CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2022

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9:10 Classes

2-adic computer science (🌶️🌶️🌶️, Eric, $\boxed{\text{TW}\Theta\text{FS}}$)
In this class you'll learn why

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is in fact an incredibly natural and sensible thing for computers to do when dealing with negative numbers (and no, that's not a typesetting error). We will explore some intersections between computer science (how computers store various types of numbers) and algebraic number theory (how number theorists store various types of numbers). We'll learn some basics about how to analyze the runtime of algorithms, and compare a few algorithms for computing x^y for very very large integers x, y. The result we're aiming towards is an absurdly efficient algorithm for performing exponentiation based around the use of p-adic numbers, which are a tool developed in algebraic number theory for combining ideas from real analysis and modular arithmetic.

This course is intended as a first introduction to a lot of computer-related concepts; if you know how floating point numbers work and you're comfortable figuring out the asymptotic runtime of some classic sorting algorithms you probably don't have much to learn from days 1 and 2 of the class. Days 3 and 4 of the class will focus on developing the p-adic numbers, and we'll tie the computer science and p-adics together on day 5.

Homework: Recommended

Prerequisites: Fluency with modular arithmetic, say at the level of the statement "x is invertible mod n iff $\gcd(x, n) = 1$." You **don't** need to know anything at all about computer science or programming.

Common continuity (🌶️, Zoe, $\boxed{\text{TW}\Theta\text{FS}}$)
Continuity is at the heart of point-set topology. A huge tool for classifying topological spaces is considering what kinds of functions on these spaces are continuous. In this class, we'll look at what makes functions continuous and why continuity is an appealing property that we like.

But! We will also look at why continuity is a *bad, unappealing* property that we don't like. We'll see why continuity is a much weaker property than we often imagine it to be, and we will carefully examine *gross* spaces and *gross* functions that nevertheless manage to be continuous.

Homework: Recommended

Prerequisites: None

Measuring fairness (🌶🌶, *Moon Duchin*, $\boxed{\text{TW}\Theta\text{FS}}$)
This class is about one very specific kind of measuring fairness. Namely, what does it mean for an electoral district to be unfairly tilted to some political party, and how would you certify, by contrast, that a districting plan is partisan-fair? We will go from zero (no assumptions that you know anything about U.S. politics, in particular!) to literal expert level. The math involved is elementary but, trust me, pretty cool.

Homework: Recommended

Prerequisites: None

Representation theory of finite groups (week 1) (🌶🌶🌶🌶, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)
It turns out that you can learn a lot about a group by studying homomorphisms from it to groups of linear transformations (if you prefer, groups of matrices). Such a homomorphism is called a representation of the group; representations of groups have been used widely in areas ranging from quantum chemistry and particle physics to the famous classification of all finite simple groups. For example, Burnside, who was one of the pioneers in this area along with Frobenius and Schur, used representation theory to show that the order of any finite simple group that is not cyclic must have at least three distinct prime factors. (The smallest example of such a group, the alternating group A_5 of order $60 = 2^2 \cdot 3 \cdot 5$, is important in understanding the unsolvability of quintic equations by radicals.) We may not get that far, but you'll definitely see some unexpected, beautiful, and important facts about finite groups in this class, along with proofs of most or all of them. With any luck, the first week of the class will get you to the point of understanding character tables, which are relatively small, square tables of numbers that encode all the information about the representations of particular finite groups; these results are quite elegant and very worthwhile, even if you go no further. In the second week, the chili level may ramp up a bit (from about $\pi + 0.4$ to a true 4) as we start introducing techniques from elsewhere in algebra (such as algebraic integers, tensor products, and possibly modules) to get more sophisticated information.

Homework: Recommended

Prerequisites: Linear algebra, group theory, and general comfort with abstraction. (The "eigenstuff" material from the week 2 class will come up after a while, but I can catch you up on that outside class time as necessary.)

