation, proofs with complicated logical structure, induction proofs, and proofs using cardinality and the pigeonhole principle. If you are unfamiliar with these proof techniques, then this class is highly recommended for you. If you have heard of these techniques, but would like to practice using them, this class is also right for you. Here are some problems that can assess your knowledge of proof writing:

- Negate the following sentence without using any negative words ("no", "not", etc.): "If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."

[1] Note that I do not say "easy," because some of these integrals are pretty hard.

- Given two sets of real numbers A and B, we say that A dominates B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two disjoint, nonempty sets A and B such that A dominates B and B dominates A.
- Prove that there are infinitely many prime numbers.
- Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective.
- Define rigorously what it means for a function to be increasing.
- What is wrong with the following argument (aside from the fact that the claim is false)? On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

  Proof. We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. □

- Mathcampers can message each other privately on Slack over the course of camp. Prove that there are two campers who messaged the same number of people throughout camp.

If you would not be comfortable writing down proofs or presenting your solutions to these problems, then you can probably benefit from this crash course. If you found this list of questions intimidating or didn't know how to begin thinking about some of them, then you should definitely take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Homework: Required

Class format: Lecture

Prerequisites: None

Formal proof verification in Lean (🌶🌶🌶, Aaron, TWΘFS)

The proof of the 4-color theorem in graph theory is really long. So long that no human could write out all the cases, or even check all of them. So how did this proof ever get written or checked?

Rather than getting written up in paragraphs, a large part of that proof is written in computer code, so that a computer can not only generate a symbolic proof of each case, but actually check all the logic. (You can see a modern version here.[2])

These days, there are several proof languages that computers understand, but in this class, we'll learn a bit of a language called Lean, where math looks rather like this:

```
theorem fermats_last_theorem (n : N) (n_gt_2 : n > 2) :
  not(exists x y z : N, (x^n + y^n = z^n) and (x > 0) and (y > 0) and (z > 0)) :=
begin
  sorry,
end
```

In the past few years, hundreds of people (including a few Mathcampers!) have translated a bunch of math[3] into Lean. Let's join them, and learn how to write computer-verifiable proofs.

Homework: Required

[2] https://github.com/coq-community/fourcolor

[3] https://leanprover-community.github.io/mathlib-overview.html

Class format: Computer-based IBL. We'll be spending most of the time at keyboards, actively coding up proofs to exercises.

Prerequisites: You should be comfortable with basic logic terms and proof techniques (and, or, not, for all, there exists, contradiction, induction).

We will be programming, but you do not need experience programming. This programming can be done in a browser, but you may want to install Lean ahead of time[4]

Jacobi sums (🌶🌶🌶🌶, *Dave Savitt*, TWΘFS)

Let's count the number of solutions (x, y) to the equation $x^3 + y^3 \equiv 1 \pmod{p}$, where p is a prime congruent to 1 modulo 3. For $p = 7$ there are 6 solutions. For $p = 13$ there are 6 solutions again. But for $p = 19$, there are 24. Here's a table for a few small values of p.

p	$\#\{(x,y) : x^3 + y^3 \equiv 1 \pmod{p}\}$
7	6
13	6
19	24
31	33
37	24
⋮	⋮
379	348

It looks like there are around p solutions, but why fewer than that for some primes p, and more for others? Here is another table of numbers. In the second column I've written $4p$ in the form $A^2 + 27B^2$ and chosen the sign of A so that $A \equiv 1 \pmod{3}$.

p	$4p$	$=$	A^2	$+27 \cdot B^2$	$p - 2 + A$
7	28	$=$	1^2	$+27 \cdot 1^2$	6
13	52	$=$	$(-5)^2$	$+27 \cdot 1^2$	6
19	76	$=$	7^2	$+27 \cdot 1^2$	24
31	124	$=$	4^2	$+27 \cdot 2^2$	33
37	148	$=$	$(-11)^2$	$+27 \cdot 1^2$	24
⋮		⋮		⋮	⋮
379	1516	$=$	$(-29)^2$	$+27 \cdot 5^2$	348

Coincidence? I think not. We will prove this observation in general, and explain what it has to do with some very special sums of roots of unity, called *Jacobi sums*.

Homework: Recommended

Class format: Lecture

Prerequisites: Comfort with modular arithmetic, including Fermat's Little Theorem and the existence of primitive roots modulo p. Comfort with complex numbers. Familiarity with basic group theory is not necessary, but some arguments will be more intuitive if you have this background.

[4] `https://leanprover-community.github.io/get_started.html#regular-install`

Learn topology with PALs! (🌶🌶, Arya, TWΘFS)
Roughly speaking, topology is the study of the "shape" of an object, except when all objects are made of play-doh! So you're allowed to mold objects into different shapes, but you're not allowed to cut the object, or glue two objects together. For example, topologically a triangle is the same as a square. But is it the same as a cube? A square with a hole punched out? A donut? In this class, we shall work through classifying one and two dimensional objects topologically. And the "objects" we work with are built using our good old PALs - polyhedra!

Homework: Required

Class format: Lecture

Prerequisites: None

Martingales (🌶🌶, Yuval, TWΘFS)
Suppose we play the following game: at every turn, I pick some number m, and then we flip a fair coin. If it comes up tails, then I give you m M&M's, but if it comes up heads, you give me m M&M's. This game is completely fair, since at every turn, we both have equal odds of winning or losing m M&M's.

However, I might implement the following strategy: on the first turn, I pick $m = 2$. If I win, then I end the game. If not, then on the second turn I pick $m = 4$, and if I win, I end the game. If not, I pick $m = 8$, and so on; on the nth turn, I set $m = 2^n$, and only stop when I finally win. I'm bound to win eventually (since there's no chance we'll keep getting tails forever), and if I win on the nth turn, then I will get 2^n M&M's on that turn. Also, if we add up how many M&M's I gave to you over all the turns I lost, we get

$$2 + 4 + 8 + \cdots + 2^{n-1} = 2^n - 2.$$

In other words, I won 2^n M&M's and only lost $2^n - 2$, meaning that I swindled you out of 2 M&M's! So how can this be a fair game?

The answer to this question can be found with martingales, which are arguably the most powerful tool in all of probability. In this class, we'll use them to solve many problems (including the one above), and see many examples where they can convert a seemingly intractable problem into a two-line computation.

Homework: Recommended

Class format: Lecture

Prerequisites: None

Colloquia

Applied game theory (*Po-Shen Loh*, Tuesday)
At first glance, it looks expensive to solve real-world problems. The scale is just enormous. There are many units of good that need to be produced, and each unit of good has net cost x to produce. There's a trick: find a way to do this where $x < 0$. However, that isn't good enough if the net cost $x = a + b$, where $a > 0$ and $b < 0$ but a is paid by Person A, and $|b|$ is earned by Person B. (Then, Person A won't want to participate.) Real world applications of game theory focus on ways to design infrastructure that produce a negative net cost for every individual participant: a win-win situation.

The speaker has invented solutions to some large scale real world problems over the past 2 years, in healthcare (`https://novid.org`) and in education (`https://live.poshenloh.com`). This talk will share stories of creating and scaling up these ideas, catered to a Mathcamp audience. It will be informal, approachable, and fun.

Three-term arithmetic progressions (Yuval, Wednesday)
$9, 11, 13$; $100, 200, 300$; $37, 52, 67$; $a, a + d, a + 2d \dots$ There can't be any interesting math about three-term arithmetic progressions, right?

Wrong! As it turns out, three-term arithmetic progressions are at the center of one of the most important and interesting mathematical stories of the 20th and 21st centuries, beginning with a question of Erdős and Turán in 1936, through remarkable results of Behrend in 1946 and Roth in 1953, and culminating (for now!) in a major breakthrough by Bloom and Sisask in 2020. All these results concern the following innocuous question: how many elements can there be in a subset of $\{1, 2, \dots, N\}$ without a three-term arithmetic progression?

In this talk, we'll see various different ways to think and prove theorems about this problem, including perspectives coming from combinatorics, number theory, probability, and high-dimensional geometry.

Counting things with bad maps (Zoe, Thursday)
Certain mathematical insights come about when mathematicians practiced in one discipline decide to take a look at mathematics from other distinct disciplines. My entirely biased favorite example of this is using the existence (or non-existence) of a combinatorial object to construct a mapping that can't exist! Aside from being my favorite, a good property of this example is that there are many questions that can be answered in this way. A few examples are fair division of rent, splitting necklaces, graph colorings, and there are many more. Come to this colloquium to hear about how to solve these questions or about the unexpected connections between combinatorics and topology.

1,2,5,14...FRIEZE 🥶 (Kayla, Friday)
It's a hot summer here in Waterville, Maine. Come hide from the heat with this colloquium on 🥶 frieze patterns 🥶! In the 70's, Conway and Coxeter showed a bijection between frieze patterns of positive integers and triangulations of polygons. After this bijection was shown, interest in frieze patterns stayed dormant for many years. But due to the birth of 🤩 cluster algebras 🤩 in the early 2000's, people have now become very interested in friezes again. Come learn about finite frieze patterns of positive integers and their generalizations! Time permitting, I can give an overview of open research problems on frieze patterns people are thinking about today (including some Mathcamp alumni).

CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2022

Contents

9:10 Classes

Algorithms for large primes (🌶🌶, *Zach Abel*, TWΘFS)
Much of modern internet security relies on a counterintuitive principle: **testing** whether large numbers are prime is fast, but **factoring** those same numbers is believed to be infeasible, even with state-of-the-art supercomputers and factoring algorithms.

For example, consider this 617-digit number n:

3049393803906409820462572243298853574672149664378108215389188696453420214699722967584199470131652491
3849210517415875076785196312119495759970859252434309129302173156352106846709170430429056753647687903
1227528692058927690483709214285585719241101990073778161131981122159963106459662254167802232291640108
9348914343202481190896533900420837116144945653222123954830825359910625724337519235659570699858976093
3034168762845787208048115384026599867498109469257288083679805389339036591501281524285494832182868787
4342301743019419306688013850612219622243010119848476991152725406666046444056748106004723607644097968
61925466465327459.

This number n is **not** prime and $n + 8$ **is** prime, and a typical laptop can **verify** both of these facts in fractions of a second. By contrast, the technology to **factor** n (and numbers like it) into primes does not yet exist, and most encrypted communications (in particular, most internet traffic) depends on this fact! The example n above is copied directly from the public certificate that protects `https://www.amazon.com`, but this security could be breached by anyone who can factor n into primes, so Amazon and all of its users rely on this not being feasible.

To factor a large number and/or test whether it is prime, the naïve "trial division" algorithm considers all potential factors individually: "is it divisible by 2? 3? 4? 5? etc." But for numbers with hundreds of digits, this is way too slow, since the universe will literally suffer heat death before this algorithm makes noticeable progress.

