WEEK 5 CLASS PROPOSALS, MATHCAMP 2022

Contents

Aaron's Classes

Electric circuits and random walks (🌶🌶, Aaron, 4 days)
In the Martingales class, Yuval showed that a frog hopping along the integer number line always gets back to where it started eventually. However, what about a person wandering a two-dimensional city grid, or a bird flying randomly in three-dimensional air?

We will tackle these problems, with electricity!

Prerequisites: Graph theory, and some optional problems will use linear algebra

Homework: Optional

Generalized lazy catering (🌶🌶–🌶🌶🌶, Aaron, 2 days)
Say we have a cake, which is a disk in $\mathbb{R}^2$, and a knife, which is an ideal line. How many pieces can we cut the cake into with n slices? The answers are known as the *lazy caterer's sequence.*

The first few terms of the sequence are 1, 2, and 4, from which you could reasonably surmise that the sequence is just the powers of two. Until, that is, you fail to cut more than 7 pieces with 3 cuts. The real answer is a quadratic polynomial.

In this class, we'll see why, even if the knife can cut the cake at bizarre shapes instead of lines, the answer is always the powers of two, or a polynomial, whose degree we can bound with the *VC-dimension*.

Prerequisites: Big O notation would be helpful.

Homework: Recommended

The Tarski–Seidenberg algorithm (🌶🌶🌶, Aaron, 2 days)
You may be aware of Gödel's Incompleteness Theorems, which say among other things that the theory of natural numbers is undecidable. However, in this respect, the real numbers are much simpler, and their theory is decidable!

In this class, we'll introduce the Tarski–Seidenberg algorithm, which evaluates any statement about the real numbers (in first-order logic).

Prerequisites: No strict prereqs, but I will define some logic terms (sentences and formulas) kind of quickly.

Homework: Recommended

The friendship theorem (🌶🌶🌶, Aaron, 2 days)
A *friendship graph* is a graph where every two distinct vertices share exactly one neighbor. It turns out that these take a very particular form—they look like windmills!

To prove this, we will first develop some linear algebra mod p, and then see what this has to do with graph theory.

Prerequisites: Linear algebra (eigenvalues), a bit of graph theory, finite fields recommended

Homework: Recommended

Transcendence (🌶🌶🌶, Aaron, 2 days)
If you want to find out if two vector spaces are isomorphic, it's simple. You just find out their dimension. If you want to know if two algebraically closed fields are isomorphic, it's only a little more complicated—you need to know their characteristic and a sort of dimension called the *transcendence degree*.

In this class, we'll see why algebraically closed fields and vector spaces are kind of the same thing, and figure out how to translate words like "linearly independent", "span", and "basis". From this, we'll see how to adapt linear algebra ideas to prove when algebraically closed fields are isomorphic.

Prerequisites: Linear algebra, ring theory

Homework: Recommended

Alan Chang and Charloutte's Classes

Counting with polynomials (🌶🌶–🌶🌶🌶, Alan Chang and Charloutte, 4 days)
Relatively recently, mathematicians discovered that polynomials are a very useful and surprisingly powerful (poggerful?) tool for counting. In 2008, Dvir solved something known as "the finite field Kakeya conjecture" using a very simple principle: you can estimate the size of a set S via the degree of a polynomial that vanishes on it (i.e., that is zero at every point in S). We won't talk about this proof. Instead we'll discuss some of the classic problems in incidence geometry that were subsequently solved using the so-called "polynomial method."

Here is a classic example of a problem in incidence geometry: given a set of lines L and a set of points P, we can define an incidence between L and P to be a pair $(p, \ell) \in P \times L$ such that $p \in \ell$. The Szemerédi–Trotter theorem gives an upper bound on the number of possible incidences, in terms of $|L|$ and $|P|$.

The polynomial method gives an elegant proof of this theorem. We'll talk about this result and more, and eventually discuss the (near) solution to Erdős distinct distance problem, which uses the polynomial method. This problem asks the following: if $x_1, \ldots, x_N$ are points in $\mathbb{R}^2$, then how small can the size of $D = \{|x_j - x_i| : 1 \leq i > j \leq N\}$ be? That is, what is the least number of distinct distances that are determined by N points? If this problem sounds familiar, it may or may not be because you read about it in a class blurb that may or may not have been cancelled (sadge)! Well, here's one more chance.

Prerequisites: you should be comfortable with multi-variate polynomails; have some familiarity with multi-variable calculus—enough to know what the gradient is; know what "linearly independent vectors" are

Homework: Recommended

Arya's Classes

Hyperbolic geometry (in 1D) (🌶🌶🌶, Arya, 3 days)
Roughly speaking, hyperbolic geometry on the plane is the geometry where lines bend away from each other, where one can have infinitely many lines through a point that are parallel to a line, where the area of a triangle is determined by its angles. Assaf made the following joke once at lunch: "What is a hyperbolic hyperbolic geometer? A mathematician who thinks that every graph is a tree." In this class, I shall explain what this joke means. In particular, we shall study properties of "hyperbolic graphs" and "hyperbolic groups" as examples of 1D objects capturing the essence of hyperbolic geometry. Hopefully, by the end of this class, we can all laugh at Assaf's joke.

Prerequisites: Definition of a group, (Day 1 of) Hyperbolic geometry. If you didn't come to hyperbolic geometry but are interested in taking this class, talk to me.

Homework: None

Hyperbolic geometry (in 3D) (🌶🌶🌶🌶, Arya, 3 days)
Roughly speaking, hyperbolic geometry on the plane is the geometry where lines bend away from each other, where one can have infinitely many lines through a point that are parallel to a line, where the area of a triangle is determined by its angles. In this class, we shall further explore a 3D model of hyperbolic geometry, study hyperbolic tetrahedra and their volume, and draw some pictures of 3-manifolds that are hyperbolic.