Schubert calculus (🌶🌶🌶, Kayla, $\boxed{\text{TW}\Theta\text{FS}}$)
Have you heard of Hilbert's iconic list of 23 problems? Have you heard of Hilbert's 15th problem? It is only partially resolved and in this class, we are going to see some of the progress that has been made. Hilbert's 15th problem has to do with enumerating intersections of subspaces in a fixed ambient space and asks to formalize Schubert's "calculus" for counting these intersections... but this calculus isn't one you've seen on an AP exam and Schubert's proof using "Principle of Conservation of Number" was quite the unfinished symphony in the math community's eyes.

Come see how the world of Schubert calculus has become a beautiful intersection of geometry, topology, algebra and combinatorics. You Schur won't be disappointed by taking *another* calculus class.

Homework: Optional

Prerequisites: Linear algebra (more specifically, row reduction of matrices and elementary row operations)

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Diophantine approximation (🌶🌶🌶, Travis, TW$\boxed{\Theta\text{FS}}$)
When judging for a Rule 2 violation, you have to see just how irrational an idea really is. And the same goes for real numbers: How well can you approximate irrational numbers by rational ones which will slip under the wary staff's radar? We'll start by proving that for every irrational number α, there are infinitely many rational numbers p/q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

We'll see why this is interesting, surprising, and useful. We'll solve a puzzle. And then we'll follow in Liouville's footsteps and generalize this result to transcend this mortal coil.

Homework: Recommended

Prerequisites: None

Integer right triangles (🌶🌶, *David Roe*, T$\boxed{\text{W}}\Theta$FS)
You probably recognize the triples (3,4,5) and (5,12,13), the first few of the infinitely many Pythagorean triples that measure triangles with integer side lengths. We will start by exploring a geometric method that generates Pythagorean triples, generating a beautiful parameterization of all possibilities. We will then focus on the area of these triangles, while allowing the edges to have rational lengths. The examples above show that 6 and 30 are possible, but what about 7 or 15? We will only scratch the surface of this question, but will dig deep enough to get a glimpse of the profound conjectures underneath.

Homework: Recommended

Prerequisites: None

Nonstandard analysis (🌶🌶🌶, Aaron, T$\boxed{\text{W}\Theta\text{FS}}$)
The early history of calculus is filled with sketchy computations about infinitesimal quantities, which George Berkeley criticized as follows:

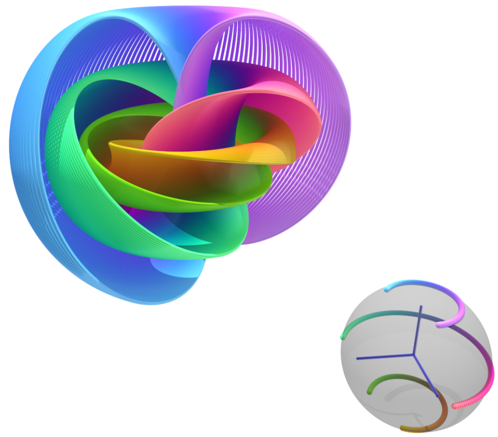
"They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?"

The rigor of calculus was later rescued with limits, but there is another approach. In this class, we will construct the hyperreals, a system of numbers that includes real numbers, infinitesimally small numbers, and infinitely large numbers. We will then see how to rigorously do calculus the infinitesimal way.

Homework: Recommended

Prerequisites: Calculus (preferably with epsilon-delta proofs)

On beyond on beyond i (🌶🌶🌶, Assaf, T$\boxed{\text{W}\Theta\text{FS}}$)
You may have heard of the quaternions (perhaps in an earlier class by a similar name, or at some colloquium about e^x), but did you know that they also encode secret and hidden geometries of 3- and 4-dimensions? Just by allowing multiple square roots of -1, we get a number system on $\mathbb{R}^4$ which allows us to rotate spheres in a snap, draw knotted vector fields in the 3-dimensional sphere S^3, and create insane images such as:



This class is about the **geometry** of the quaternions, not the algebra. As such, **"On beyond i" is not a required prerequisite.**

Homework: Recommended

Prerequisites: Linear algebra, knowing what a dot product is would be helpful

Special relativity (🌶🌶, Nic, TWΘFS)
Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but if you took the equations literally they implied some bizarre things about the structure of space and time: depending on their relative velocities, different observers could disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

For a long time, a lot of creative excuses were invented for why we *shouldn't* take the equations literally (including one with the incredibly Victorian name "luminiferous aether") but, in what was probably the second most unsettling event in early twentieth-century physics, all of them failed. The physics community was left with only one viable conclusion: space and time really do behave that way!