So how is it possible to conclude that a large number (like n) is composite *without* factoring it? How can we be sure that a large number (like $n+8$) is prime *without* testing all of its possible prime factors? We'll explore clever algorithms that enable efficient tests like these, and the elegant underlying number theory. Topics may include: primality certificates; probable vs provable primes; the Great Internet Mersenne Prime Search; generating large primes; the AKS primality test.

Homework: Optional

Class format: Lecture

Prerequisites: Modular arithmetic: should understand modular inverses and Fermat's Little Theorem. I plan **not** to assume or use any knowledge of abstract algebra.

Extremal graph theory (🌶🌶🌶🌶, Yuval, TWΘFS)
A basic fact in graph theory is that every tree with n vertices has exactly $n-1$ edges. Said differently, if an n-vertex graph has no cycles, then it has at most $n-1$ edges.

What if, rather than excluding *all* cycles, we only exclude the triangle C_3? Then suddenly we can put in a lot more edges: the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ has $\lfloor n^2/4\rfloor$ edges and no C_3. This really is a *lot* more edges—rather than growing linearly in n as before, it now grows quadratically.

Can we put in any more edges? It turns out we can't: $\lfloor n^2/4\rfloor$ is the most edges a C_3-free n-vertex graph can have. We'll prove this on the first day of class.

What if, instead of excluding C_3, we exclude the five-cycle C_5? It turns out that again, the most edges we can have is $\lfloor n^2/4\rfloor$. This result is way too hard to prove in a Mathcamp class, though we will prove a slightly weaker version of it. The exact same thing is true if instead we exclude C_7, C_9, or any other odd cycle.

What if, instead, we exclude C_4? Somewhat shockingly, the answer is suddenly a lot smaller: the maximum number of edges in a C_4-free n-vertex graph is around $n^{3/2}$, which grows much more slowly than the quadratic behavior we saw before. We'll prove this on the second day of class. Similar techniques allow one to show that if we exclude C_6, the answer grows like $n^{4/3}$.

So now we know the answer if we exclude C_3, C_4, C_5, C_6, or C_7. What happens if we exclude C_8? ***No one has any idea.***

This class will be an introduction to the wild wild world of extremal graph theory, where the problems are simple, the techniques are beautiful, the results are deep and powerful, and there are a million natural questions that seem completely impossible to answer.

Homework: Recommended

Class format: Lecture

Prerequisites: Graph theory

On beyond i (🌶🌶, Steve, TWΘFS)
There is a nice progression of number systems, $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$: we start with the natural numbers, and at each stage we fix some problem. So, for example, we go from $\mathbb{Q}$ to $\mathbb{R}$ to "fill in the holes," and we go from $\mathbb{R}$ to $\mathbb{C}$ so that equations like $x^2+1=0$ will have solutions. Once we get to $\mathbb{C}$, though, we seem to be done: there are no holes as in the case of $\mathbb{Q}$, and the fundamental theorem of algebra tells us that every polynomial which is not constant already has a root over $\mathbb{C}$. So there's no need to keep going.

So let's keep going! Having only *one* square root (up to $\pm$) is boring. We want more! There are number systems past the complex numbers—strange things like the quaternions, octonions, and sedenions—which satisfy this perfectly normal craving. In this class, we'll begin by playing around with these systems, and then turn to the underlying bit of abstract mathematics which lets us build these and many others. Oh, and we'll also look at reasons why someone might be interested in these systems *other* than curiosity and a love of the bizarre.

Homework: Recommended

Class format: Lecture

Prerequisites: Complex numbers (but not complex analysis!), knowing the definition of a ring is helpful but not necessary

Ring theory (🌶🌶, Kayla, TWΘFS)
If you like it, you should ideally put a ring on it! When we first learn about number systems, we learn the basic operations: addition, subtraction, multiplication, division. If we lose the context of strictly looking at integers, real or complex numbers, for which sets can we still do these operations? Rings are algebraic structures in which addition and multiplication exist and act as we'd expect. This abstract

way of thinking about algebraic structures is the backbone of many other interesting topics (and classes to come at camp!) such as commutative algebra, algebraic geometry, representation theory, field and Galois theory. In this class, we will see a quick introduction to the beautiful world of ring theory.

Homework: Recommended

Class format: Lecture

Prerequisites: None

Required for: Commutative algebra and algebraic geometry (W3); A curious connection between p-adic distances and triangulations of a square (W4); Finite fields (W4); Introduction to Galois theory (W4)

The residue theorem (🌶🌶🌶, Kevin, TWΘFS)
Complex analysis studies functions whose input and output are both complex numbers $z = a+bi$ rather than real numbers. Many of the same concepts that come up in calculus extend to the complex setting, but miraculous things start to happen! For example, the residue theorem says that integrals around a closed curve in the complex plane can be evaluated simply by studying the function's behavior near the points inside the curve where it's undefined. This result is tremendously important not only in complex analysis, but also in other fields of math from combinatorics to number theory. It even helps us to evaluate real integrals! In this class, we'll start with the definition of complex differentiation and build our way up to this remarkable theorem and several related results and applications.

Homework: Recommended

Class format: Lecture. The (recommended but not required) homework will guide you through some proofs that we won't cover in detail during lecture: you'll be able to follow the lectures without doing the homework, but the homework will be necessary if you want to prove everything we cover.

Prerequisites: Single-variable calculus (derivatives, integrals, and power series); the multivariable calculus that we need will be covered in the class/homework.

10:10 Classes

Bonus group theory part 2 (🌶🌶🌶, Ben, TWΘFS)
In Susan's group theory class[1], you learned about Lagrange's theorem, which says that if H is a subgroup of a finite group G, then the size of H divides the size of G.

We might wonder about the converse of this—suppose we have a group G of order, say, $12 = 2^2 \cdot 3$. Does G necessarily have subgroups of orders $1, 2, 3, 4, 6$, and 12? If it does, how many of these subgroups can it have?

In this class, we'll discuss the Sylow theorems, which (among other things) tell us that our G has to have subgroups of order $2, 3$, and 4. (If you're wondering about 6, it turns out there is a group of order 12 with no subgroup of order 6.)

These theorems are also useful for some classification problems that we'll investigate, for instance:

- How many groups of order $15 = 3 \cdot 5$ are there?
- Why is this different than the number of groups of order $21 = 3 \cdot 7$?
- And why do I keep writing out prime factorizations?

Homework: Recommended

Class format: Lecture

Prerequisites: Group theory—Susan's class covers all of the needed background (groups, subgroups, Lagrange's theorem, normal subgroups)

[1] Or in whatever class you first met groups in, I'm willing to bet.

Equidistribution (🌶🌶, Viv, T WΘFS)
What does the sequence

$$0.1415..., 0.2831..., 0.4247..., 0.5663..., 0.7079..., 0.8495..., 0.9911..., 0.1327..., ...$$

have that the sequence

$$0.6666..., 0.3333..., 0, 0.6666..., 0.3333..., 0, ...$$

doesn't?

Well, a lot more terms, for one! The first sequence above is the portion after the decimal point of multiples of π, and the second is the same thing for multiples of $\frac{2}{3}$. If we kept going, we'd find that the decimal portions of multiples of π defy simple categorization: as a set, they don't stay in specific sections of the interval $[0, 1)$ (or on specific points like $0, 0.3333...$ and $0.6666...$), but rather move all over the place, a lot. In fact, this sequence seems to love every part of the interval $[0, 1)$ equally! This property is known as *equidistribution.* In this class, we'll define and build up intuition for the concept of equidistribution of sequences mod 1. We'll prove a beautifully simple way of checking that a sequence is equidistributed, and then we'll use it to show that the sequence $(n\alpha \bmod 1)_{n\geq 1}$ is equidistributed mod 1 if and only if α is irrational.

Homework: Recommended

Class format: Lecture

Prerequisites: Single-variable calculus (especially limits, sequences, and series), complex numbers (i.e. comfort with i and $e^{2\pi i\theta}$ being the unit circle, but no more).

Fractal geometry (🌶🌶, Steve, T WΘFS)
The usual three dimensions are fun and all, but they get kind of boring after a while. One way to liven things up is to add more dimensions; billion-dimensional shapes are probably super cool! But you know what I like even more than big numbers? *Wrong numbers.* I want a two-and-a-half-dimensional shape. Or a π-dimensional shape. Or a shape with a decent number of dimensions, but for terrible reasons.

It turns out that we can make this happen! The answer is *fractals*, a particularly weird and beautiful kind fo shape. Fractals crop up throughout mathematics in all sorts of weird ways, and have lots of fascinating properties *besides* just being dimensionally weird. This class will be about what dimensions are, why fractals have silly numbers of them, and how awesome that is.

Homework: Optional

Class format: Lecture

Prerequisites: None

Lehmer factor stencils (🌶, Aaron and Eric, T WΘF S)
In the 1920s, one of the fastest known ways to factor large numbers was with Lehmer Factor Stencils. If you wanted to factor, say, 1229209, you could go to one of a few libraries, borrow a massive set of paper stencils, and then start doing calculations with an adding machine. After a while, your calculations might pop out the numbers $-21, -5, 11, -2, 103, 3$. You lay stencils labelled $-21, -5, 11, -2, 103, 3$ on top of each other, and amid the grid of numerically-labelled holes, light shines through exactly one, labelled 827. You check if 1229209 is divisible by 827, it is not divisible, and you conclude that 1229209 is prime.

Obviously this is not the fastest way to factor large numbers anymore, but in this class, we'll go back in time, grab our stencils, and factor away. Along the way, we'll learn how to create our own set of stencils, and how holes in paper can know so much about factoring numbers.

Homework: Optional

Class format: IBL—we will do proof exercises to understand why the stencils work, derive an algorithm to use them efficiently, and factor numbers using the stencils (and probably a 4-function calculator, it's not the 20s anymore).

Prerequisites: Modular arithmetic

The Hales–Jewett theorem (🌶🌶🌶🌶, Misha, T$\boxed{\text{W}\Theta\text{FS}}$)
The Hales–Jewett theorem is a classic result in Ramsey theory that, informally, says that "high-dimensional tic-tac-toe can never end in a draw." It is known for (1) many applications to other problems, and (2) eeeeenormous upper bounds. We will see two proofs of this theorem, and also visit exciting locales such as hypergraphs, arithmetic progressions, and point constellations.

If you think inequalities like

$$r\text{-Fun}(t) \leq \underbrace{r\text{-HJ}^{r\text{-HJ}^{r\text{-HJ}^{\cdot^{\cdot^{\cdot^{r\text{-HJ}(2)}}}}(2)}(2)}(2)}_{rt \text{ levels}} \leq \underbrace{r^{4^{r^{4^{\cdot^{\cdot^{\cdot^{r^{4^r}}}}}}}}}_{2rt-1 \text{ levels}}$$

are fun, think about taking this class!