Prerequisites: Hyperbolic geometry

Homework: None

Teichmüller theory of not the torus (🌶🌶–🌶🌶🌶, Arya, 4 days)
In week 2, we studied the Teichmüller space (space of all markings) of the torus, the moduli space (space of all metrics), the Farey graph (a graph of all simple closed curves) and Möbius transformations (transformations that change markings but fix the metric). A lot of the ideas we discussed for the torus generalise to other surfaces; however, it is very non-trivial to do so. In this class, we shall spend some time discussing the obstructions and ideas in generalising the torus results to other surfaces. We shall not go into much rigour, but draw lots of pictures.

Prerequisites: Teichmüller theory of the torus; having attended hyperbolic geometry is useful but not necessary.

Homework: None

Tracking pants (🌶🌶🌶–🌶🌶🌶🌶, Arya, 3–4 days)
Topologically, a pair of pants is a sphere with three holes cut out. Most surfaces can be built by gluing together several copies of pairs of pants together. There are several possible ways to do so, and studying these relates to the study of non-intersecting curves on the surface. In this class, we shall study pants decompositions of the surface, the pants graph, and parametrise the various pants decompositions using gadgets called "train tracks."

Prerequisites: Some familiarity with topology/classification of surfaces

Homework: None

Arya, Assaf et al.'s Classes

T-shirt talks (🌶, Arya, Assaf et al., 3–4 days)
A t-shirt is a topological sphere with 4 disks removed. Campers and mentors alike use these devices everyday for various purposes. Campers occasionally ask the mentors ludicrous questions like "hey, what's that math on your t-shirt?". To put an end to this line of inquiry, a few brave mentors shall present themselves along with these t-shirts, and talk about the math on the t-shirts. Campers shall sit back, relax and ask the occasional "what's your favourite bird?".

Prerequisites: None

Homework: None

Assaf's Classes

Degen's eight square identity (🌶, Assaf, 1 day)
Degen's eight square identity is the following:

$$\begin{aligned}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2+a_8^2)(b_1^2+b_2^2+b_3^2+b_4^2+b_5^2+b_6^2+b_7^2+b_8^2)\\
=&(a_1b_1-a_2b_2-a_3b_3-a_4b_4-a_5b_5-a_6b_6-a_7b_7-a_8b_8)^2+\\
&(a_1b_1-a_2b_2-a_3b_3-a_4b_4-a_5b_5-a_6b_6-a_7b_7-a_8b_8)^2+\\
&(a_1b_2+a_2b_1+a_3b_4-a_4b_3+a_5b_6-a_6b_5-a_7b_8+a_8b_7)^2+\\
&(a_1b_2+a_2b_1+a_3b_4-a_4b_3+a_5b_6-a_6b_5-a_7b_8+a_8b_7)^2+\\
&(a_1b_3-a_2b_4+a_3b_1+a_4b_2+a_5b_7+a_6b_8-a_7b_5-a_8b_6)^2+\\
&(a_1b_3-a_2b_4+a_3b_1+a_4b_2+a_5b_7+a_6b_8-a_7b_5-a_8b_6)^2+\\
&(a_1b_4+a_2b_3-a_3b_2+a_4b_1+a_5b_8-a_6b_7+a_7b_6-a_8b_5)^2+\\
&(a_1b_4+a_2b_3-a_3b_2+a_4b_1+a_5b_8-a_6b_7+a_7b_6-a_8b_5)^2+\\
&(a_1b_5-a_2b_6-a_3b_7-a_4b_8+a_5b_1+a_6b_2+a_7b_3+a_8b_4)^2+\\
&(a_1b_5-a_2b_6-a_3b_7-a_4b_8+a_5b_1+a_6b_2+a_7b_3+a_8b_4)^2+\\
&(a_1b_6+a_2b_5-a_3b_8+a_4b_7-a_5b_2+a_6b_1-a_7b_4+a_8b_3)^2+\\
&(a_1b_6+a_2b_5-a_3b_8+a_4b_7-a_5b_2+a_6b_1-a_7b_4+a_8b_3)^2+\\
&(a_1b_7+a_2b_8+a_3b_5-a_4b_6-a_5b_3+a_6b_4+a_7b_1-a_8b_2)^2+\\
&(a_1b_7+a_2b_8+a_3b_5-a_4b_6-a_5b_3+a_6b_4+a_7b_1-a_8b_2)^2+\\
&(a_1b_8-a_2b_7+a_3b_6+a_4b_5-a_5b_4-a_6b_3+a_7b_2+a_8b_1)^2\end{aligned}$$

There's a very elegant, one line proof of it using division algebras and octonions, but in this class, we'll prove it the old fashioned way. By expanding it all out, and equating the terms. If there's time, we will do the octonionic one-line proof.

Prerequisites: None

Homework: None

Don't @ me, Eric (🌶🌶🌶🌶, Assaf, 4 days)
In his inflammatory colloquium, Eric pretty much called me a toddler of topology who can't even observe that the torus has a group structure. To that, I say that I may indeed be a toddler of topology, but I sure can define a group structure on the torus. In this class, I will do just that, using the Weierstrass $\wp$-function.

In kindergarten, we learned about the trigonometric functions and their roles in parametrizing the circle. By grade 1, after mastering calculus, we also learned about the definitions of the inverses of these functions in terms of integrals, and showed that they are derivatives of each other. In this class, we will go up a dimension and discuss the complex analogue of a trigonometric function, called the Weierstrass $\wp$ function. This doubly-periodic meromorphic function behaves like a trigonometric function in parametrizing complex tori in $\mathbb{C}P^2$. Along the way, we'll pass through series of meromorphic functions, the complex projective space, and prove the addition formula for elliptic curves.

Prerequisites: Complex analysis/Residue theorem is a must, topology is helpful

Homework: Required

Geometric group theory (🌶🌶🌶, Assaf, 4–5 days)
Geometric group theory is, broadly speaking, the study of groups by their actions on metric spaces[1]. One such metric space is the Cayley graph, which is a well-defined geometric object up to quasi-isometry, which is like an isometry, but allows for finite amounts of scaling and fudge factors.