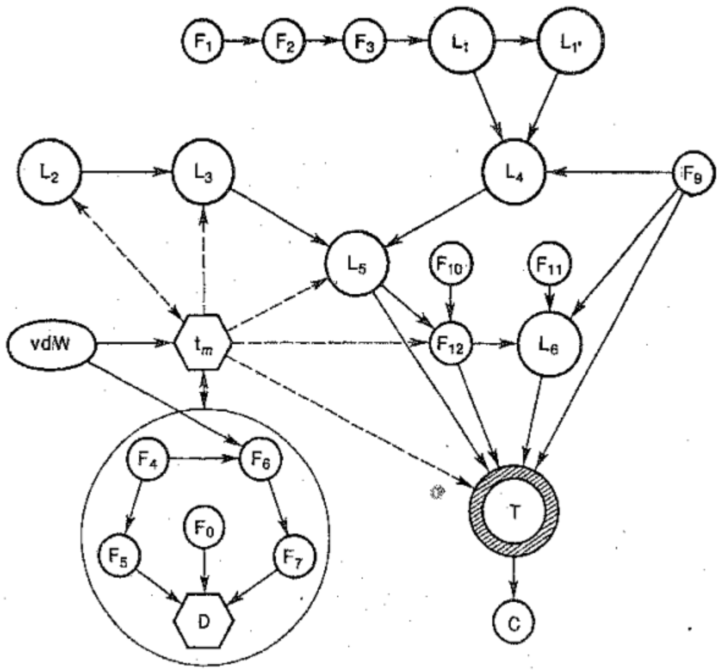
In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics had to be rebuilt to accommodate them. We will see how, as the physicist Hermann Minkowski said, "Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." At the end, we'll also briefly look at how to revise the classical definitions of momentum and energy and see why we should believe that $E = mc^2$.

Homework: Optional

Prerequisites: Enough physics to know what momentum and kinetic energy are, but no more than that. No prior exposure to special relativity expected!

Szemerédi's {theorem, regularity lemma} (🌶🌶🌶🌶, Yuval, TWΘFS)
In 1975, Szemerédi proved that if $\varepsilon > 0$ is fixed and N is sufficiently large, then any set $A \subseteq \{1, 2, \ldots, N\}$ of size $|A| \geq \varepsilon N$ contains an arbitrarily long arithmetic progression. Although this is a number-theoretic statement, Szemerédi's proof was entirely combinatorial. At the time, this was perhaps the most complicated and intricate combinatorial proof ever devised; the following diagram from Szemerédi's original paper shows merely the *logical structure* of the argument, and not even any of the ideas.

202 E. Szemerédi



The diagram represents an approximate flow chart for the accompanying proof of Szemerédi's theorem. The various symbols have the following meanings: F_k = Fact k, L_k = Lemma k, T = Theorem, C = Corollary, D = Definitions of B, S, P, α, β, etc., t_m = Definition of t_m, vdW = van der Waerden's theorem, F_0 = "If $f: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is subadditive then $\lim\limits_{n\to\infty} \dfrac{f(n)}{n}$ exists".

In proving this theorem, Szemerédi introduced a tool (now called Szemerédi's regularity lemma, found in the circle L_1 above), which is now itself one of the most important tools in all of graph theory. Roughly speaking, it says that all "big" graphs look "the same", and that we can actually more or less forget about where the edges of our graphs are. This may sound stupid, but its power cannot be overstated.