Homework: Recommended

Class format: Lecture

Prerequisites: None

The Ra(n)do(m) graph (🌶🌶, Travis, TWΘF$\boxed{\text{S}}$)
Take a collection of vertices and draw an edge between each of them with probability 1/2. If the collection of vertices is the natural numbers, it turns out that there's only one graph that results from this process. It's called the Rado graph, and this is only the first of its super-cool properties. We'll talk about this and as many other Rad(o) facts as we can.

Homework: Optional

Class format: Lecture

Prerequisites: You should know what a graph is. You should also know why, if I flip three coins, the probability that they all end up heads is $\frac{1}{8}$. If you know those things, you're good.

11:10 Classes

Counter? I hardly know 'er! (🌶, Narmada and Travis, $\boxed{\text{TW}\Theta\text{FS}}$)
Turns out there's more to counting than using your fingers. In this class, we'll introduce some of the techniques used to sneakily count things that don't want to be counted. (Topics will include basic counting techniques, bijective proofs, formula discovery, and recurrences. If you've seen this before, this class may not be for you. If you haven't, it'll be oodles and oodles of fun. (First counting lesson: that's two oodles (AKA one pair of oodles (AKA one poodle)).))

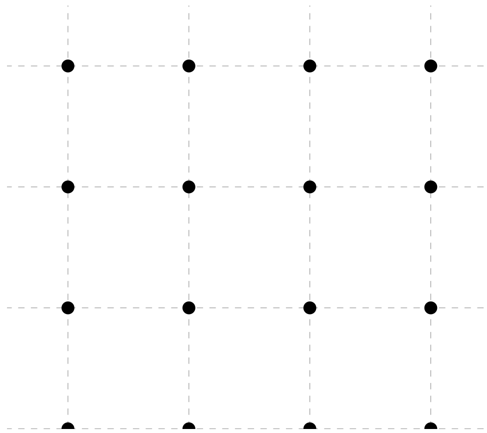
Homework: Recommended

Class format: Mostly group work with some lecture

Prerequisites: None

Erdős' distinct distance problem in the plane (🌶🌶🌶, *Neeraja Kulkarni*, $\boxed{\text{TW}\Theta\text{FS}}$)
If P is a set of N distinct points in the plane, the set of distances between points in P is called the distance set $\Delta(P)$. The size of the distance set is at most $\binom{N}{2}$ and we can find examples where this upper bound is realized (for instance, choose the endpoints of a scalene triangle). A much more difficult

question is to ask how small the distance set can be. For example, $P = \{(1,0),(2,0),\dots,(N,0)\}$ gives $|\Delta(P)| = N$. Paul Erdős discovered a better example by taking his points in a square lattice, that is, taking all points with integer coordinates between 0 and $\sqrt{N}$:



For this set, $|\Delta(P)|$ works out to be about $N/\sqrt{\log N}$. Based on the lattice example, Paul Erdős conjectured in 1946 that $|\Delta(P)| \geq N/\sqrt{\log N}$. This conjecture was proved by Guth and Katz in 2015 (or rather almost proved, as they showed a lower bound of $N/\log N$). In this course we will look at their proof, which uses topological tools such as the polynomial ham sandwich theorem, algebraic geometry tools and clever incidence geometry arguments.

Homework: Recommended

Class format: Lecture

Prerequisites: Comfort with vector geometry. Familiarity with critical points of a function would be very helpful.

My two favourite type of sets: Cantor sets and Kakeya sets (🌶🌶🌶, Charlotte, TWΘFS)
KAKEYA SETS are sets in the plane that contain a unit line segment in every single direction. Seems like they'd be large, eh?

CANTOR SETS are sets that are constructed iteratively. The standard Cantor set is constructed by starting with the unit interval, dividing it into three subintervals, and throwing away the middle one. Then we divide our remaining two intervals into three parts, and again throw away the middle ones. We do this forever.

These two types of sets are very interesting in their own right, but in this class, we will discuss a very cool connection between the two. In particular, we will use a Cantor set to construct a Kakeya set with zero(!) area.

Homework: Recommended

Class format: Lecture

Prerequisites: Know what an open set is and what a limit is. Have experience working with proofs involving epsilons.

Teichmüller theory of the torus (🌶🌶, Arya and Assaf, TWΘFS)
Take a paper square, and glue opposite sides. If done correctly (i.e., in $\mathbb{R}^4$), you will get a torus which is flat—just like the paper you used to create it. In this class, we will study the geometry of this type of construction. We will look at the "space of all flat tori" (Teichmüller space) and study it using Lattices (in $\mathbb{R}^2$), Loops (on the torus), and Linear algebra. Along the way, we'll meet some beautiful

critters like the curve graph, the Farey tessalation of the circle, and Möbius transformations in the upper-half plane.

Be warned—this class *will* involve some divison by zero, under staff supervision.

Homework: Recommended

Class format: Lecture

Prerequisites: Linear algebra (2-dimensional vector addition, multiplication, and 2×2 matrices), complex numbers will be helpful

The continuum hypothesis (week 1) (🌶🌶🌶🌶, Susan, TWΘFS)

How do you prove that a statement is unprovable? Well, that sort of depends on *why* the statement is unprovable. If it's unprovable because it's false, you can prove its negation—done! But what if it's neither true nor false? There's a huge class of mathematical statements that are actually independent of our standard collection of mathematical axioms (the Zermelo-Fraenkl axioms with choice, or ZFC for short). One excellent example of an independent statement is the continuum hypothesis.

The continuum hypothesis is a famous conjecture about the nature of infinity. A lot of the early exploration of infinite sets was done by Georg Cantor in the late 1800s. Cantor discovered the somewhat surprising fact that there are different sizes of infinity. Some familiar infinite sets turn out to be the same size, like the naturals and the rationals (which in and of itself is a bit surprising if you're used to thinking of the rationals as "bigger," but hey, that's infinity for you). In 1874, Cantor published a proof that the real numbers were a strictly larger size of infinity than the natural numbers.

The obvious followup question is: are there any infinities in between? The continuum hypothesis is the statement that, no, the size of the continuum (the real numbers) is the very next size of infinity. However, this question remained open for nearly ninety years, until 1963, when Paul Cohen proved that the continuum hypothesis is independent of ZFC. His technique was essentially to build two miniature set theoretic universes—one in which the continuum hypothesis was true, and one in which it was false.

In this class, we'll take a fast march through the proof of the independence of the continuum hypothesis. There are no prerequisites beyond a basic familiarity with cardinality, but be prepared to move fast!

Homework: Recommended

Class format: This will be a standard lecture class. Homework problems are not required, but you should be prepared to go over your notes and ask me questions in between classes.

Prerequisites: None

1:10 Classes

Computer-aided design (🌶, *Elizabeth Chang-Davidson*, TWΘFS)

Computers are awesome! They can do so many cool things! In particular, if you can imagine some shape or machine, you can make a computer draw it in 3D. Once the computer knows what it is, then it can show you what it would like from any angle, and you can tweak it without having to redraw the whole thing. You can also turn it colors and zoom in on small details. Basically, anything you can do in your head, you can show to other people, with the computer.

In addition, once you have told the computer about it, the computer can print out pictures or files that let machinists or machines make the part in real life. Computer aided design is useful for all kinds of things, from making robots to race cars to mathematical art.

Homework: Recommended

Class format: A lot of time to work on your own projects

Prerequisites: None

Eigenstuff! (🌶🌶🌶, Mark, [T]WΘFS)
If after a sunny day, the next day has an 80% probability of being sunny and a 20% probability of being rainy, while after a rainy day, the next day has a 60% probability of being sunny and a 40% probability of being rainy, and if today is sunny, how can you (without taking 365 increasingly painful steps of computation) find the probability that it will be sunny exactly one year from now?

If you are given the equation $8x^2 + 6xy + y^2 = 19$, how can you quickly tell whether this represents an ellipse, a hyperbola, or a parabola, and how can you then (without technology) get an accurate sketch of the curve?

These are two of many problems that can be solved rather efficiently using "eigenstuff" – more formally, eigenvalues and eigenvectors of square matrices. In this class we'll define what those are and quickly look at a few examples of the cool things that can be done with them. They will also come up in the representation theory class in weeks 3 and 4; however, they won't come up at the very beginning of that class, and if you don't make it to the "eigenstuff" class but you want to take representation theory, I'm willing to try to get you caught up on them early in week 3.

Homework: Optional

Class format: Lecture

Prerequisites: Linear algebra (specifically, linear transformations and their matrices, the idea of a basis, and matrix multiplication)

Grammatical group generation (🌶, Eric, TWΘ[FS])
Do you like silly word games? Normal subgroups and presentations of groups got you down? Come to this extremely light-hearted romp through the world of grammatically generated groups! In this class, based on a real actual published math paper, we will use group theory to understand how many homophones and anagrams the English language has. If you think this sounds silly, that's because it is silly. But we'll do it anyways, and see some cool group theory along the way! Be prepared for terrible jokes and words you will never see used in any other context.

Homework: Optional

Class format: 50/50 mixture of interactive lecture and small group/solo work

Prerequisites: Group theory: familiarity with what normal subgroups and quotient groups are.

Hyperplane arrangements (🌶🌶, Emily, [TWΘFS])
They sound fancy, but hyperplane arrangements are pretty simple to define. In $\mathbb{R}^2$, they are collections of lines; in $\mathbb{R}^3$, they are collections of planes (and we can keep going into higher dimensions!). For example, cutting a pizza into slices produces a hyperplane arrangement, where the cuts are the hyperplanes. We will discuss how to classify the different pieces of hyperplane arrangements, and how to do operations on them. Another thing that we will explore is how to count the number of slices that hyperplanes cut $\mathbb{R}^n$ into. This is obviously very easy in the case of a pizza, but in general it is not always so nice (especially when we are constructing arrangements that we cannot easily visualize). Some tools that we will use are posets, the Möbius function, and characteristic polynomials.

Homework: Recommended

Class format: Lecture with some group work

Prerequisites: None

Information theory (🌶🌶, Linus, [TWΘ]FS)
Exactly how much does learning today's weather tell you about tomorrow's? Approximately how many possible Tweets are there, if we restrict to reasonable English Tweets only? These questions can be answered using *entropy*, a notion of the amount of information contained in a random variable.

In this class, we'll introduce entropy and use it to give slick proofs of a few theorems in discrete math, such as upper bounding the number of $n^2 \times n^2$ Sudoku puzzles.

Homework: Recommended

Class format: Lecture

Prerequisites: None

Maximally colorful mathematics (🌶🌶, Zoe, TWΘFS)
Brouwer's fixed point theorem gives us such fun facts as the existence of a cake cutting such that everyone having cake is happy with their slice, and many more! This theorem has several equivalent statements and switching statements makes for easier proofs in different settings. In particular, some of these statements allow for the ease of "colorful generalizations!" With more and more colorful generalizations we get to see more and more connections to different types of mathematics including combinatorics and discrete geometry.