In this class, we will explore this further, and prove the fundamental theorem of geometric group theory: the Svarc–Milnor lemma, and other related concepts.

Prerequisites: Group theory, metric spaces, some point-set topology is helpful

Homework: Recommended

Proving the fundamental theorem of algebra using Laplace's Equation (🌶🌶🌶🌶, Assaf, 3 days)
The fundamental theorem of algebra says that any nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$. This theorem is oftentimes proven in a complex analysis course. This should confuse you. Why should the fundamental theorem of ALGEBRA be proven using complex ANALYSIS? Wouldn't it make more sense to prove it using a field closer to algebra, like partial differential equations, maybe?

Of course it makes sense! In this class, we'll see how.

Prerequisites: Complex analysis, multivariable calculus, topology can be helpful

Homework: Recommended

Reading Leibniz's notes in Latin (🌶, Assaf, 1 day)
Are your math notes messy? Do you scribble things out in the margins? Do you write theorems that are objectively wrong? Do you think that in ten years, you'd be unable able to decipher what you wrote in a margin of scrap paper even it if ended up being the foundation of calculus? If you answered yes to any of thee questions, then you should know that Leibniz did too, only his scribbles are in Latin and were written in the 1600s. In this class, we will be trying to piece together some of Leibniz's old manuscripts and translate them to see how foundational identities in calculus were discovered. We may even find some mistakes in Leibniz's notes!

Prerequisites: None!

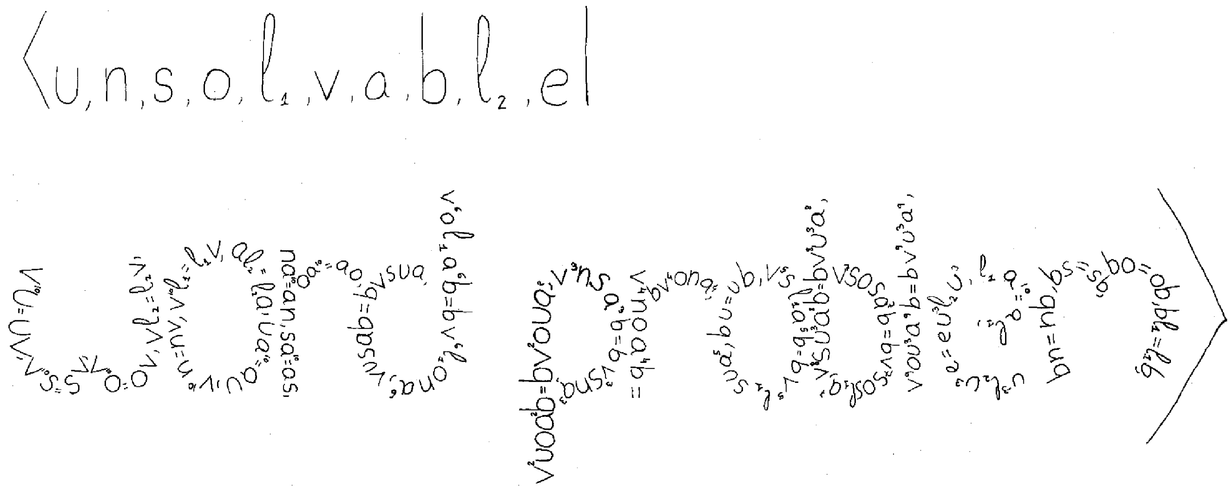
Homework: None

The word problem for groups (🌶🌶🌶–🌶🌶🌶🌶, Assaf, 3 days)
A group can be thought of as a collection of reversible operations, with some rules about how they relate to each other. Such a way of thinking is called a *group presentation*, and as an example, we have that the cyclic group of order 3 can be written as $\langle x | x^3 = e \rangle$, where x is a generator, and $x^3 = e$ is a relation. Given a group presentation with a finite set of generators and relations, does there always exist an algorithm to tell if a word in the generators is the identity? This is called the Word Problem, and it was posed by Max Dehn in 1911.

The surprising answer, shown by Pyotr Novikov in 1955, is no—there does not exist such an algorithm. In this class, we will prove this result by studying how we can embed "uncodable" sets in groups. This will give us a candidate for a "bad" group for which a solution to the word problem would contradict the "uncodablity" of the set.

Additionally, this class comes with a really sweet t-shirt design that we should totally reprint this year, and which contains a group presentation that has unsolvable word problem (we will not prove that this specific presentation does not have solvable word problem in this class):

[1] the "geometric" in geometric group theory referring to metric spaces



Prerequisites: Group theory
Homework: Required

Ben's Classes

Bonus bonus group threeory: part 3 (🌶🌶–🌶🌶🌶, Ben, 2–4 days)
In Bonus group theory: part 2, we saw a number of weird results that had some interesting consequences. For example, we proved that there is only one group of order 15. We already know that there are at least 2 groups of order 6—what's the difference? Are there any other secret groups of order 6? What about 21? More complicated orders?

Depending on how long the class runs, we can also investigate other Bonus Group Theory content—how many groups are there of order 12, and why is this a lot harder? What if we decide that nonabelian groups are not our friends, and we want to only care about abelian groups? And what is so simple about simple groups?

Prerequisites: Introductory group theory. In particular, I will not assume Bonus group theory: part 2 material, or any previous familiarity with group actions! We'll use some results from Bonus Group Theory: Part 2, but we'll review anything we use.

Homework: Recommended

Chaos and the intermediate value theorem (🌶🌶🌶–🌶🌶🌶🌶, Ben, 4 days)
In 1975, the mathematicians Li and Yorke proved an interesting result about dynamical systems that is commonly referred to as "period three implies chaos." This is a result with a few parts that we're not actually proving in this class, because I will not be telling a story about Li and Yorke. Part of their result showed that if a map $f : \mathbb{R} \to \mathbb{R}$ has a point of exact period 3, it also has points of exact period n, for all n.