In this class, we'll approach Szemerédi's theorem from the perspective of Szemerédi's regularity lemma. We'll focus primarily on the graph theory, but we'll see how graph-theoretic insights can yield number-theoretic results, including seeing a complete proof of Roth's theorem (the case of three-term arithmetic progressions). Along the way, we'll see other applications of the regularity lemma; for example, the Erdős–Stone–Simonovits theorem, the main result of my Extremal graph theory class, will be a simple homework exercise once we understand the regularity lemma.

Homework: Recommended

Prerequisites: Graph theory

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Arrow's impossibility theorem (🌶🌶, Ben, TWΘFS)
If you've heard of Arrow's Impossibility Theorem before, it might have in some form like "a good voting system doesn't exist," which leaves a bit to be desired, as a theorem statement. What do we mean by "good," or by "a … voting system," or by "a good voting system," for that matter?

Our first mission in this class is to clear up what, exactly, Arrow's Impossibility Theorem says. Our third mission is to prove it. If you're wondering about the second mission—it's to define and briefly discuss ultrafilters, which turn out to be useful for that "prove it" mission we just mentioned.

Time permitting, we might also get to talk about the Gibbard–Satterthwaite Theorem, which says that there's always[1] some voter who shouldn't vote for who they want to win.

Homework: Recommended

Prerequisites: None

Buffon's needle (🌶, Ben, TWΘ$\boxed{\text{FS}}$)

Suppose that I draw a bunch of lines parallel to each other, spaced one inch apart, and drop my standard-issue one-inch needle onto them randomly. How likely is it that the needle will cross one of the lines? More generally, what if the needle is longer, shorter, or is actually a squiggly piece of uncooked pasta?

There are, broadly speaking, two approaches to this. One of them involves setting up some number of integrals to figure out these probabilities. This approach is entirely valid and will work, but there's another approach that just relies on probability theory.

So, in this class we will compute no integrals, and use no calculus[2]. Instead, we will see a marvelous display of the glorious power called "linearity of expectation," and that's all we'll need.

Homework: Recommended

Prerequisites: None

Commutative algebra and algebraic geometry (🌶🌶🌶, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)

In its classical form, algebraic geometry is the study of sets in n-dimensional space that can be described by polynomial equations (in n variables). This is both a very old and a quite active branch of mathematics, and for over a century now it has relied heavily on commutative algebra—that is, on the properties of commutative rings and related objects. We'll start by looking at some of those, including prime and maximal ideals and a review of quotient rings, and we'll see how the algebra can be used to give us information about the geometric sets. For instance, we'll use the algebra to show that if a set can be given by polynomial equations, then a finite number of such equations will do. We may also see how to translate the idea of dimension into the language of algebra. There may well be cameo appearances by the axiom of choice (in the guise of Zorn's lemma) and a bit of point-set topology (on a space whose points are ideals!), but you don't need to know any of those things going in. It's quite possible that the TBD class listed in the week 4 schedule will be a continuation of this class—that depends on how much interest there seems to be. If that happens, I hope, among other things, to prove Hilbert's famous Nullstellensatz ("Theorem of the Zeros"), arguably the starting point for modern algebraic geometry, at least for the case of two variables. (The theorem will presumably be stated and used in the first week.)

Homework: Recommended

Prerequisites: Familiarity with polynomial rings, ideals, and quotient rings.

Curves that classify geometry problems (🌶🌶, *J-Lo*, $\boxed{\text{TW}\Theta\text{FS}}$)

In this class we will turn geometry problems into curves. For example, consider the following three problems:

(0) How many triangles have integer side lengths and a 60° angle?