With fabulously far reaching consequences, come work through these elegant techniques towards approaching problems with more than enough styles of proofs to keep us happy for a week.

Homework: Recommended

Class format: IBL

Prerequisites: None

The category of sets (🌶🌶🌶, Nic, TWΘFS)
If you've heard of category theory before, there's a decent chance you heard that it has a reputation for being horribly abstract and impossible to understand, and that you need to know a lot of math before you start learning it. This class is my attempt to convince you that almost none of that is true.

While it's kinda true that category theory is abstract, that's only because it's so widely applicable; the ideas show up in almost every corner of modern math! In this class, we'll explore category theory in a simple, familiar setting: that of sets and functions between them. It turns out that if we build up the core concepts of set theory by focusing on *functions* rather than *elements*, then these definitions—emptiness, products, unions, intersections, power sets, and more—will have generalizations to a wide variety of other mathematical objects. (Plus, it's also just a fun way to think about set theory.) Once you see how it works for sets, my hope is that category theory will feel much more approachable.

Homework: Required

Class format: Lecture

Prerequisites: Some facility with basic set theory ideas like union, intersection, product, and so on; no prior exposure to category theory is expected. Group theory and linear algebra will pop up in a couple optional, totally skippable exercises.

The probabilistic method (🌶🌶🌶, Yuval, TWΘFS)
A set A of integers is called *sum-free* if there do not exist $x, y, z \in A$ satisfying $x + y = z$. Erdős proved the following amazing theorem: given any set S of integers, there is a sum-free subset $A \subseteq S$ with $|A| \geq |S|/3$. In other words, given *any* set of integers, you can pick out one-third of them so that no two numbers you've picked out sum to a third.

This is a theorem about numbers, so it ought to have a number-theoretic proof. But the only known proof uses almost nothing about numbers. Instead, the only way we know how to prove this theorem is by using *randomness*.

This is the heart of the probabilistic method, which is one of the most powerful techniques in modern combinatorics. Rather than proving something "directly," you impose some kind of randomness, and then show that your desired result holds with positive probability. In this class, we'll see a few examples of this idea in action, including a quick proof of the result above.

Homework: Optional

Class format: Lecture

Prerequisites: It would be helpful to know basic probability concepts like random variables and expectation; the first day of Martingales is more than enough.

Colloquia

Project selection (Staff, Tuesday)
This is not a colloquium.

Many Mathcampers enjoy working on some kind of long-term project throughout camp: on their own, or in groups, and possibly with guidance from a staff member. These projects range from reading math papers to folding origami to doing original research to baking. They can take lots of time every day or just some planning once or twice a week. If this sounds appealing to you, and you have a project you'd like to work on, just talk to any of the Mathcamp staff about it! We'd be happy to help out. If this sounds appealing to you, but you don't have a project in mind yet, then come to this event: the project selection fair! Staff have many of their own project ideas for you to sign up for.

Exploring extreme x in e^x (Assaf, Wednesday)
This expository expo expounds experiments explicitly expanding exponents. The expanded expression expels explosive exploration of $\exp(x)$ and explains its expansive exploits. Explicitly, experience how exponentiation exports expanses to groups and exposes exploitable expressways to solving ODE and PDEs. Before expiring, expect explicit explanations of extreme examples of $\exp(x)$.

In English: Using calculus, we can write $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. But what if we took that to be the **definition** of e^x? If that's the case, then we can define e^x for some very weird x's ranging from complex numbers to strange algebra systems to matrices to derivatives... whatever that means. In this colloquium, we will find out what this means!

Map coloring tourism (Misha, Thursday)
When it comes to coloring maps, a so-called "four color theorem" may have been proven by computer, but the situation changes when countries impose their own political[2] constraints on the process.

In this colloquium, we will survey the recent history of three famous[3] countries in the Nonspecific Ocean in search of answers. We will go on a tour of the canals and cantons of Circlevania; we will color between the valleys and chasms of Carstenland; we will visit the 68 baronies, 20 counties, and 4 duchies of The Mirzakhanate. We will get four different answers to the question "How many colors do we *really* need to color[4] a map?"

Fruit math memes (Eric, Friday)
You may be familiar with the fruit math meme: an image of some equations where the variables are fruits, accompanied by the claim that "95% of people can't solve this" or some similar figure. Usually these exist to troll people. Occasionally though, a fruit math meme can teach us about the frontiers of number theory! We'll conquer some fruit math memes by learning about elliptic curves, and along the way we'll encounter a Hilbert problem, some Millennium problems, and more.

[2] Mathematical.
[3] Fictional.
[4] List color.

CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2022

Contents

9:10 Classes

2-adic computer science (🌶🌶🌶, Eric, TWΘFS)
In this class you'll learn why

1111111111111111111111111111111111111111111111111111111111111

is in fact an incredibly natural and sensible thing for computers to do when dealing with negative numbers (and no, that's not a typesetting error). We will explore some intersections between computer science (how computers store various types of numbers) and algebraic number theory (how number theorists store various types of numbers). We'll learn some basics about how to analyze the runtime of algorithms, and compare a few algorithms for computing x^y for very very large integers x, y. The result we're aiming towards is an absurdly efficient algorithm for performing exponentiation based around the use of p-adic numbers, which are a tool developed in algebraic number theory for combining ideas from real analysis and modular arithmetic.

This course is intended as a first introduction to a lot of computer-related concepts; if you know how floating point numbers work and you're comfortable figuring out the asymptotic runtime of some classic sorting algorithms you probably don't have much to learn from days 1 and 2 of the class. Days 3 and 4 of the class will focus on developing the p-adic numbers, and we'll tie the computer science and p-adics together on day 5.

Homework: Recommended

Prerequisites: Fluency with modular arithmetic, say at the level of the statement "x is invertible mod n iff $\gcd(x, n) = 1$." You **don't** need to know anything at all about computer science or programming.

Common continuity (🌶, Zoe, TWΘFS)
Continuity is at the heart of point-set topology. A huge tool for classifying topological spaces is considering what kinds of functions on these spaces are continuous. In this class, we'll look at what makes functions continuous and why continuity is an appealing property that we like.

But! We will also look at why continuity is a *bad, unappealing* property that we don't like. We'll see why continuity is a much weaker property than we often imagine it to be, and we will carefully examine *gross* spaces and *gross* functions that nevertheless manage to be continuous.

Homework: Recommended

Prerequisites: None

Measuring fairness (🌶🌶, *Moon Duchin*, TWΘFS)
This class is about one very specific kind of measuring fairness. Namely, what does it mean for an electoral district to be unfairly tilted to some political party, and how would you certify, by contrast, that a districting plan is partisan-fair? We will go from zero (no assumptions that you know anything about U.S. politics, in particular!) to literal expert level. The math involved is elementary but, trust me, pretty cool.

Homework: Recommended

Prerequisites: None

Representation theory of finite groups (week 1) (🌶🌶🌶🌶, Mark, TWΘFS)
It turns out that you can learn a lot about a group by studying homomorphisms from it to groups of linear transformations (if you prefer, groups of matrices). Such a homomorphism is called a representation of the group; representations of groups have been used widely in areas ranging from quantum chemistry and particle physics to the famous classification of all finite simple groups. For example, Burnside, who was one of the pioneers in this area along with Frobenius and Schur, used representation theory to show that the order of any finite simple group that is not cyclic must have at least three distinct prime factors. (The smallest example of such a group, the alternating group A_5 of order $60 = 2^2 \cdot 3 \cdot 5$, is important in understanding the unsolvability of quintic equations by radicals.) We may not get that far, but you'll definitely see some unexpected, beautiful, and important facts about finite groups in this class, along with proofs of most or all of them. With any luck, the first week of the class will get you to the point of understanding character tables, which are relatively small, square tables of numbers that encode all the information about the representations of particular finite groups; these results are quite elegant and very worthwhile, even if you go no further. In the second week, the chili level may ramp up a bit (from about $\pi + 0.4$ to a true 4) as we start introducing techniques from elsewhere in algebra (such as algebraic integers, tensor products, and possibly modules) to get more sophisticated information.

Homework: Recommended

Prerequisites: Linear algebra, group theory, and general comfort with abstraction. (The "eigenstuff" material from the week 2 class will come up after a while, but I can catch you up on that outside class time as necessary.)

Schubert calculus (🌶🌶🌶, Kayla, TWΘFS)
Have you heard of Hilbert's iconic list of 23 problems? Have you heard of Hilbert's 15th problem? It is only partially resolved and in this class, we are going to see some of the progress that has been made. Hilbert's 15th problem has to do with enumerating intersections of subspaces in a fixed ambient space and asks to formalize Schubert's "calculus" for counting these intersections. . . but this calculus isn't one you've seen on an AP exam and Schubert's proof using "Principle of Conservation of Number" was quite the unfinished symphony in the math community's eyes.

Come see how the world of Schubert calculus has become a beautiful intersection of geometry, topology, algebra and combinatorics. You Schur won't be disappointed by taking *another* calculus class.

Homework: Optional

Prerequisites: Linear algebra (more specifically, row reduction of matrices and elementary row operations)

10:10 Classes

Diophantine approximation (🌶🌶🌶, Travis, TWΘFS)
When judging for a Rule 2 violation, you have to see just how irrational an idea really is. And the same goes for real numbers: How well can you approximate irrational numbers by rational ones which will slip under the wary staff's radar? We'll start by proving that for every irrational number α, there are infinitely many rational numbers p/q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

We'll see why this is interesting, surprising, and useful. We'll solve a puzzle. And then we'll follow in Liouville's footsteps and generalize this result to transcend this mortal coil.

Homework: Recommended

Prerequisites: None

Integer right triangles (🌶🌶, *David Roe*, TWΘFS)
You probably recognize the triples (3,4,5) and (5,12,13), the first few of the infinitely many Pythagorean triples that measure triangles with integer side lengths. We will start by exploring a geometric method that generates Pythagorean triples, generating a beautiful parameterization of all possibilities. We will then focus on the area of these triangles, while allowing the edges to have rational lengths. The examples above show that 6 and 30 are possible, but what about 7 or 15? We will only scratch the surface of this question, but will dig deep enough to get a glimpse of the profound conjectures underneath.

Homework: Recommended

Prerequisites: None

Nonstandard analysis (🌶🌶🌶, Aaron, TWΘFS)
The early history of calculus is filled with sketchy computations about infinitesimal quantities, which George Berkeley criticized as follows:

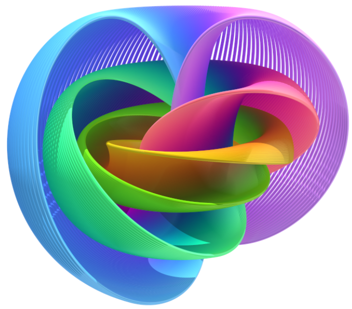
"They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?"