That is the part we care about, because it is a massively special case of a theorem of Oleksander Sharkovsky, proven in 1964. This theorem tells us much more exact conditions on how periodic points behave—for instance, it tells us that a map can have a point of period 5 without having period 3, or a point of period 245902 without having a point of period 123453.

Where did these numbers come from? What's going on here? And why do we only need the intermediate value theorem to prove these crazily intricate results? Learn all this and more!

Prerequisites: Know the intermediate value theorem. Past experience with dynamical systems is not necessary, but may be helpful

Homework: Recommended

Gibbard–Satterthwaite's impossibility theorem (🌶–🌶🌶, Ben, 1 day)

"This principle of voting makes an election more of a game of skill than a real test of the wishes of the electors."

So said the Reverend Charles Dodgson[2] when discussing *strategic voting*, the phenomenon when a voter has an incentive to falsely represent their true desires. Back in my week 3 class on Arrow's impossibility theorem, I touched on this idea a few times—this class will return to the topic with a vengeance!

In particular, we'll prove that, no matter how we pick winners of an election, our system will either be (a) nonunanimous (a word that here means "very, very bad"), (b) a dictatorship, or (c) a voting system where some voters want to strategically vote, at least some of the time.

If this looks vaguely like the statement of Arrow's impossibility theorem, it is for good reason—we shall, in fact, use Arrow's theorem to prove the Gibbard–Satterthwaite Theorem!

Prerequisites: None; having seen Arrow's impossibility theorem (e.g. in Week 3) will be somewhat helpful

Homework: Recommended

Overly convoluted drawings (🌶🌶, Ben, 2 days)

(Despite the name, this has no connection to "Overly convoluted plans" except that both are analysis-flavored.)

In the 1800s, mathematicians started discovering that things like "the real number line" and "the plane $\mathbb{R}^2$" and "functions into or out of these spaces" are a lot weirder than they had previously imagined. Functions with no derivatives, circumstances where addition doesn't commute, and other monsters began to haunt the dreams of analysts everywhere.

One of the more surprising examples is a **space-filling curve**—a continuous function with domain $[0, 1] \subset \mathbb{R}$ (that is, a curve, something with a one-dimensional domain) whose image takes up the entire unit square $[0, 1] \times [0, 1]$. Let me say that again. There is a continuous function, with a one-dimensional domain, whose image takes up a two-dimensional area.

The first of these was discovered by Peano, in 1890, but in this class I'd present a simpler construction of Hilbert, a year later. We'll use a few facts about analysis to show that we can build up a space-filling curve—but our main tool, as the course title suggests, is a few complicated sketches.

Prerequisites: Knowing what uniform convergence is

Homework: Recommended

Ultrafilters and nonmeasurable sets (🌶🌶🌶, Ben, 2–3 days)

We all like ultrafilters! Nonprincipal ultrafilters are especially great for all kinds of things—Aaron likes them for having hyperreals be things! Ben likes them for not having dictators! Nic likes that they're as unprincipled as he is! Steve likes them for [INSERT STEVE'S REASONS HERE]! Zoe likes doing Ramsey theory with them! Susan says they're delicious!

Nobody likes nonmeasurable sets! They make people sad because they don't have any reasonable notion of length.

One fun fact is that both the existence of nonprincipal ultrafilters, and the existence of nonmeasurable sets, are consequences of the axiom of choice. On the other hand, some Ultrafilter Enjoyers may

[2]Perhaps better known as Lewis Carroll, author of *Alice's Adventures in Wonderland.*

be aware that you don't need the *full awesome power* of the axiom of choice to get ultrafilters. You might be wondering—is there some sneaky axiom we can use that will let us have all the wonderful ultrafiltery nonsense we want, without any of these gross nonmeasurable sets.

There's not! A lovely theorem of Sierpinski demonstrates that, if there is a nonprincipal ultrafilter on $\mathbb{N}$, there is a nonmeasurable subset of $\mathbb{R}$.

Come learn about measure, ultrafilters, and how to smash them into each other until something breaks!

Prerequisites: Know what a nonprincipal ultrafilter is

Homework: Recommended

Charloutte's Classes

Predict the future with the axiom of choice! (🌶–🌶🌶, Charloutte, 2 days)
You may have heard of "the infinite hat problem" (or a variant of it). A countably infinite number of prisoners are lined up and randomly given a black or white hat; each prisoner can see the the hat colours of each peer ahead of them. Each prisoner needs to guess the colour of their own hat in order to be freed. Is there a way for the prisoners to guarantee that all but a finite number of them will be freed?

It turns out, yes, there is, if the prisoners are allowed to use the axiom of choice. If this result seems preposterous to you, then enjoy this even more preposterous statement: using the axiom of choice, we can predict the future!!!

Prerequisites: Ideally, you'd be familiar with well-orders, and the equivalence of the axiom of choice and the well-ordering theorem.

Homework: Recommended

Emily's Classes

Block designs (🌶🌶, Emily, 2 days)
Suppose you are running a scrabble tournament and 13 people show up to play. You wish to structure the tournament so that each game consists of four people, and each pair of people plays against each other in some game exactly once. Is this structure possible? If so, how many games must be played?

Now suppose the year is 1850 and you are Thomas Kirkman. You wish to solve the following problem: "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."

Both of these problems can be understood using block designs: a set together with a collection of subsets that satisfy certain conditions. We will explore some properties of block designs and how they can be constructed. This will involve combinatorics and in some cases projective planes.

Prerequisites: None

Homework: None

Destination permutation (🌶🌶, Emily, 1 day)
Cayley's theorem says that every group can be expressed as a permutation group; that is, as a subgroup of a symmetric group. That's cool and all, but how do we actually *find* this permutation group? In this class, we will explore how this is done using coset enumeration, which is an algorithm heavily dependent on group actions.

Prerequisites: Know what a group is, in particular the symmetric group. You do not need to have seen group actions before.