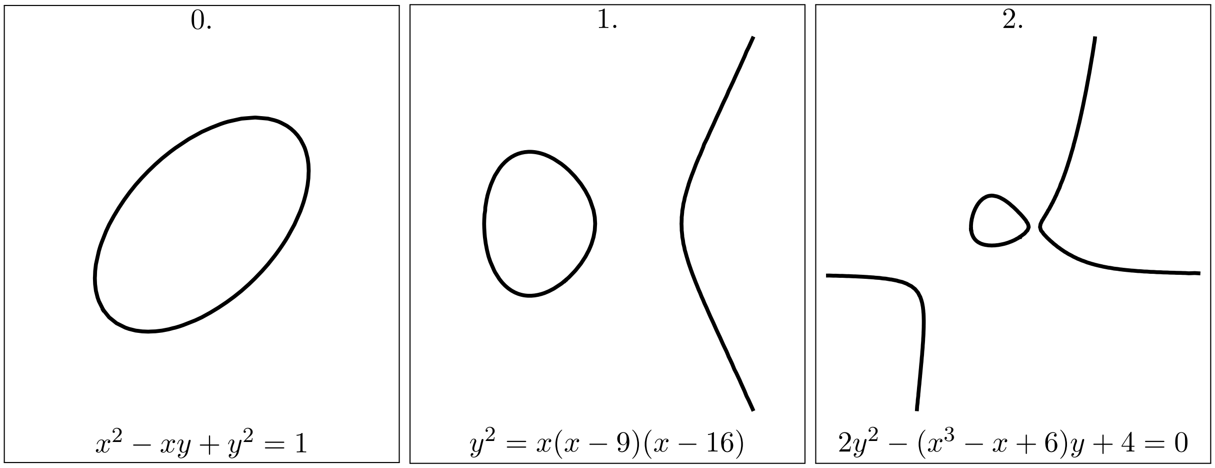
[1] OK, not literally always, just usually

[2] . . . OK we'll want to think a little bit about limits but I promise that's all

(1) How many triangles have integer side lengths and integer area, with two of the sides in a ratio of 3 to 4?

(2) How many pairs of triangles with integer side lengths, one right and one isosceles, have equal area and equal perimeter?

(Do not count scaled solutions separately; for example, in problem 0, you should only count one equilateral triangle.) Each problem can be solved by finding rational points (points (x, y) with x and y rational numbers) on a certain curve:

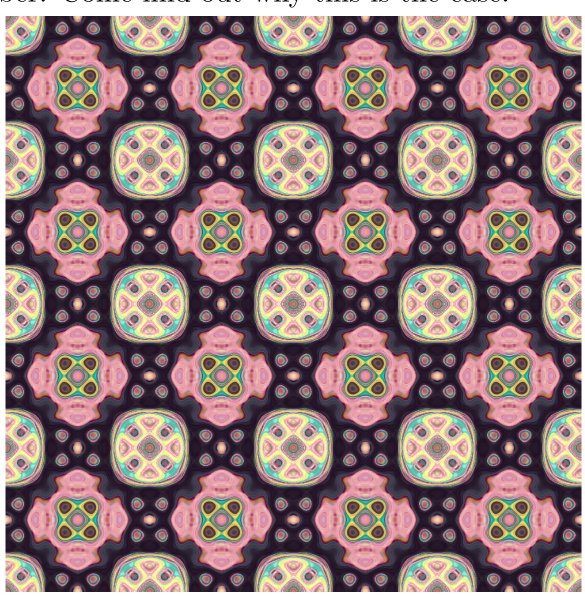


Come to this class to learn how to convert geometry problems into curves, to explore some tools we can sometimes use (such as stereographic projection and elliptic curve addition) to find the points on these curves that we're looking for, and to discuss some unsolved problems about right triangles.

Homework: Recommended

Prerequisites: None

The 17 wallpaper patterns (, Emily, TWΘFS)
No, I'm not talking about the wallpaper at your grandma's house—I'm talking about mathematical wallpaper! These are two dimensional repetitive patterns which are distinguished based upon their symmetries, such as the example below. To each of these we can assign a group which consists of the transformations of the plane which preserve the pattern. It turns out there are exactly 17 of these—what a strange number! Come find out why this is the case.