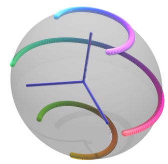
The rigor of calculus was later rescued with limits, but there is another approach. In this class, we will construct the hyperreals, a system of numbers that includes real numbers, infinitesimally small numbers, and infinitely large numbers. We will then see how to rigorously do calculus the infinitesimal way.

Homework: Recommended

Prerequisites: Calculus (preferably with epsilon-delta proofs)

On beyond on beyond i (🌶🌶🌶, Assaf, TWΘFS)
You may have heard of the quaternions (perhaps in an earlier class by a similar name, or at some colloquium about e^x), but did you know that they also encode secret and hidden geometries of 3- and 4-dimensions? Just by allowing multiple square roots of -1, we get a number system on $\mathbb{R}^4$ which allows us to rotate spheres in a snap, draw knotted vector fields in the 3-dimensional sphere S^3, and create insane images such as:





This class is about the **geometry** of the quaternions, not the algebra. As such, **"On beyond i" is not a required prerequisite.**

Homework: Recommended

Prerequisites: Linear algebra, knowing what a dot product is would be helpful

Special relativity (🌶🌶, Nic, TWΘFS)
Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but if you took the equations literally they implied some bizarre things about the structure of space and time: depending on their relative velocities, different observers could disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

For a long time, a lot of creative excuses were invented for why we *shouldn't* take the equations literally (including one with the incredibly Victorian name "luminiferous aether") but, in what was probably the second most unsettling event in early twentieth-century physics, all of them failed. The physics community was left with only one viable conclusion: space and time really do behave that way!

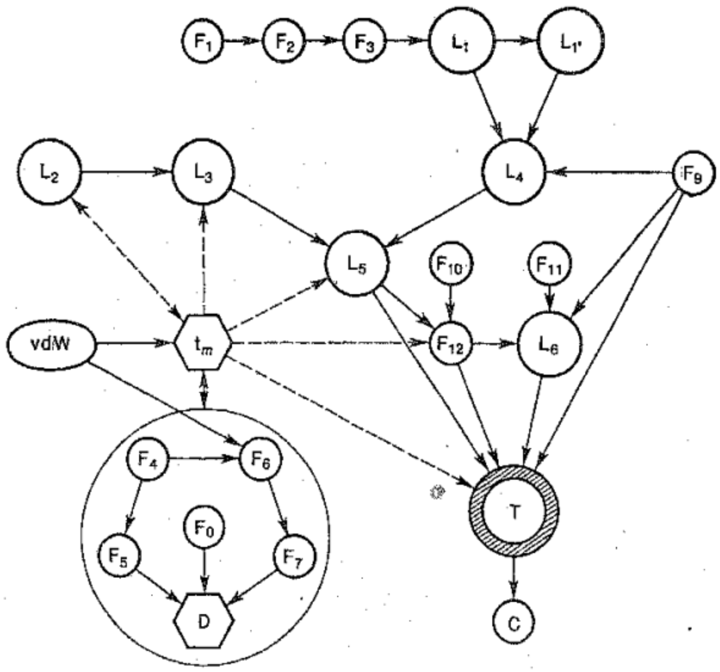
In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics had to be rebuilt to accommodate them. We will see how, as the physicist Hermann Minkowski said, "Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." At the end, we'll also briefly look at how to revise the classical definitions of momentum and energy and see why we should believe that $E = mc^2$.

Homework: Optional

Prerequisites: Enough physics to know what momentum and kinetic energy are, but no more than that. No prior exposure to special relativity expected!

Szemerédi's {theorem, regularity lemma} (🌶🌶🌶🌶, Yuval, TWΘFS)
In 1975, Szemerédi proved that if $\varepsilon > 0$ is fixed and N is sufficiently large, then any set $A \subseteq \{1, 2, \dots, N\}$ of size $|A| \geq \varepsilon N$ contains an arbitrarily long arithmetic progression. Although this is a number-theoretic statement, Szemerédi's proof was entirely combinatorial. At the time, this was perhaps the most complicated and intricate combinatorial proof ever devised; the following diagram from Szemerédi's original paper shows merely the *logical structure* of the argument, and not even any of the ideas.

202 E. Szemerédi



The diagram represents an approximate flow chart for the accompanying proof of Szemerédi's theorem. The various symbols have the following meanings: F_k = Fact k, L_k = Lemma k, T = Theorem, C = Corollary, D = Definitions of B, S, P, α, β, etc., t_m = Definition of t_m, vdW = van der Waerden's theorem, F_0 = "If $f: \mathbf{R}^+ \to \mathbf{R}^+$ is subadditive then $\lim_{n\to\infty} \frac{f(n)}{n}$ exists".

In proving this theorem, Szemerédi introduced a tool (now called Szemerédi's regularity lemma, found in the circle L_1 above), which is now itself one of the most important tools in all of graph theory. Roughly speaking, it says that all "big" graphs look "the same", and that we can actually more or less forget about where the edges of our graphs are. This may sound stupid, but its power cannot be overstated.

In this class, we'll approach Szemerédi's theorem from the perspective of Szemerédi's regularity lemma. We'll focus primarily on the graph theory, but we'll see how graph-theoretic insights can yield number-theoretic results, including seeing a complete proof of Roth's theorem (the case of three-term arithmetic progressions). Along the way, we'll see other applications of the regularity lemma; for example, the Erdős–Stone–Simonovits theorem, the main result of my Extremal graph theory class, will be a simple homework exercise once we understand the regularity lemma.

Homework: Recommended

Prerequisites: Graph theory

11:10 Classes

Arrow's impossibility theorem (🌶🌶, Ben, TWΘFS)
If you've heard of Arrow's Impossibility Theorem before, it might have in some form like "a good voting system doesn't exist," which leaves a bit to be desired, as a theorem statement. What do we mean by "good," or by "a … voting system," or by "a good voting system," for that matter?

Our first mission in this class is to clear up what, exactly, Arrow's Impossibility Theorem says. Our third mission is to prove it. If you're wondering about the second mission—it's to define and briefly discuss ultrafilters, which turn out to be useful for that "prove it" mission we just mentioned.

Time permitting, we might also get to talk about the Gibbard–Satterthwaite Theorem, which says that there's always[1] some voter who shouldn't vote for who they want to win.

Homework: Recommended

Prerequisites: None

Buffon's needle (🌶️, Ben, TWΘ FS)

Suppose that I draw a bunch of lines parallel to each other, spaced one inch apart, and drop my standard-issue one-inch needle onto them randomly. How likely is it that the needle will cross one of the lines? More generally, what if the needle is longer, shorter, or is actually a squiggly piece of uncooked pasta?

There are, broadly speaking, two approaches to this. One of them involves setting up some number of integrals to figure out these probabilities. This approach is entirely valid and will work, but there's another approach that just relies on probability theory.

So, in this class we will compute no integrals, and use no calculus[2]. Instead, we will see a marvelous display of the glorious power called "linearity of expectation," and that's all we'll need.

Homework: Recommended

Prerequisites: None

Commutative algebra and algebraic geometry (🌶️🌶️🌶️, Mark, TWΘFS)

In its classical form, algebraic geometry is the study of sets in n-dimensional space that can be described by polynomial equations (in n variables). This is both a very old and a quite active branch of mathematics, and for over a century now it has relied heavily on commutative algebra—that is, on the properties of commutative rings and related objects. We'll start by looking at some of those, including prime and maximal ideals and a review of quotient rings, and we'll see how the algebra can be used to give us information about the geometric sets. For instance, we'll use the algebra to show that if a set can be given by polynomial equations, then a finite number of such equations will do. We may also see how to translate the idea of dimension into the language of algebra. There may well be cameo appearances by the axiom of choice (in the guise of Zorn's lemma) and a bit of point-set topology (on a space whose points are ideals!), but you don't need to know any of those things going in. It's quite possible that the TBD class listed in the week 4 schedule will be a continuation of this class—that depends on how much interest there seems to be. If that happens, I hope, among other things, to prove Hilbert's famous Nullstellensatz ("Theorem of the Zeros"), arguably the starting point for modern algebraic geometry, at least for the case of two variables. (The theorem will presumably be stated and used in the first week.)

Homework: Recommended

Prerequisites: Familiarity with polynomial rings, ideals, and quotient rings.

Curves that classify geometry problems (🌶️🌶️, *J-Lo*, TWΘFS)

In this class we will turn geometry problems into curves. For example, consider the following three problems:

(0) How many triangles have integer side lengths and a 60° angle?

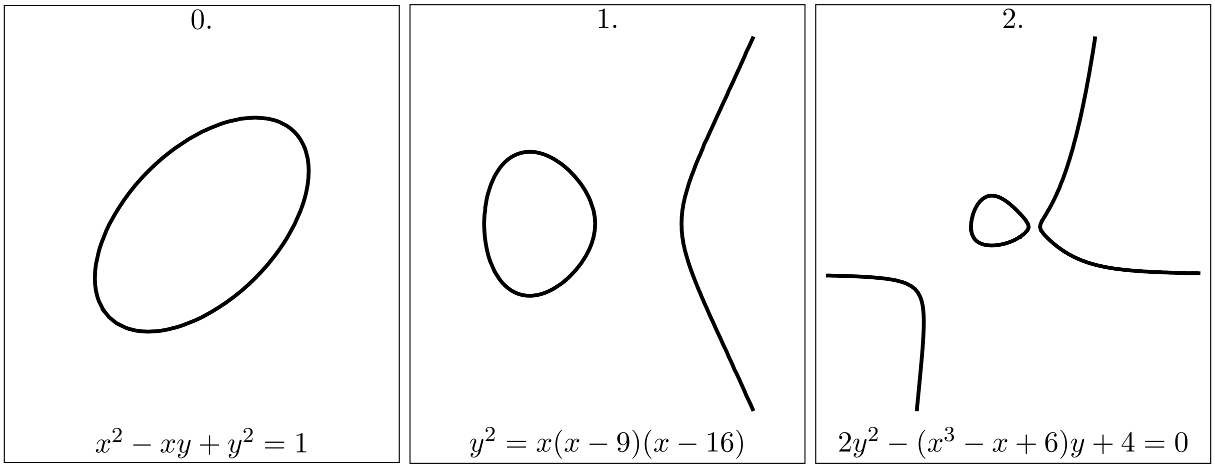
[1]OK, not literally always, just usually

[2]. . . OK we'll want to think a little bit about limits but I promise that's all

(1) How many triangles have integer side lengths and integer area, with two of the sides in a ratio of 3 to 4?

(2) How many pairs of triangles with integer side lengths, one right and one isosceles, have equal area and equal perimeter?