Homework: Optional

Primes in the Eisenstein integers (🌶🌶🌶, Emily, 2–3 days)
The Eisenstein integers aren't like the integers that we are used to—they are complex numbers which form a triangular lattice on the complex plane. However, they behave quite similarly to the integers! In particular, we can classify which numbers are prime, some of which are primes from the regular old integers and many who are not. Furthermore, we can talk about when two Eisenstein integers are *relatively* prime, i.e. have a GCD of 1. This allows us to define an Euler phi function, which is an important function from number theory.

Prerequisites: Familiarity with complex numbers, modular arithmetic, and quotient groups.

Homework: Optional

Eric's Classes

Evil floating point bit level hacking (🌶🌶🌶, Eric, 1 day)
This is a thematic follow up to my 2-adic computer science class, though this class will be entirely self-contained! You'll totally be able to follow this class even if you didn't attend 2-adic.

Computers approximate real numbers using a technique called "floating point numbers," which is basically scientific notation in binary. This has some downsides, in that many computations with floats are not quite exact; but the upside is that there's a ton of cool tricks you can do with this format. In this class we'll learn about one of the most famous pieces of code ever written, a shockingly simple algorithm for approximating $\frac{1}{\sqrt{x}}$ using "evil floating point bit level hacking." As was the case with my 2-adic computer science class, the secret to understanding what's going on will be logarithms!

Prerequisites: It'll help if you've seen logarithms and any programming language before, though neither is strictly necessary

Homework: Optional

The linear algebra class that wasn't (🌶🌶–🌶🌶🌶, Eric, 1–3 days)
In week 1 I was supposed to teach introduction to linear algebra, but then I got Covid and Misha valiantly stepped in to teach the course. In this week 5 class I'll present a sequel to Misha's class which covers some interesting topics around linear algebra: it might be eigenthings similar to Mark's week 2 class, maybe characteristic polynomials, maybe a bit on infinite dimensions, maybe a bit on fields other than $\mathbb{R}$.

Prerequisites: Introduction to linear algebra

Homework: Recommended

The sound of proof (🌶, Eric, 1 day)
Can you hear what a proof sounds like? I'll present five proofs from Euclid's Elements, and then play (recordings of) five pieces of music written to capture each proof in sound. You'll get to try and work out which piece of music lines up with which proof, and then we'll dissect how a couple of the compositions "sonify" the proofs. All of the material I'm drawing on is from an art piece entitled "The Sound of Proof" by mathematician Marcus du Sautoy and composer Jamie Perera at the Royal Northern College of Music in Manchester.

Prerequisites: None

Homework: None

JOÃO'S CLASSES

The Littlewood–Richardson rule (🌶🌶🌶–🌶🌶🌶🌶, João, 2–3 days)
What do representation theory and algebraic geometry have in common? They both are much more fun if you make them combinatorics! In this class we will learn about partitions, tableaux, and Schur polynomials in order to prove the Littlewood–Richardson rule—a beautiful result that gives a combinatorial interpretation to the coefficients of the decomposition of products of Schur polynomials in terms of the Schur basis for symmetric functions; and that shows up in several areas of math.

Prerequisites: Some familiarity with group actions helps, but not required

Homework: Optional

JUSTIN LEE'S CLASSES

Sets that can tile lattices (🌶🌶🌶🌶, Justin Lee, 1 day)
Given a set of natural numbers, $\mathcal{S} = \{a_1, a_2, \ldots, a_k\}$, call sets of the form $\{a_1 n, a_2 n, \ldots, a_k n\}$ "scalings" of $\mathcal{S}$. In this class, we will analyze the structure of sets $\mathcal{S}$ that have the property that $\mathbb{N}$ can be partitioned into a set of scalings of $\mathbb{S}$.

For instance, the set $\{1, 2\}$ satisfies this property: scaling the set by every odd number, then every number of the form $4\times$ odd, then $16\times$ odd, and so forth, results in a partition of $\mathbb{N}$.

Let's call sets with this property "tileable" sets. (The name will make sense after the start of class.) It turns out that tileable sets are very structured; for instance, we will show that every element of a tileable set divides the largest element. Yet, surprisingly complicated (and difficult to describe) sets are tileable. The set $\{1, 2, 100, 200\}$ works but the set $\{1, 2, 1000, 2000\}$ does not.

The problem can be rephrased as partitioning natural numbers *additively* instead of multiplicatively—and we will work not over $\mathbb{N}$, but rather $\mathbb{N}^d$. Which sets of lattice points can tile $\mathbb{N}^d$ with translations? What properties do they have? We will answer these questions using a combinatorial approach, and then see how some (but not all!) of the same conclusions can be arrived at with a neat algebraic trick.

Prerequisites: None

Homework: None

LINUS'S CLASSES

Getting your money's worth from a 4-function calculator (🌶🌶🌶🌶, Linus, 2 days)
You have a pocket calculator with the usual four functions ($+$, $-$, $\times$, $\div$ (rounding down), exponentiation, and bitwise-AND). Unfortunately it lacks many useful buttons such as "factorial" and "primality test."

Fortunately internet users Dennis et al. found that n is prime iff

$$\frac{(4^n+1)^n \mod 4^{n^2}}{n} \,\&\, \frac{2^{2n^2+n}}{2^n+1} = 0.$$

What other functions can you compute with a fixed sequence of button presses? This system is *not* Turing-complete! (Why?) In this class you will solve a series of puzzles leading to a full classification.

Prerequisites: Know how to compute the bitwise-AND of two numbers. Be familiar with basic programming (any language).

Homework: Optional

My thesis defense (🌶🌶🌶🌶, Linus, 1 day)
Linus delivers his thesis defense verbatim. Here is the original math department announcement:

Thesis title: Applications and limits of convex optimization

Talk abstract: To word it as confusingly as possible, the Paulsen problem asks: if a set of vectors is close to isotropic and equal-norm, then is it close to an isotropic equal-norm set of vectors? This question gained notoriety as "one of the most intractable problems in [operator theory]." In 2017, it was finally resolved by Kwok et al. in a triumphant 103 pages of smoothed analysis and operator scaling. Unfortunately we won't have enough time to deliver their proof, so I will instead exhibit a new one. It is short enough to present unabridged, and still have time left over to graze on two other problems related to convex optimization: graphical model structure learning, and the impossibility of accelerated gradient descent in hyperbolic space.

