CLASS DESCRIPTIONS—WEEK 4, MATHCAMP 2022

Contents

9:10 Classes

A curious connection between p-adic distances and triangulations of a square (, Charlotte, $\boxed{\text{TW}\Theta\text{FS}}$)
If you're given a square, you could find a way to divide it into an even number of triangles of equal area. Now try dividing it into an odd number of triangles of equal area!

Well, you probably didn't, because you can't, a fact which is known as Monsky's theorem. What's lovely about the proof of Monsky's theorem is that it is entirely unexpected: its main tool is algebraic, the 2-adic valuation (which is closely related to the 2-adic numbers, and gives a different way of measuring "distance" between points). We'll use 2-adic valuations to coulour the plane, and see some slick combinatorial arguments.

Homework: Recommended

Prerequisites: Familiarity with metrics, and the definitions and basic properties of groups, rings, subrings, invertible elements of rings, and quotients of groups

Ancient Greek mathematics (, Yuval, $\boxed{\text{TW}\Theta\text{FS}}$)
You may have heard some crazy stories about ancient Greek mathematicians:

- Pythagoras proved the Pythagorean theorem, but also hated beans. Also, he killed someone for figuring out that $\sqrt{2}$ is irrational.
- Euclid once sassed King Ptolemy I by telling him "there is no royal road to geometry".
- Archimedes shouted "Eureka!" in the bathtub, ran down the streets of Syracuse naked, and later invented a giant mirror to burn attacking Roman ships.
- The Greeks tried really hard to square the circle, but never could. In 1882, von Lindemann proved that squaring the circle is impossible.

Sadly, probably none of these stories is true.

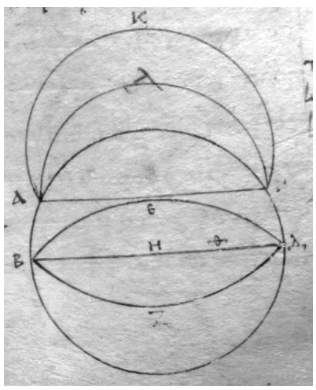
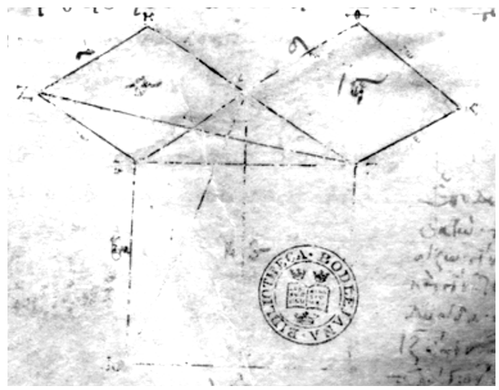
Wait, what? *None* of them is true? What about Pythagoras proving the Pythagorean theorem? Turns out that probably didn't happen.

But the truths about ancient Greek mathematics are, perhaps, even crazier than the myths.

- Rather than being sad about irrational numbers, the Greeks loved and were obsessed with them.
- Eudoxus essentially invented Dedekind cuts, and thus the formal theory of the real numbers, 2250 years before Dedekind.

- Euclid *sort of* proved the infinitude of the primes, but his proof only really implies that there are at least four primes.
- The Greeks were actually *really good* at squaring the circle! They came up with three different methods, von Lindemann's "proof of impossibility" notwithstanding.
- Archimedes essentially invented integrals, and used them to compute areas and volumes of crazy shapes (parabolas, spirals, spheres) 1900 years before the "real" invention of calculus. He also invented systems for expressing huge numbers, initiated the field of mathematical physics, and proved perhaps the most difficult and complicated theorem proved until the 19th century.

This class is all about ancient Greek mathematics. We'll learn both about what mathematics they did—including some shockingly difficult, complicated, and beautiful proofs—and about how they *thought* about mathematics. In many cases, they thought about mathematics in more or less the same way we do (and, indeed, our mathematics continues a tradition directly inherited from the Greeks), but in other cases, it can feel like speaking to an alien. For example, would you have guessed that these pictures depict, respectively, two circles and a proof of the Pythagorean theorem?

Homework: Recommended

Prerequisites: None

Baire necessities for Banach–Tarski (🌶️🌶️🌶️🌶️, Narmada, $\boxed{\text{TW}\Theta\text{FS}}$)

If you give a mathematician a proof of the Banach–Tarski paradox, she will tear it apart into finitely many pieces and reassemble it into *two* proofs of the paradox. We'll look at the first proof of the Banach–Tarski paradox (which was mostly due to Hausdorff) that manipulates the free group. Then we'll look at the *poggers* proof that uses graph theory to prove an even stronger version of the paradox: you can force the pieces in your decomposition to be topologically nice. What's with the Baire in the title? Come to class to find out!