(Do not count scaled solutions separately; for example, in problem 0, you should only count one equilateral triangle.) Each problem can be solved by finding rational points (points (x, y) with x and y rational numbers) on a certain curve:

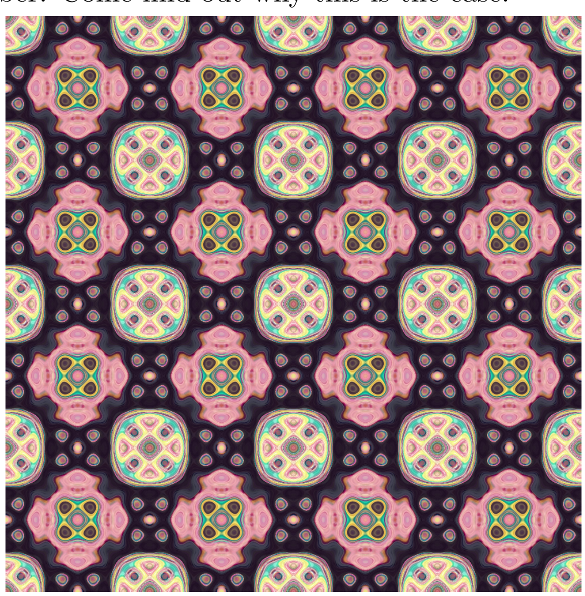


Come to this class to learn how to convert geometry problems into curves, to explore some tools we can sometimes use (such as stereographic projection and elliptic curve addition) to find the points on these curves that we're looking for, and to discuss some unsolved problems about right triangles.

Homework: Recommended

Prerequisites: None

The 17 wallpaper patterns (🌶🌶🌶, Emily, TWΘFS)
No, I'm not talking about the wallpaper at your grandma's house—I'm talking about mathematical wallpaper! These are two dimensional repetitive patterns which are distinguished based upon their symmetries, such as the example below. To each of these we can assign a group which consists of the transformations of the plane which preserve the pattern. It turns out there are exactly 17 of these—what a strange number! Come find out why this is the case.



Homework: Recommended

Prerequisites: Group theory, linear algebra

Ultrafilters and combinatorics (🌶🌶🌶🌶, Steve, TWΘFS)
Combinatorics is full of results saying that functions on an infinite set are well-behaved "a lot" of the time. An easy example of this is the Pigeonhole Principle: given a function $f : \mathbb{N} \to X$ with X finite, no matter how crazy f is there is always some infinite set $S \subseteq \mathbb{N}$ on which f is constant. A slightly trickier instance of this is Infinite Ramsey's Theorem (for pairs): any 2-coloring of pairs of natural numbers has an infinite homogeneous subset. (If you haven't seen this before, don't worry, we'll prove it in class.)

However, what if "infinite" just isn't big enough? For example, for a function $f : \mathbb{N} \to X$ with X finite, maybe we want f to be constant on a set which is not only infinite but *closed under (finite) sums.* Can we always find such a set? If so, what's the most ridiculous way we can prove it?

In this class we'll do combinatorics using ultrafilters—bizarre, beautiful objects from the mysterious land of set theory! Ultrafilters cannot even be proved to exist without the axiom of choice, but that won't stop us from using them to build big homogeneous sets. Oh, and we'll also need to say the words "compact space," "topological semigroup," and "idempotent" a few times.

Homework: Required

Prerequisites: None

Zero knowledge proofs (🌶🌶🌶, *Dan Zaharopol*, TWΘFS)
Picture this: You want to convince someone that you know something is true, but you don't want that person to actually be able to reconstruct the proof themselves (or to have *any* advantage in doing so). For example, maybe you want to prove that a graph has a Hamiltonian cycle, but you want to give absolutely no information to the other person that would allow them to find the cycle themselves—you just want to convince them that it exists!

You might think, "Surely, that isn't possible!" It sounds outlandish that you could prove to someone that you know a cycle without showing it to them. And yet you can; doing so is called a zero-knowledge proof, and besides being really cool it also has applications all across different areas of cryptography.

In this class, we'll see two things: how to accomplish certain zero-knowledge proofs, and how to give a rigorous definition of them. In particular, it's not just interesting that we can do it, but also that we can write down precisely what it means to prove something without sharing any knowledge. This class will be a chance to explore how that works and to get more insight into how both computer science and cryptography are formalized mathematically.

Homework: Recommended

Prerequisites: None, but some knowledge of graph theory or big-O notation will be helpful. If you've seen formalized language around computer science, the class will be easier.

1:10 Classes

Hyperbolic geometry (🌶🌶, Arya, TWΘFS)
In normie Euclidean geometry, the sum of angles of a triangle is always equal to 180 degrees, areas are computed by actually multiplying two lengths, and inverting across circles does spooky things. Imagine drawing a line through a point parallel to some given line, and NOT being able to draw a second one? Dealing differently with triangles that are clearly similar, just because SoMeOnE mAdE a triangle bIg? Drop the blindfolds of Cartesian coordinates and join this class to free your imagination and learn some hyperbolic geometry! We shall talk about different models of hyperbolic

spaces, isometries, hyperbolic trigonometry, analogues of theorems from Euclidean geometry, and why "hyperbolic metrics are the natural geometric structures on almost all surfaces."

Homework: Recommended

Prerequisites: Knowing what complex numbers are, some familiarity with sine rule and cosine rule.

In-fun-ite groups (🌶🌶🌶, Narmada, TWΘFS)
If you've ever thought "There's no way groups can be this nice," then this is the perfect class for you. Halloween has come early this year and we're going to be looking at some truly monstrous groups. It all started when we let them be infinite...

We'll start by seeing that even the free group on two generators can't be trusted. Then, we'll look at how the Axiom of Choice plays into the structure of free groups and free abelian groups. For the grand finale, we'll study a (still unsolved!) problem that has been called the equivalent of Fermat's last theorem for group theory.

Homework: Optional

Prerequisites: Group theory: talk to me if you have questions!

Machine learning (NOT neural networks) (🌶🌶🌶, Linus, TWΘFS)
Netflix wants to recommend me TV shows that I will like. To do this, they analyze a giant matrix of people and their ratings of TV shows and movies. When a user rates a movie, Netflix learns one entry of this matrix; their goal is to find patterns in their dataset and use them to predict the unknown entries.

In 2006, Netflix offered $1000000 to the first team that could beat their internal prediction algorithm by 10%. This problem embodies the second era of machine learning, *linear algebra on big data*. In this class, we'll show off (a simplified version of) a winning algorithm.

We'll also explore the first era of machine learning, *classical PAC-learning algorithms*, full of sharp combinatorial algorithms with strong provable guarantees.

We won't touch on the third era, the *neural network jamboree*, unless I go crazy on Day 5. (Why not? Because they can't prove anything...)

Homework: Recommended

Prerequisites: Linear algebra. Also it'll help to have seen linearity of expectation.

Problem solving: graph theory (🌶🌶🌶, Misha, TWΘFS)
In this class, we will solve problems. Some of these will come from math competitions, some of these I made up myself, and some of these I found "in the wild."

Some of our problems will be questions about graphs, and some of them are questions that we can model—and solve—with graph theory. There are few ideas from graph theory that will be especially important:

- Using the handshake lemma and Euler's formula.
- Matchings in bipartite graphs and Hall's theorem.
- Trees, their properties, and what they tell us about connected graphs.

We will work through problems together every day in class and see some of the key ideas involved in solving them. If you want to get the most out of this class, you should work on the remaining problems on your own during TAU!

Homework: Recommended

Prerequisites: Basic familiarity with graphs. Ideally, nothing I mentioned in the blurb should scare you; if it does, but you still want to take the class, talk to me!

The continuum hypothesis (week 2) (🌶🌶🌶🌶, Susan, TWΘFS)
This week we proved that Martin's Axiom implies the existence of a dominating function for uncountable-but-not-continuum-sized sets of functions from $\mathbb{N}$ to $\mathbb{N}$. We also proved the Löwenheim–Skolem theorem, showing that we can create a little tiny countable set theoretic universe, and saw the Mostowsky collapse construction for making sure that countable universe is transitive. We're now deep in the weeds, trying to figure out how to formally adjoin filters to set theoretic universes. Want to see the rest? Tune in to the exciting conclusion of the continuum hypothesis!

Homework: Recommended

Prerequisites: The continuum hypothesis, week 1

Colloquia

Everyone hates analysis (Charlotte, Tuesday)
It is a truth universally acknowledged that everyone hates analysis. Nowhere is this more true than at Mathcamp. When Mathcampers see a class tagged "analysis," they quickly avert their eyes, in fear that they will be turned to stone lest they read the blurb in full. In this colloquium, I will make you absolutely miserable for approximately fifty minutes. In fact, you will almost surely leave this colloquium hating analysis even more than you do right now.

Hyperspheres (*David Roe*, Wednesday)
Draw a square, divide it into four equal squares, and then inscribe a circle within each. Within those four circles, you can fit another smaller circle. We can draw this setup on a blackboard, and Euclidean geometry gives us the tools to compare the sizes of all the objects involved. But what happens when we increase the dimension? This thought experiment has an interesting answer, and will lead us into the broader world of sphere packing, an area of mathematics with connections to error correcting codes, chemistry, number theory, hyperbolic geometry and string theory.

High-dimensional oranges (Travis, Thursday)
You know oranges; you might even know *and* love them. Maybe they're your favorite fruit that can occasionally be found in the dining hall. As a rule, they're not very interesting: They just sit there and maybe roll around a bit, awaiting their eventual end. But put on a pair of Hi-Tek Xtra-Dimensional Power Goggles™ (patent pending), and you'll find that oranges contain multitudes. We'll plunge the depths of this hidden knowledge to learn about the geometry of spheres with many, many dimensions and how the measurement of volume is a much more tricksy concept than you may have been led to believe.

Heisenberg geometry (*Moon Duchin*, Friday)
I will tell you all about a world where walking around in a circle causes you to involuntarily levitate. This is nilpotent geometry, with the innocuous-looking 3-dimensional Heisenberg group as the gateway drug. Geometry meets groups meets analysis meets robots. . . . . .

CLASS DESCRIPTIONS—WEEK 4, MATHCAMP 2022

Contents

9:10 Classes

A curious connection between p-adic distances and triangulations of a square (🌶🌶, Charlotte, TWΘFS)
If you're given a square, you could find a way to divide it into an even number of triangles of equal area. Now try dividing it into an odd number of triangles of equal area!

Well, you probably didn't, because you can't, a fact which is known as Monsky's theorem. What's lovely about the proof of Monsky's theorem is that it is entirely unexpected: its main tool is algebraic, the 2-adic valuation (which is closely related to the 2-adic numbers, and gives a different way of measuring "distance" between points). We'll use 2-adic valuations to coulour the plane, and see some slick combinatorial arguments.