Prerequisites: Linear algebra. I will use without proof the fact that a symmetric matrix M has an orthonormal eigenbasis (i.e. orthonormal vectors $v_1, v_2, \ldots, v_n$ such that $Mv_i = \lambda_i v_i$ for all i.)

Homework: None

Mark's Classes

Multiplicative functions (🌶🌶–🌶🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, and/or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or two, you should consider this class.

Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* required.)

Homework: Optional

Quadratic reciprocity (🌶🌶🌶, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is q a square modulo p?"
(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly ingenious proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Prerequisites: Some basic number theory (if you know Fermat's little theorem, you should be OK)

Homework: Optional

Systems of differential equations (🌶🌶🌶, Mark, 3–4 days)
Many models have been devised to try to capture the essential features of phenomena in economics, ecology, and other fields using systems of differential equations. One classic example is given by the

Volterra–Lotka equations from the 1920s:

$$\frac{dx}{dt} = -k_1\, x + k_2\, xy\,,$$
$$\frac{dy}{dt} = k_3\, y - k_4\, xy\,,$$

in which x, y are the sizes of a predator and a prey population, respectively, at time t, and k_1 through k_4 are constants. There are two obvious problems with such models. Often, the equations are too hard to solve (except, perhaps, numerically); more importantly, they are not actually correct (they can only hope to approximate what really goes on). On the other hand, if we're approximating anyway and we have a system

$$\frac{dx}{dt} = f(x,y),\ \frac{dy}{dt} = g(x,y),$$

why not approximate it by a linear system such as

$$\frac{dx}{dt} = px + qy\,,\ \frac{dy}{dt} = rx + sy\,?$$

Systems of that last form can be solved using eigenvalues and eigenvectors, and usually (but not always) the general behavior of the solutions is a good indication of what actually happens for the original (nonlinear) system if you look near the right point(s). If this sounds interesting, come find out about concepts like trajectories, stationary points, nodes, saddle points, spiral points, and maybe Lyapunov functions. Expect plenty of pictures, as well as a bit of computation.

Prerequisites: Eigenvectors and eigenvalues (the one-day "eigenstuff" class should be enough), calculus, a little bit of multivariable calculus (equation of tangent plane).

Homework: Optional

The Cayley–Hamilton theorem (🌶🌶🌶, Mark, 1 day)
Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Prerequisites: Linear algebra, and a solid grasp of determinants (if the "Magic of determinants" class has happened first, that would certainly be enough)

Homework: None

The Prüfer correspondence (🌶🌶, Mark, 1 day)
Suppose you have n points around a circle, with every pair of points connected by a line segment. (If you like, you have the complete graph K_n). Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments, but in only one way. (This tree will be a spanning tree for K_n). How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll go through a combinatorial proof that is especially cool!

Prerequisites: None

Homework: None

The Riemann zeta function (🌶🌶–🌶🌶🌶, Mark, 3–4 days)
Many highly qualified people believe that the most important open question in pure mathematics is the

Riemann hypothesis, a conjecture about the zeros of the Riemann zeta function. Having been stated in 1859, the conjecture has outlived not only Riemann and his contemporaries, but a few generations of mathematicians beyond, and not for lack of effort! So what's the zeta function, and what's the conjecture? By the end of this class you should have a pretty good idea. You'll also have seen a variety of related cool things, such as the probability that a "random" positive integer is not divisible by a perfect square (beyond 1) and the reason that $-691/2730$ is a useful and interesting number.

Prerequisites: Some single-variable calculus (including integration by parts) and some familiarity with complex numbers and infinite series; in particular, geometric series.

Homework: Optional

The magic of determinants (🌶🌶🌶, Mark, 3 days)
You may have taken a class on linear algebra (at Mathcamp or somewhere else) that barely touched on determinants in general. If that left you feeling dissatisfied, either about not really having a good definition (by the way, using the Laplace expansion, while it is computationally often handy, leads to a miserable definition—with no intuitive basis at all) or about not having seen many of the properties that determinants have, you may enjoy this class. If all goes well, we'll give a definition of determinant that's both motivated and rigorous, and there will be proofs of all its main properties (such as Laplace expansion). There may also be a few applications, such as general formulas for the inverse of a matrix and for the solution of n linear equations in n unknowns ("Cramer's Rule").

Prerequisites: Some linear algebra, including linear transformations, matrix multiplication, and preferably 2-by-2 determinants

Homework: Optional

Wedderburn's theorem (🌶🌶🌶, Mark, 1 day)
You may well have seen the quaternions, which form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$ with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j.$$

Have you seen any examples of *finite* division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Prerequisites: Some group theory; knowing what the words "ring" and "field" mean. Familiarity with complex roots of unity would help.

Homework: None

Mark Takken's Classes

Spherical trigonometry with applications (🌶🌶, Mark Takken, 1 day)
If you *really* wanted to fly to Cheyenne, Wyoming (41.1°N, 104.8°W) from Portland, Maine (43.7°N, 70.3°W) overnight with a staff member, in what direction would the plane start flying, and how many miles long would the trip be? What is the time and direction of sunset today? How could we construct an accurate sundial for Waterville, Maine at this time of year? Planar trigonometry on its own will not get us anywhere, since the Earth and sky are not flat[*citation needed*], so instead we will turn to the forgotten science of spherical trigonometry. We will start by developing all of the right-triangle formulas, among which we will discover a beautiful and surprising symmetry, and then extend them

to the spherical equivalents of the Law of Sines and Cosines. Finally, we will use what we've learned to solve the above navigational and astronomical problems!