Homework: Optional

Prerequisites: Know what graphs and groups are

Problem solving: cheating in geometry (🌶️🌶️🌶️ → 🌶️🌶️🌶️🌶️, Zack, $\boxed{\text{TW}\Theta\text{FS}}$)

Geometry is hard. Sometimes you can bash geometry problems with algebra, but algebra is hard too. Everything would really be a lot nicer if geometry were easy, like if every pair of lines intersected or if every circle passed through the same two points. Helpfully, projective geometry (motto: "what if geometry were better") exists! In projective geometry, everything is great, lines and curves behave how they should, and geometry is easy.[1] We'll build some intuition for projective space through examples, and discover some geometric and algebraic tools which will sometimes allow us to solve hard geometry problems quickly and easily, in particular the somewhat infamous "method of moving points." Side

[1] OK, maybe not easy, but at least there aren't any angles.

effects may include, but are not limited to: an inability to return to thinking about angles and lengths, a tendency to write solutions that will make your graders sad, and following every sentence with "you know, this would really be a lot nicer over $\mathbb{CP}^2 \ldots$"

Homework: Required

Prerequisites: None—in particular, no experience with olympiad geometry will be assumed.

The distribution of prime numbers (, Viv, $\boxed{\text{TW}\Theta\text{FS}}$)
What is the distribution of prime numbers?

This question is really vague, and encompasses a lot of other questions. Questions like:

- How many prime numbers are there? For a fixed $x > 0$, how many primes are there less than x? How precisely can we count them?
- How many twin primes are there (that is, primes p where $p + 2$ is also prime)? Are there infinitely many?
- What is the biggest gap between two primes that are less than x? How frequently is the gap between two consecutive primes small? How frequently is it big?
- What is the distribution of final digits of primes? What about final digits of pairs of consecutive primes?

Many of these questions are... hard. Like, really *really* hard. For example, Wikipedia says that the Twin Primes Conjecture, which states that there are infinitely many primes, "has been one of the great open questions in number theory for many years." Instead of trying to answer these questions, we'll do our best to understand what the answers *should* be, and why. Along the way, we'll develop and evaluate a random model for prime numbers, and discuss my favorite conjectures (for some definition of favorite).

Homework: Recommended

Prerequisites: Some basic number theory is helpful; specifically, being comfortable with modular arithmetic is helpful, as well as the Chinese Remainder Theorem.

10:10 Classes

Algebraic topology: homology (, Zoe, T$\boxed{\text{W}\Theta\text{FS}}$)
Whenever faced with real wonky situations in mathematics, our usual end goal is to try to get a comparison to a situation we actually know things about. Homology takes whatever weird space one could think of, and gives us a way to measure how close that space is to any n-dimensional hole. In this class, you will learn efficient ways to compute homology as well what homology can give us when analyzing a problem.

Homework: Recommended

Prerequisites: Group theory and Linear algebra (not strict prereqs, feel free to talk to me about what exactly is needed).

Commutative algebra and algebraic geometry (week 2) (→ , Mark, T$\boxed{\text{W}\Theta\text{FS}}$)
This class, which was originally announced as "TBD", will be a continuation of the week 3 class. If you didn't take the class last week and you would like to join now, it's probably a good idea to consult with Mark first.

Homework: Recommended

Prerequisites: Commutative algebra and algebraic geometry (week 1)

High-dimensional potatoes (🌶🌶🌶, Travis, TWΘFS)
Have you ever looked at a potato? Like, *really* looked at it? Did you then think that they would be cooler if they had a few more dimensions, like maybe 573 of them? Perfect. We'll take a deep dive into high-dimensional potatoes, answering such questions as: When can potatoes intersect? How hard is it to specify a point inside a potato? Is it always possible to split them in half? The mysteries abound!

If you want to see what happens in high dimensions without needing any integrals, switch your diet from oranges to potatoes!

Homework: Recommended

Prerequisites: Linear algebra (familiarity with the real vector space $\mathbb{R}^d$ and linear independence)

The abc's of polynomialand (🌶, Eric, TWΘFS)
Constants. Irreducibles. Squares. Monics. Long ago, the elements of Polynomialand lived together in harmony. Then, everything changed when Queen Polynomia went missing. Only the Wronskian, which could mediate to a degree between the feuding factions of abecedarians and radicals, could keep Polynomialand stable in her absence, but when the polynomials needed it most, they forgot about how it worked. A hundred years passed, and Mathcampers re-discovered the Wronskian. And although their understanding of integers is great, they have a lot to learn before they're ready to save any polynomials. But I believe that Mathcampers can save Polynomialand!