Homework: Recommended

Prerequisites: Familiarity with metrics, and the definitions and basic properties of groups, rings, subrings, invertible elements of rings, and quotients of groups

Ancient Greek mathematics (🌶, Yuval, TWΘFS)
You may have heard some crazy stories about ancient Greek mathematicians:

- Pythagoras proved the Pythagorean theorem, but also hated beans. Also, he killed someone for figuring out that $\sqrt{2}$ is irrational.
- Euclid once sassed King Ptolemy I by telling him "there is no royal road to geometry".
- Archimedes shouted "Eureka!" in the bathtub, ran down the streets of Syracuse naked, and later invented a giant mirror to burn attacking Roman ships.
- The Greeks tried really hard to square the circle, but never could. In 1882, von Lindemann proved that squaring the circle is impossible.

Sadly, probably none of these stories is true.

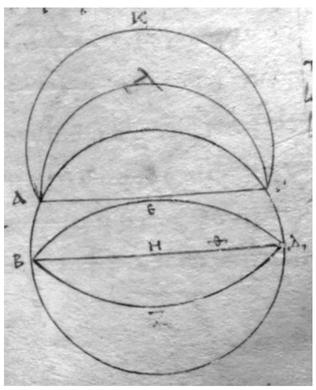
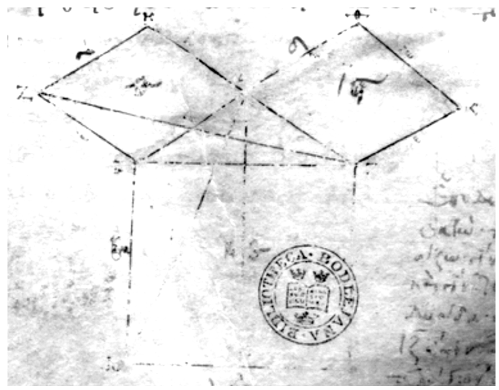
Wait, what? *None* of them is true? What about Pythagoras proving the Pythagorean theorem? Turns out that probably didn't happen.

But the truths about ancient Greek mathematics are, perhaps, even crazier than the myths.

- Rather than being sad about irrational numbers, the Greeks loved and were obsessed with them.
- Eudoxus essentially invented Dedekind cuts, and thus the formal theory of the real numbers, 2250 years before Dedekind.

- Euclid *sort of* proved the infinitude of the primes, but his proof only really implies that there are at least four primes.
- The Greeks were actually *really good* at squaring the circle! They came up with three different methods, von Lindemann's "proof of impossibility" notwithstanding.
- Archimedes essentially invented integrals, and used them to compute areas and volumes of crazy shapes (parabolas, spirals, spheres) 1900 years before the "real" invention of calculus. He also invented systems for expressing huge numbers, initiated the field of mathematical physics, and proved perhaps the most difficult and complicated theorem proved until the 19th century.

This class is all about ancient Greek mathematics. We'll learn both about what mathematics they did—including some shockingly difficult, complicated, and beautiful proofs—and about how they *thought* about mathematics. In many cases, they thought about mathematics in more or less the same way we do (and, indeed, our mathematics continues a tradition directly inherited from the Greeks), but in other cases, it can feel like speaking to an alien. For example, would you have guessed that these pictures depict, respectively, two circles and a proof of the Pythagorean theorem?

Homework: Recommended

Prerequisites: None

Baire necessities for Banach–Tarski (🌶️🌶️🌶️🌶️, Narmada, TWΘFS)

If you give a mathematician a proof of the Banach–Tarski paradox, she will tear it apart into finitely many pieces and reassemble it into *two* proofs of the paradox. We'll look at the first proof of the Banach–Tarski paradox (which was mostly due to Hausdorff) that manipulates the free group. Then we'll look at the *poggers* proof that uses graph theory to prove an even stronger version of the paradox: you can force the pieces in your decomposition to be topologically nice. What's with the Baire in the title? Come to class to find out!

Homework: Optional

Prerequisites: Know what graphs and groups are

Problem solving: cheating in geometry (🌶️🌶️🌶️ → 🌶️🌶️🌶️🌶️, Zack, TWΘFS)

Geometry is hard. Sometimes you can bash geometry problems with algebra, but algebra is hard too. Everything would really be a lot nicer if geometry were easy, like if every pair of lines intersected or if every circle passed through the same two points. Helpfully, projective geometry (motto: "what if geometry were better") exists! In projective geometry, everything is great, lines and curves behave how they should, and geometry is easy.[1] We'll build some intuition for projective space through examples, and discover some geometric and algebraic tools which will sometimes allow us to solve hard geometry problems quickly and easily, in particular the somewhat infamous "method of moving points." Side

[1] OK, maybe not easy, but at least there aren't any angles.

effects may include, but are not limited to: an inability to return to thinking about angles and lengths, a tendency to write solutions that will make your graders sad, and following every sentence with "you know, this would really be a lot nicer over $\mathbb{CP}^2 \ldots$"

Homework: Required

Prerequisites: None—in particular, no experience with olympiad geometry will be assumed.

The distribution of prime numbers (🌶🌶, Viv, TWΘFS)
What is the distribution of prime numbers?

This question is really vague, and encompasses a lot of other questions. Questions like:

- How many prime numbers are there? For a fixed $x > 0$, how many primes are there less than x? How precisely can we count them?
- How many twin primes are there (that is, primes p where $p + 2$ is also prime)? Are there infinitely many?
- What is the biggest gap between two primes that are less than x? How frequently is the gap between two consecutive primes small? How frequently is it big?
- What is the distribution of final digits of primes? What about final digits of pairs of consecutive primes?

Many of these questions are... hard. Like, really *really* hard. For example, Wikipedia says that the Twin Primes Conjecture, which states that there are infinitely many primes, "has been one of the great open questions in number theory for many years." Instead of trying to answer these questions, we'll do our best to understand what the answers *should* be, and why. Along the way, we'll develop and evaluate a random model for prime numbers, and discuss my favorite conjectures (for some definition of favorite).

Homework: Recommended

Prerequisites: Some basic number theory is helpful; specifically, being comfortable with modular arithmetic is helpful, as well as the Chinese Remainder Theorem.

10:10 Classes

Algebraic topology: homology (🌶🌶, Zoe, TWΘFS)
Whenever faced with real wonky situations in mathematics, our usual end goal is to try to get a comparison to a situation we actually know things about. Homology takes whatever weird space one could think of, and gives us a way to measure how close that space is to any n-dimensional hole. In this class, you will learn efficient ways to compute homology as well what homology can give us when analyzing a problem.

Homework: Recommended

Prerequisites: Group theory and Linear algebra (not strict prereqs, feel free to talk to me about what exactly is needed).

Commutative algebra and algebraic geometry (week 2) (🌶🌶🌶 → 🌶🌶🌶🌶, Mark, TWΘFS)
This class, which was originally announced as "TBD", will be a continuation of the week 3 class. If you didn't take the class last week and you would like to join now, it's probably a good idea to consult with Mark first.

Homework: Recommended

Prerequisites: Commutative algebra and algebraic geometry (week 1)

High-dimensional potatoes (🌶🌶🌶, Travis, T[WΘFS])
Have you ever looked at a potato? Like, *really* looked at it? Did you then think that they would be cooler if they had a few more dimensions, like maybe 573 of them? Perfect. We'll take a deep dive into high-dimensional potatoes, answering such questions as: When can potatoes intersect? How hard is it to specify a point inside a potato? Is it always possible to split them in half? The mysteries abound!

If you want to see what happens in high dimensions without needing any integrals, switch your diet from oranges to potatoes!

Homework: Recommended

Prerequisites: Linear algebra (familiarity with the real vector space $\mathbb{R}^d$ and linear independence)

The abc's of polynomialand (🌶, Eric, T[WΘFS])
Constants. Irreducibles. Squares. Monics. Long ago, the elements of Polynomialand lived together in harmony. Then, everything changed when Queen Polynomia went missing. Only the Wronskian, which could mediate to a degree between the feuding factions of abecedarians and radicals, could keep Polynomialand stable in her absence, but when the polynomials needed it most, they forgot about how it worked. A hundred years passed, and Mathcampers re-discovered the Wronskian. And although their understanding of integers is great, they have a lot to learn before they're ready to save any polynomials. But I believe that Mathcampers can save Polynomialand!

(This is still a math class! It's about the *abc* conjecture from a 🌶 perspective: what it is, why it's hard, and mostly why the polynomial version is more straightfoward. This class will just also be . . . silly in the way described above.)

Homework: Recommended

Prerequisites: You should be comfortable with unique factorization and the Euclidean algorithm for integers; Mark's intro number theory class is more than enough.

The satisfiability problem (🌶🌶🌶, Misha, T[WΘFS])
Questions like

- "Does this Sudoku have a solution?"
- "Is there a red-blue coloring of $\{1, 2, \ldots, 9\}$ with no monochromatic 3-term arithmetic progression?"
- "Does this 2048-bit integer factor into two 1024-bit integers?"

have one thing in common. Each one can be expressed as a formula whose variables are not numbers but *Boolean* values: true or false. The Boolean satisfiability problem is to choose the values of these variables to satisfy the formula: make it true.

This problem is notoriously difficult—it is the first problem proven to be NP-complete. (This means that if we find a polynomial-time algorithm to solve it, we get a million dollars.) Most computer scientists are happy to say that there is no known algorithm significantly better than the $O(2^n)$ algorithm that tries all possible values of n variables.

But "significantly better" can have multiple meanings. An $O(n\cdot(\sqrt{3})^n)$ algorithm is still exponential, but it can sometimes mean the difference between solving a problem in minutes or in hours. And (spoiler alert!) we'll be able to do better than $O(n\cdot(\sqrt{3})^n)$ by the end of this class.

Homework: Recommended

Prerequisites: None

11:10 Classes

Cantor before set theory (🌶🌶🌶🌶, Ben, [TWΘFS])
If you've ever looked into the history of set theory, you might have read that it came about because

of Georg Cantor's investigations into the infinite, motivated by his work in real analysis. One might wonder—what question was Cantor trying to answer, that made him start thinking about the nature of infinite sets?

In the 1800s, trigonometric series became a major area of study due to the work of Joseph Fourier. A lot of this work centered on what kinds of functions could be written in the form

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

One of Cantor's colleagues[2] asked a different question—if you already know that a function can be represented by a trigonometric series, could there be more than one? For example, we can represent the function $f(x) = 0$ as the trivial trigonometric series where all of the a_n and b_n are taken to be 0. Is there another way?

In this course, we'll not only learn the answer to this question, but also see how investigating it sent Cantor along the road to set theory—into investigations of the infinite.

Homework: Recommended

Prerequisites: Know the difference between uniform and pointwise convergence, know how to take integrals and derivatives.

Finite fields (🌶🌶, Aaron, TWΘFS)
Fields are everywhere in math, but usually we encounter infinite fields such as $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$.