Homework: Recommended

Prerequisites: Group theory, linear algebra

Ultrafilters and combinatorics (🌶🌶🌶🌶, Steve, TWΘFS)
Combinatorics is full of results saying that functions on an infinite set are well-behaved "a lot" of the time. An easy example of this is the Pigeonhole Principle: given a function $f : \mathbb{N} \to X$ with X finite, no matter how crazy f is there is always some infinite set $S \subseteq \mathbb{N}$ on which f is constant. A slightly trickier instance of this is Infinite Ramsey's Theorem (for pairs): any 2-coloring of pairs of natural numbers has an infinite homogeneous subset. (If you haven't seen this before, don't worry, we'll prove it in class.)

However, what if "infinite" just isn't big enough? For example, for a function $f : \mathbb{N} \to X$ with X finite, maybe we want f to be constant on a set which is not only infinite but *closed under (finite) sums.* Can we always find such a set? If so, what's the most ridiculous way we can prove it?

In this class we'll do combinatorics using ultrafilters—bizarre, beautiful objects from the mysterious land of set theory! Ultrafilters cannot even be proved to exist without the axiom of choice, but that won't stop us from using them to build big homogeneous sets. Oh, and we'll also need to say the words "compact space," "topological semigroup," and "idempotent" a few times.

Homework: Required

Prerequisites: None

Zero knowledge proofs (🌶🌶🌶, *Dan Zaharopol*, TWΘFS)
Picture this: You want to convince someone that you know something is true, but you don't want that person to actually be able to reconstruct the proof themselves (or to have *any* advantage in doing so). For example, maybe you want to prove that a graph has a Hamiltonian cycle, but you want to give absolutely no information to the other person that would allow them to find the cycle themselves—you just want to convince them that it exists!

You might think, "Surely, that isn't possible!" It sounds outlandish that you could prove to someone that you know a cycle without showing it to them. And yet you can; doing so is called a zero-knowledge proof, and besides being really cool it also has applications all across different areas of cryptography.

In this class, we'll see two things: how to accomplish certain zero-knowledge proofs, and how to give a rigorous definition of them. In particular, it's not just interesting that we can do it, but also that we can write down precisely what it means to prove something without sharing any knowledge. This class will be a chance to explore how that works and to get more insight into how both computer science and cryptography are formalized mathematically.

Homework: Recommended

Prerequisites: None, but some knowledge of graph theory or big-O notation will be helpful. If you've seen formalized language around computer science, the class will be easier.

1:10 Classes

Hyperbolic geometry (🌶🌶, Arya, TWΘFS)
In normie Euclidean geometry, the sum of angles of a triangle is always equal to 180 degrees, areas are computed by actually multiplying two lengths, and inverting across circles does spooky things. Imagine drawing a line through a point parallel to some given line, and NOT being able to draw a second one? Dealing differently with triangles that are clearly similar, just because SoMeOnE mAdE a triangle bIg? Drop the blindfolds of Cartesian coordinates and join this class to free your imagination and learn some hyperbolic geometry! We shall talk about different models of hyperbolic

spaces, isometries, hyperbolic trigonometry, analogues of theorems from Euclidean geometry, and why "hyperbolic metrics are the natural geometric structures on almost all surfaces."

Homework: Recommended

Prerequisites: Knowing what complex numbers are, some familiarity with sine rule and cosine rule.

In-fun-ite groups (🌶🌶🌶, Narmada, TWΘFS)
If you've ever thought "There's no way groups can be this nice," then this is the perfect class for you. Halloween has come early this year and we're going to be looking at some truly monstrous groups. It all started when we let them be infinite…

We'll start by seeing that even the free group on two generators can't be trusted. Then, we'll look at how the Axiom of Choice plays into the structure of free groups and free abelian groups. For the grand finale, we'll study a (still unsolved!) problem that has been called the equivalent of Fermat's last theorem for group theory.

Homework: Optional

Prerequisites: Group theory: talk to me if you have questions!

Machine learning (NOT neural networks) (🌶🌶🌶, Linus, TWΘFS)
Netflix wants to recommend me TV shows that I will like. To do this, they analyze a giant matrix of people and their ratings of TV shows and movies. When a user rates a movie, Netflix learns one entry of this matrix; their goal is to find patterns in their dataset and use them to predict the unknown entries.