(This is still a math class! It's about the *abc* conjecture from a 🌶 perspective: what it is, why it's hard, and mostly why the polynomial version is more straightfoward. This class will just also be …silly in the way described above.)

Homework: Recommended

Prerequisites: You should be comfortable with unique factorization and the Euclidean algorithm for integers; Mark's intro number theory class is more than enough.

The satisfiability problem (🌶🌶🌶, Misha, TWΘFS)
Questions like

- "Does this Sudoku have a solution?"
- "Is there a red-blue coloring of $\{1, 2, \ldots, 9\}$ with no monochromatic 3-term arithmetic progression?"
- "Does this 2048-bit integer factor into two 1024-bit integers?"

have one thing in common. Each one can be expressed as a formula whose variables are not numbers but *Boolean* values: true or false. The Boolean satisfiability problem is to choose the values of these variables to satisfy the formula: make it true.

This problem is notoriously difficult—it is the first problem proven to be NP-complete. (This means that if we find a polynomial-time algorithm to solve it, we get a million dollars.) Most computer scientists are happy to say that there is no known algorithm significantly better than the $O(2^n)$ algorithm that tries all possible values of n variables.

But "significantly better" can have multiple meanings. An $O(n\cdot(\sqrt{3})^n)$ algorithm is still exponential, but it can sometimes mean the difference between solving a problem in minutes or in hours. And (spoiler alert!) we'll be able to do better than $O(n \cdot (\sqrt{3})^n)$ by the end of this class.

Homework: Recommended

Prerequisites: None

11:10 Classes

Cantor before set theory (🌶🌶🌶🌶, Ben, TWΘFS)
If you've ever looked into the history of set theory, you might have read that it came about because

of Georg Cantor's investigations into the infinite, motivated by his work in real analysis. One might wonder—what question was Cantor trying to answer, that made him start thinking about the nature of infinite sets?

In the 1800s, trigonometric series became a major area of study due to the work of Joseph Fourier. A lot of this work centered on what kinds of functions could be written in the form

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

One of Cantor's colleagues[2] asked a different question—if you already know that a function can be represented by a trigonometric series, could there be more than one? For example, we can represent the function $f(x) = 0$ as the trivial trigonometric series where all of the a_n and b_n are taken to be 0. Is there another way?

In this course, we'll not only learn the answer to this question, but also see how investigating it sent Cantor along the road to set theory—into investigations of the infinite.

Homework: Recommended

Prerequisites: Know the difference between uniform and pointwise convergence, know how to take integrals and derivatives.

Finite fields (🌶🌶, Aaron, TWΘFS)
Fields are everywhere in math, but usually we encounter infinite fields such as $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$.

In this class, we'll explore the *finite* fields! These are useful all over math, both in their own right and as a miniature test case for the infinite fields you already know. We'll construct them, see what sizes they can have, characterize their additive and multiplicative behavior, and see how they fit together.

Homework: Recommended

Prerequisites: Ring theory, linear algebra, with optional group theory.

Knot theory (🌶🌶, Emily and Kayla, TWΘFS)
Contrary to popular belief, knot theory is not "not theory." Specifically, it teaches you how you know when two knots are not the same, and when they are not not the same. Certain knavish knots defy classification, however, so knowledgeable methods and theoretical theories must be used to distinguish such gnarly knots. Know naught about knot theory yet? Worry not, for this class is an introduction!

(This blurb was written by Nathan S.)

Homework: Recommended

Prerequisites: None

Mathematical billiards (🌶, Arya, TWΘFS)
Suppose you have a point-sized ball gliding on a billiard table with a frictionless surface. The trajectory ends if it goes into a hole, and if it hits the boundary of the table, the ball follows the standard laws of reflection (the angle of incidence is the same as the angle of reflection). Depending on the shape of the table, we can ask several questions—how many times can the ball hit a wall before it goes into a hole? Can it come back to where it started, and keep looping its path in a periodic motion? Is the trajectory of the ball dense inside your shape? In this class, we shall try to answer some of these questions and discuss some related open questions.

Homework: Recommended

Prerequisites: None

[2] Eduard Heine; if you've heard of e.g. the Heine–Borel Theorem, it's that guy.

Representation theory of finite groups (week 2) (, Mark, TWΘFS)
This is a continuation of last week's class of the same name. If you didn't take the class last week and you would like to join now, it's probably a good idea to consult with Mark first.