In this class, we'll explore the *finite* fields! These are useful all over math, both in their own right and as a miniature test case for the infinite fields you already know. We'll construct them, see what sizes they can have, characterize their additive and multiplicative behavior, and see how they fit together.

Homework: Recommended

Prerequisites: Ring theory, linear algebra, with optional group theory.

Knot theory (🌶🌶, Emily and Kayla, TWΘFS)
Contrary to popular belief, knot theory is not "not theory." Specifically, it teaches you how you know when two knots are not the same, and when they are not not the same. Certain knavish knots defy classification, however, so knowledgeable methods and theoretical theories must be used to distinguish such gnarly knots. Know naught about knot theory yet? Worry not, for this class is an introduction!

(This blurb was written by Nathan S.)

Homework: Recommended

Prerequisites: None

Mathematical billiards (🌶, Arya, TWΘFS)
Suppose you have a point-sized ball gliding on a billiard table with a frictionless surface. The trajectory ends if it goes into a hole, and if it hits the boundary of the table, the ball follows the standard laws of reflection (the angle of incidence is the same as the angle of reflection). Depending on the shape of the table, we can ask several questions—how many times can the ball hit a wall before it goes into a hole? Can it come back to where it started, and keep looping its path in a periodic motion? Is the trajectory of the ball dense inside your shape? In this class, we shall try to answer some of these questions and discuss some related open questions.

Homework: Recommended

Prerequisites: None

[2] Eduard Heine; if you've heard of e.g. the Heine–Borel Theorem, it's that guy.

Representation theory of finite groups (week 2) (🌶🌶🌶🌶, Mark, TWΘFS)
This is a continuation of last week's class of the same name. If you didn't take the class last week and you would like to join now, it's probably a good idea to consult with Mark first.

Homework: Recommended

Prerequisites: Representation theory of finite groups (week 1), Group theory, Linear algebra

1:10 Classes

Algebraic solutions to Painlevé VI (🌶🌶🌶🌶, *Aaron Landesman*, TWΘFS)
In 1902, Painlevé introduced six differential equations, the most difficult of which was the so-called "Painlevé VI." The algebraic solutions to Painlevé VI were only classified recently in 2014. It turns out these algebraic solutions correspond to finding certain canonical triples of 2 by 2 matrices. In the class, we will search for collections of these canonical tuples of matrices. Our search will lead us to discover a sequence of beautiful connections between group theory, geometry, topology, representation theory, and algebraic geometry.

Homework: Required

Prerequisites: Linear algebra, Group theory

Chaotic dynamics and elephant drawing (🌶🌶🌶, Ben, TWΘFS)
In the study of dynamical systems, we have some rule for extrapolating what "things tomorrow" look like, given what "things today" look like. A practical example of this is the weather; we can consider this as a dynamical system. But while the weather tomorrow is fairly predictable, and modern weather forecasting can even extrapolate a week out pretty well, long-term weather forecasting is right out—is it going to snow in Toronto on 16 December 2022? We won't know for a while.

This motivates the definition of chaotic dynamical systems, in which small changes to present conditions may cause large changes in the future (the so-called "butterfly effect"). We will aim to show that some easily-described discrete dynamical systems are chaotic.

Time permitting, we'll also use our chaotic dynamical systems for a practical[3] purpose: overfitting data! We'll see how we can carefully pick a two parameter "model" that can fit any data set almost perfectly. Our model will be based on a specific dynamical system, and its marvelous overfitting powers? Are based on the fact that it is chaotic.

Homework: Recommended

Prerequisites: None

Conway's soldiers (🌶🌶, Misha, TWΘFS)
Let's play checkers! Except the pieces jump horizontally and vertically instead of diagonally. Also, the checkerboard is infinitely large and the opponent is MIA. How far forward can our set of soldiers step?

You might think that with infinite pieces, through a clever series of jumps, we should be able to travel infinitely far forward, but in fact the best we can do is exactly 0% of that. Proving this will reunite us with an old irrational friend and take us through a world of monovariants and power series that will make you say *no (Con)way!*

Then, we will go one step further than that—literally. We'll find out what can happen when are given the power to do infinitely many things in a finite length of time.

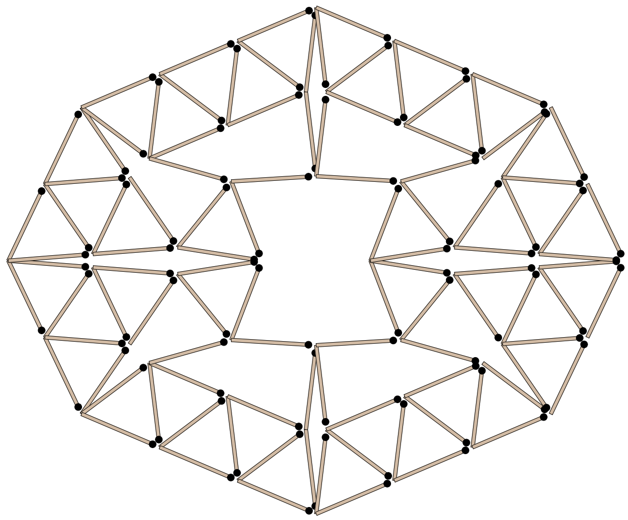
(This blurb was co-written with Lucas.)

Homework: None

[3] OK, maybe not practical. But fun!

Prerequisites: None

Electric charge on matchsticks (🌶🌶, Misha, TWΘF[S])
The graph below is called the Harborth graph:



Its edges are matchsticks all of the same length. They never cross, and four of them meet at every vertex. (We don't know if it's the smallest graph of this type.)

We do know that we can't make a graph like this with five matchsticks meeting at every vertex. We will prove this by putting electric charges on such a graph, moving them around, and showing that the graph's existence would violate conservation of energy.

Then we'll see what else can be done with the so-called "discharging method"!

Homework: None

Prerequisites: You should know what vertices, edges, and faces of a graph are, and know or be willing to accept on faith that $|V| - |E| + |F| = 2$.

Game theory, traffic, and the price of anarchy (🌶, Assaf, [TWΘ]FS)
Some schools[4] of libertarian and capitalistic thought say that if everyone does what is best for themselves, this will be best for society, since each person maximizes their own happiness in the context of their surroundings. This perspective is the Nash-equilibrium solution to humanity, and as mathematicians, we can always ask: "can we do better?" The answer is: "sometimes" and the difference between the *best* course of action and the *anarchist* course of action is called *the Price of Anarchy.*

This class is an intro to game theory class, where we will talk about combinatorial games, pure and mixed strategies, and Nash equilibria (and use Brouwer's fixed point theorem to show that one always exists!). We will then turn our attention to some real-life examples of Nash equilibria that are not ideal scenarios, and brainstorm *game-changing* ways to turn a selfish decision into a decision that is best for all of the players.

Homework: Recommended

Prerequisites: None

Introduction to Galois theory (🌶🌶🌶, Sim, [TWΘFS])
The Fundamental Theorem of Algebra states that all roots of polynomials with rational coefficients

[4] but definitely not all!

lie in the complex numbers $\mathbb{C}$. This feels like a pretty "continuous" result, but what if I told you that group theory and field theory could prove it?

In this class, we will honor Évariste Galois' legacy by exploring his namesake field: Galois theory. We'll cover field extensions, automorphism groups, and just what makes some field extensions special enough to be Galois. We'll think about what these tell us about solutions to polynomial equations, and how it can prove the Fundamental Theorem of Algebra. Finally, we'll cover the Fundamental Theorem of Galois Theory, which beautifully summarizes the relation between fields and Galois groups.

Along the way, we'll also learn the story of Évariste Galois' life, one full of trial, tribulation, love, and death (he died at age 20 under suspicious and miserable circumstances).

Homework: Recommended

Prerequisites: Group theory, Ring theory

Metric spaces (🌶🌶🌶, Steve, TWΘFS)
A *metric space* is just a set X of "points" together with a *distance function*, d, which behaves the way distance should: the distance between any two points is zero iff they are actually the same point; the distance between x and y is the distance between y and x; and it is never more efficient to go from x to y to z than to just go from x to z. The standard examples of metric spaces are things like $\mathbb{R}$ (or the various $\mathbb{R}^n$s) with the appropriate Euclidean metric.

However, this is not remotely the end of the story! A metric space can be extremely structurally complicated, with "points" being interesting objects in their own right. For instance, the set $C_0[0,1]$ of continuous functions from $[0,1]$ to $[0,1]$ forms a metric space with the distance function $d(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\}$. We can also form metric spaces whose points are closed sets in some other metric space—there is even a metric space *of metric spaces*!

In the first half of this class, we'll develop the basic notions of metric space theory: completeness (and completions), compactness, and various other common ideas and results. In the second half we'll look at a few particularly bizarre metric spaces, such as the one alluded to two sentences prior and (time permitting) a kind of "line" that cannot be cut into two smaller "lines!"

Homework: Recommended

Prerequisites: None

Colloquia

Pure mathematics as applied physics (*Tadashi Tokieda*, Tuesday)
Humans tend to be better at physics than at mathematics. When an apple falls from a tree, there are more people who can catch it—we know physically how the apple moves—than people who can compute its trajectory from a differential equation. Applying physical ideas to discover and establish mathematical results is therefore natural, even if it has seldom been tried in the history of science. (The exceptions include Archimedes, some old Russian sources, a recent book by Mark Levi, as well as my articles.) This lecture presents a diversity of examples, and tries to make them easy for imaginative beginners and difficult for seasoned researchers.

Graph on, graph off (Narmada, Wednesday)
Way back in the old days of 2004, two Hungarian mathematicians published a paper that changed the world of graph theory forever. They asked the simple yet powerful question: what if sequences of graphs could converge? (Actually they asked more complicated questions about statistical physics and quasirandomness, but those magically transformed into this question.) I will draw several colorful pictures to convince you that the limit of a sequence of graphs is not a graph at all, but a graph*on*.

Join me as I navigate the treacherous waters of the combinatorics of graph homomorphisms to emerge, unscathed, in a world of integration and measure theory.

Killing the Cookie Monster (Arya, Thursday)
Every TAU, the Cookie Monster shows up bearing cookies and carrots. The Cookie Monster is a monstrous being with possibly several heads connected to a single body. A camper, fed up with this practice of snacking, decides to cut off one of the heads of the Cookie Monster. But behold! Two new heads pop out. Suppose the camper is adamant, and keeps chopping off heads, while the Cookie Monster keeps popping new heads. Will Nic receive his 3 carrots and a singular Chip-Ahoy, or be saddened by the demise of the multi-headed messenger? Come find out!

(Don't try this at home; Rule 0 might be broken.)

Future of Mathcamp (Staff, Friday)
Do you have opinions about what would make Mathcamp better? Then come to this event for brainstorming and discussion in groups about what we can change in the future.