In 2006, Netflix offered $1000000 to the first team that could beat their internal prediction algorithm by 10%. This problem embodies the second era of machine learning, *linear algebra on big data*. In this class, we'll show off (a simplified version of) a winning algorithm.

We'll also explore the first era of machine learning, *classical PAC-learning algorithms*, full of sharp combinatorial algorithms with strong provable guarantees.

We won't touch on the third era, the *neural network jamboree*, unless I go crazy on Day 5. (Why not? Because they can't prove anything…)

Homework: Recommended

Prerequisites: Linear algebra. Also it'll help to have seen linearity of expectation.

Problem solving: graph theory (🌶🌶🌶, Misha, TWΘFS)
In this class, we will solve problems. Some of these will come from math competitions, some of these I made up myself, and some of these I found "in the wild."

Some of our problems will be questions about graphs, and some of them are questions that we can model—and solve—with graph theory. There are few ideas from graph theory that will be especially important:

- Using the handshake lemma and Euler's formula.
- Matchings in bipartite graphs and Hall's theorem.
- Trees, their properties, and what they tell us about connected graphs.

We will work through problems together every day in class and see some of the key ideas involved in solving them. If you want to get the most out of this class, you should work on the remaining problems on your own during TAU!

Homework: Recommended

Prerequisites: Basic familiarity with graphs. Ideally, nothing I mentioned in the blurb should scare you; if it does, but you still want to take the class, talk to me!

The continuum hypothesis (week 2) (🌶🌶🌶🌶, Susan, $\boxed{\text{TW}\Theta\text{FS}}$)
This week we proved that Martin's Axiom implies the existence of a dominating function for uncountable-but-not-continuum-sized sets of functions from $\mathbb{N}$ to $\mathbb{N}$. We also proved the Löwenheim–Skolem theorem, showing that we can create a little tiny countable set theoretic universe, and saw the Mostowsky collapse construction for making sure that countable universe is transitive. We're now deep in the weeds, trying to figure out how to formally adjoin filters to set theoretic universes. Want to see the rest? Tune in to the exciting conclusion of the continuum hypothesis!

Homework: Recommended

Prerequisites: The continuum hypothesis, week 1

Colloquia

Everyone hates analysis (Charlotte, Tuesday)
It is a truth universally acknowledged that everyone hates analysis. Nowhere is this more true than at Mathcamp. When Mathcampers see a class tagged "analysis," they quickly avert their eyes, in fear that they will be turned to stone lest they read the blurb in full. In this colloquium, I will make you absolutely miserable for approximately fifty minutes. In fact, you will almost surely leave this colloquium hating analysis even more than you do right now.

Hyperspheres (*David Roe*, Wednesday)
Draw a square, divide it into four equal squares, and then inscribe a circle within each. Within those four circles, you can fit another smaller circle. We can draw this setup on a blackboard, and Euclidean geometry gives us the tools to compare the sizes of all the objects involved. But what happens when we increase the dimension? This thought experiment has an interesting answer, and will lead us into the broader world of sphere packing, an area of mathematics with connections to error correcting codes, chemistry, number theory, hyperbolic geometry and string theory.

High-dimensional oranges (Travis, Thursday)
You know oranges; you might even know *and* love them. Maybe they're your favorite fruit that can occasionally be found in the dining hall. As a rule, they're not very interesting: They just sit there and maybe roll around a bit, awaiting their eventual end. But put on a pair of Hi-Tek Xtra-Dimensional Power Goggles™ (patent pending), and you'll find that oranges contain multitudes. We'll plunge the depths of this hidden knowledge to learn about the geometry of spheres with many, many dimensions and how the measurement of volume is a much more tricksy concept than you may have been led to believe.

Heisenberg geometry (*Moon Duchin*, Friday)
I will tell you all about a world where walking around in a circle causes you to involuntarily levitate. This is nilpotent geometry, with the innocuous-looking 3-dimensional Heisenberg group as the gateway drug. Geometry meets groups meets analysis meets robots……