Homework: Recommended

Prerequisites: Representation theory of finite groups (week 1), Group theory, Linear algebra

1:10 Classes

Algebraic solutions to Painlevé VI (, *Aaron Landesman*, TWΘFS)
In 1902, Painlevé introduced six differential equations, the most difficult of which was the so-called "Painlevé VI." The algebraic solutions to Painlevé VI were only classified recently in 2014. It turns out these algebraic solutions correspond to finding certain canonical triples of 2 by 2 matrices. In the class, we will search for collections of these canonical tuples of matrices. Our search will lead us to discover a sequence of beautiful connections between group theory, geometry, topology, representation theory, and algebraic geometry.

Homework: Required

Prerequisites: Linear algebra, Group theory

Chaotic dynamics and elephant drawing (, Ben, TWΘFS)
In the study of dynamical systems, we have some rule for extrapolating what "things tomorrow" look like, given what "things today" look like. A practical example of this is the weather; we can consider this as a dynamical system. But while the weather tomorrow is fairly predictable, and modern weather forecasting can even extrapolate a week out pretty well, long-term weather forecasting is right out—is it going to snow in Toronto on 16 December 2022? We won't know for a while.

This motivates the definition of chaotic dynamical systems, in which small changes to present conditions may cause large changes in the future (the so-called "butterfly effect"). We will aim to show that some easily-described discrete dynamical systems are chaotic.

Time permitting, we'll also use our chaotic dynamical systems for a practical[3] purpose: overfitting data! We'll see how we can carefully pick a two parameter "model" that can fit any data set almost perfectly. Our model will be based on a specific dynamical system, and its marvelous overfitting powers? Are based on the fact that it is chaotic.

Homework: Recommended

Prerequisites: None

Conway's soldiers (, Misha, TWΘFS)
Let's play checkers! Except the pieces jump horizontally and vertically instead of diagonally. Also, the checkerboard is infinitely large and the opponent is MIA. How far forward can our set of soldiers step?

You might think that with infinite pieces, through a clever series of jumps, we should be able to travel infinitely far forward, but in fact the best we can do is exactly 0% of that. Proving this will reunite us with an old irrational friend and take us through a world of monovariants and power series that will make you say *no (Con)way!*

Then, we will go one step further than that—literally. We'll find out what can happen when are given the power to do infinitely many things in a finite length of time.

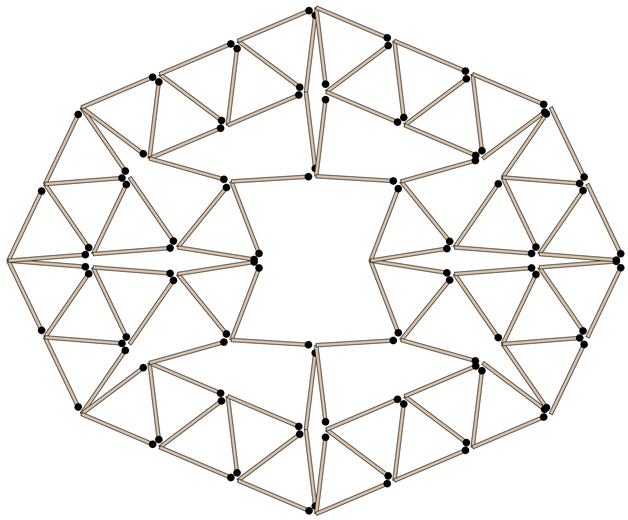
(This blurb was co-written with Lucas.)

Homework: None

[3] OK, maybe not practical. But fun!

Prerequisites: None

Electric charge on matchsticks (🌶🌶, Misha, TWΘF$\boxed{\text{S}}$)
The graph below is called the Harborth graph:



Its edges are matchsticks all of the same length. They never cross, and four of them meet at every vertex. (We don't know if it's the smallest graph of this type.)

We do know that we can't make a graph like this with five matchsticks meeting at every vertex. We will prove this by putting electric charges on such a graph, moving them around, and showing that the graph's existence would violate conservation of energy.

Then we'll see what else can be done with the so-called "discharging method"!

Homework: None

Prerequisites: You should know what vertices, edges, and faces of a graph are, and know or be willing to accept on faith that $|V| - |E| + |F| = 2$.

Game theory, traffic, and the price of anarchy (🌶, Assaf, $\boxed{\text{TW}\Theta}$FS)
Some schools[4] of libertarian and capitalistic thought say that if everyone does what is best for themselves, this will be best for society, since each person maximizes their own happiness in the context of their surroundings. This perspective is the Nash-equilibrium solution to humanity, and as mathematicians, we can always ask: "can we do better?" The answer is: "sometimes" and the difference between the *best* course of action and the *anarchist* course of action is called *the Price of Anarchy.*

This class is an intro to game theory class, where we will talk about combinatorial games, pure and mixed strategies, and Nash equilibria (and use Brouwer's fixed point theorem to show that one always exists!). We will then turn our attention to some real-life examples of Nash equilibria that are not ideal scenarios, and brainstorm *game-changing* ways to turn a selfish decision into a decision that is best for all of the players.