Prerequisites: None

Homework: None

Mia's Classes

Extreme extremal graph theory (🌶🌶, Mia, 1 day)
A typical question in extremal graph theory asks, given a graph G with n vertices, how many edges does G need to guarantee that H is a subgraph? But what if I want not one graph G, but MANY? What if I want ALL of the cycles C_k, up to some fixed k? This class will look at a delightful proof of Bondy's theorem, which gives conditions that guarantee not one cycle, but all of them.

Prerequisites: Graph theory

Homework: None

The Zarankiewicz problem (🌶🌶–🌶🌶🌶, Mia, 1 day)
If you took Yuval's extremal graph theory class, you saw the Erdős-Stone Theorem, which is a *rad* theorem in extremal graph theory. "Why is it rad?", you might ask. Well, suppose you take your favorite graph H and you want to know how many edges are needed to guarantee that any n-vertex graph contains H as a subgraph. Well, then it says that the number of edges required is $\big(\frac{\chi(H)-2}{\chi(H)-1} + o(1)\big)\frac{n^2}{2}$. Pretty rad, right?!

Unfortunately, there is one part that is not so rad... it give us little information when H is bipartite. Whomp.

But progress has been made for specific types of bipartite graphs! In this class, we'll examine the Zarankiewicz problem, which considers the case when H is a complete bipartite graph.

Prerequisites: Graph theory, extremal graph theory *not* required

Homework: None

Misha's Classes

How to avoid taking APs: fun with finite fields (🌶🌶🌶, Misha, 1 day)
Van der Waerden's theorem says that for all r, t there is an n such that if we color the elements of $\{1, 2, \dots, n\}$ by r colors, we are guaranteed a t-term arithmetic progression whose elements are all the same color.

We often want to know: what is the smallest n that works? (In week 2, we denoted this number by r-Van(t).)

In this class, we'll prove some lower bounds on r-Van(t) using number theory and abstract algebra, by playing around with polynomials over finite fields.

Prerequisites: Linear algebra; familiarity with rings or fields

Homework: None

How to avoid taking APs: the blame game (🌶🌶🌶🌶, Misha, 1 day)
Van der Waerden's theorem says that for all r, t there is an n such that if we color the elements of $\{1, 2, \dots, n\}$ by r colors, we are guaranteed a t-term arithmetic progression whose elements are all the same color.

We often want to know: what is the smallest n that works? (In week 2, we denoted this number by r-Van(t).)

In this class, we'll prove some lower bounds on r-Van(t) by coloring randomly, then teaching the arithmetic progressions to blame each other when things inevitably go wrong.

Prerequisites: Comfort with expected values

Homework: None

Packing permutation patterns (🌶🌶🌶, Misha, 2 days)
Prepare by picking a permutation π and a pattern P. Probabilistically pick $|P|$ pieces of π: perhaps putting them together produces P? Let $\rho_P(\pi)$ be the probability of producing P.

To pack P in π, puff up this probability, making P as plentiful as possible. We will ponder the packing problem for $P = 132$ (and plenty of its pals) using a progression of powerful problem-solving procedures.

Prerequisites: None, but we'll only narrowly escape ring theory and linear algebra.

Homework: Recommended

Problem solving: linear algebra (🌶🌶, Misha, 1 day)
Most high school math contests (the IMO included) do not use any topic considered to be too advanced for high school, such as linear algebra. This is a shame, because there have been many beautiful problems about linear algebra in undergraduate contests such as the Putnam Math Competition.

In this class, we will look at linear algebra from a new perspective and use it to solve olympiad problems.

Prerequisites: Linear algebra (the week 1 class is sufficient)

Homework: None

Problem solving: quick and quirky questions (🌶, Misha, 1 day)
Lots of problem-solving classes at Mathcamp are focused on the "respectable" kinds of olympiad problems: the ones where you sit down for a while and write a proof.

In this class, we'll instead look at relays-style problems that you have to solve under time pressure.

Prerequisites: None

Homework: None

Sorry, it turns out SAT is hard after all (🌶🌶🌶, Misha, 2 days)
So, uh… I talked to some mentors, and they told me that actually, the Boolean satisfiability problem is really hard. They said something like "NP-complete"?? This is embarrassing, because I just spent a whole week teaching you about why solving SAT is easy.

By way of apology, let me teach you why we think that solving SAT is hard, and why it is that if we solved SAT, we'd be able to solve all other problems in NP.

Prerequisites: None (in particular, "The satisfiability problem" from Week 4 is not a prerequisite.)

Homework: Optional

Narmada's Classes

I ♡ functional analysis (🌶🌶🌶, Narmada, 2 days)
Let's put the FUN in FUNctional analysis. I'm going to tell you all about how much I personally ♡ analysis. I prove the Riesz representation theorem for ℓ^p spaces every day, for FUN. I prove the

Hahn–Banach theorem every day, for FUN. I think about separation of convex sets every day, for FUN. Join me![3]

Prerequisites: Know what a continuous function is. Some linear algebra would be good.

Homework: Recommended

Super Deluxe Extra Bonus Group Theory (🌶🌶🌶🌶, Narmada, 2 days)
All the cool kids on the block are talking about the CFSG: Classification of Finite Simple Groups. In this class, I'll teach you how to be one of the cool kids. We'll prove simplicity for two large classes of finite simple groups, and discuss constructions of the sporadic ones (like the baby monster, and the regular monster). Expect a mixture of proofs and hand-waving, but also expect to leave this class with a better understanding of why people care about FSGs and what techniques were used to prove the CFSG.

Prerequisites: Susan's Intro to group theory

Homework: Optional

The Golay code (🌶🌶, Narmada, 2 days)
The theory of linear error-correcting codes is interesting in its own right, but the one code that's super cool is the binary Golay code. Did you know that the Golay code is the unique nontrivial binary 3-error correcting code? (We'll learn what that means.) You can construct it using the adjacency matrix of the icosahedron, or a combinatorial block design, or one of the other well-known Hamming codes. Also, its automorphism group is a sporadic simple group, and the automorphism group of its extended code is *another* sporadic simple group.