Homework: Recommended

Prerequisites: None

Introduction to Galois theory (🌶🌶🌶, Sim, $\boxed{\text{TW}\Theta\text{FS}}$)
The Fundamental Theorem of Algebra states that all roots of polynomials with rational coefficients

[4]but definitely not all!

lie in the complex numbers $\mathbb{C}$. This feels like a pretty "continuous" result, but what if I told you that group theory and field theory could prove it?

In this class, we will honor Évariste Galois' legacy by exploring his namesake field: Galois theory. We'll cover field extensions, automorphism groups, and just what makes some field extensions special enough to be Galois. We'll think about what these tell us about solutions to polynomial equations, and how it can prove the Fundamental Theorem of Algebra. Finally, we'll cover the Fundamental Theorem of Galois Theory, which beautifully summarizes the relation between fields and Galois groups.

Along the way, we'll also learn the story of Évariste Galois' life, one full of trial, tribulation, love, and death (he died at age 20 under suspicious and miserable circumstances).

Homework: Recommended

Prerequisites: Group theory, Ring theory

Metric spaces (, Steve, $\boxed{\text{TW}\Theta\text{FS}}$)
A *metric space* is just a set X of "points" together with a *distance function*, d, which behaves the way distance should: the distance between any two points is zero iff they are actually the same point; the distance between x and y is the distance between y and x; and it is never more efficient to go from x to y to z than to just go from x to z. The standard examples of metric spaces are things like $\mathbb{R}$ (or the various $\mathbb{R}^n$s) with the appropriate Euclidean metric.

However, this is not remotely the end of the story! A metric space can be extremely structurally complicated, with "points" being interesting objects in their own right. For instance, the set $C_0[0,1]$ of continuous functions from $[0,1]$ to $[0,1]$ forms a metric space with the distance function $d(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\}$. We can also form metric spaces whose points are closed sets in some other metric space—there is even a metric space *of metric spaces*!

In the first half of this class, we'll develop the basic notions of metric space theory: completeness (and completions), compactness, and various other common ideas and results. In the second half we'll look at a few particularly bizarre metric spaces, such as the one alluded to two sentences prior and (time permitting) a kind of "line" that cannot be cut into two smaller "lines!"

Homework: Recommended

Prerequisites: None

Colloquia

Pure mathematics as applied physics (*Tadashi Tokieda*, Tuesday)
Humans tend to be better at physics than at mathematics. When an apple falls from a tree, there are more people who can catch it—we know physically how the apple moves—than people who can compute its trajectory from a differential equation. Applying physical ideas to discover and establish mathematical results is therefore natural, even if it has seldom been tried in the history of science. (The exceptions include Archimedes, some old Russian sources, a recent book by Mark Levi, as well as my articles.) This lecture presents a diversity of examples, and tries to make them easy for imaginative beginners and difficult for seasoned researchers.

Graph on, graph off (Narmada, Wednesday)
Way back in the old days of 2004, two Hungarian mathematicians published a paper that changed the world of graph theory forever. They asked the simple yet powerful question: what if sequences of graphs could converge? (Actually they asked more complicated questions about statistical physics and quasirandomness, but those magically transformed into this question.) I will draw several colorful pictures to convince you that the limit of a sequence of graphs is not a graph at all, but a graph*on*.

Join me as I navigate the treacherous waters of the combinatorics of graph homomorphisms to emerge, unscathed, in a world of integration and measure theory.

Killing the Cookie Monster (Arya, Thursday)
Every TAU, the Cookie Monster shows up bearing cookies and carrots. The Cookie Monster is a monstrous being with possibly several heads connected to a single body. A camper, fed up with this practice of snacking, decides to cut off one of the heads of the Cookie Monster. But behold! Two new heads pop out. Suppose the camper is adamant, and keeps chopping off heads, while the Cookie Monster keeps popping new heads. Will Nic receive his 3 carrots and a singular Chip-Ahoy, or be saddened by the demise of the multi-headed messenger? Come find out!

(Don't try this at home; Rule 0 might be broken.)

Future of Mathcamp (Staff, Friday)
Do you have opinions about what would make Mathcamp better? Then come to this event for brainstorming and discussion in groups about what we can change in the future.