In this class, we'll learn enough about error-correcting codes to understand why the Golay code is valuable in the first place. Then you can play with your favorite construction to see how it works.

Prerequisites: Know that $\mathbb{Z}_p$ is a field when p is prime

Homework: Recommended

The axiom of choice (🌶🌶, Narmada, 2 days)
Why do mathematicians get so fussy about the axiom of choice? We'll talk a little bit about why the axiom of choice isn't just obviously true. We'll look at some obviously fake statements that are equivalent to the axiom of choice. We'll see why math without the axiom of choice might be sad sometimes. And by the end of this class, *you* get to be a mathematician who's fussy about the axiom of choice!

Prerequisites: None

Homework: Optional

Natali's Classes

Graph algorithms (🌶🌶, Natali, 1 day)
We will go through some of the important graph algorithms, including Floyd–Warshall and Dijkstra's algorithms. Knowing Python (at least very basics) is required.

Prerequisites: Python or some other programming language, at least basics

Homework: Optional

[3] For FUN!

Number fields (🌶🌶–🌶🌶🌶, Natali, 1–2 days)
Have you ever liked numbers so much that you wanted to make fields out of them? If so, you are lucky! There are number fields! This class will be an introduction to number fields, defining bunch of things in the process, and proving an interesting theorem about something something characteristic polynomial something something minimal polynomial in the end. We will use abstract and linear algebra but if you know basics, should be enough.

Prerequisites: Very basics of abstract algebra and linear algebra

Homework: Optional

Nic's Classes

Breaking, unbreaking, and using Bézout's theorem (🌶🌶, Nic, 2–3 days)
An *algebraic plane curve of degree* d is the set of points in the plane where some polynomial of degree d is equal to zero. For example, a circle is an algebraic plane curve of degree 2, because it's the set of points where $x^2 + y^2 - 1 = 0$. There's a classic theorem in algebraic geometry called *Bézout's theorem*, which says that if I have two algebraic plane curves, one of degree d and one of degree e, then they intersect in exactly de points.

Bézout's theorem is beautiful, deep, and—as I've stated it just now—clearly false. It turns out, though, that there's a way to make it true. You just have to slightly modify the definitions of "plane" and "point"! I think this is a great demonstration of a principle that shows up all over mathematics: if you have a result that feels like it ought to be true but isn't, sometimes you can make it true by changing your *definitions* rather than changing the theorem statement.

In this class we'll spend some time exploring what these changes are and what Bézout's theorem looks like with them in place. While we won't be able to give a proof of this new version of the theorem, we'll explore some fun geometric consequences of the theorem, and you'll get a good taste for the sorts of things algebraic geometers like to think about.

Prerequisites: None

Homework: Optional

On beyond on beyond on beyond i (🌶🌶🌶, Nic, 4 days)
You might be familiar with a geometric trick that you can play with a belt and a heavy object like a book. Take one end of the belt—the end without the buckle—and stick it under the book so it won't move. Then, without allowing the buckle to rotate, move it in a circle around the book until you it comes back to its original position. You should find that after doing this once, the belt can't be untwisted without rotating the buckle, but after doing it *twice*, it can! Why is this? Does something analogous happen in higher dimensions?

In three dimensions, you can define a cross product between two vectors, which is another vector that's perpendicular to the first two. The cross product can be used to write down formulas involving rotating vectors, but it seems like it's very specific to three dimensions: that's the only dimension where there's exactly a line's worth of vectors perpendicular to a given pair. Rotations certainly exist in other dimensions though! Is there a higher-dimensional analogue of a cross product?

This class will be about a mathematical object called a *Clifford algebra*, which gives a very nice geometric perspective on these questions and lots of others. (If you've ever heard of "geometric algebra," that's another name for the same thing.) We'll see how Clifford algebras allow us to talk about scalars, vectors, and higher-dimensional analogues of vectors in a unified language, and how they can give us a way to drastically simplify some computations in linear algebra, especially ones involving rotations.

Prerequisites: Linear algebra is definitely required, and some ring theory will also be very helpful; ask me if you're not sure. **Neither "On beyond i" nor "On beyond on beyond i" is a prerequisite for this class!**

Homework: Recommended

Thermodynamics and statistical mechanics (🌶🌶, Nic, 4 days)
Most of the physical systems you think about in high school physics classes only involve keeping track of a few objects at a time. You learn rules for predicting the behavior of physical systems that work pretty well in this situation, but which become unusable pretty fast as the number of objects gets large. Unfortunately, outside of physics class there are lots of things, like a hundred at the very least. So to study the sorts of physical systems we actually encounter in the real world, we need techniques that don't require us to know where every single atom is all the time.

In other words, we need statistical mechanics! In this class we'll study how to use probability to extract information about macroscopic physical systems from the microscopic laws of motion. Why is it that the behavior of large systems is often reasonably predictable even when we know that the atoms that make it up are bouncing off each other in every direction all the time? Why can many physical processes only happen in one direction, while the underlying microscopic laws of physics are completely reversible? And what really is temperature on a physical level? Come to this class to find out!

Prerequisites: Enough physics to know about conversation of energy and momentum

Homework: Optional

Rawin Hidalgo's Classes

Lattices and invariant theory (🌶🌶, Rawin Hidalgo, 1 day)
What are the "smallest" points that can be used to generate a given integer lattice of the form $ax + by \equiv 0 \mod p$? What about higher dimensional lattices? These questions (and the precise definition of "smallest") arise from studying the degree of generation of certain polynomial invariants. In this class, we'll briefly discuss the invariant theory motivation, but we'll mostly focus on studying these lattices to find bounds on this degree of generation.

Prerequisites: Some terms from abstract algebra will be used to describe the invariant theory motivation, but most of the class will focus on work with lattices, which has no prerequisites.

Homework: None

Ruthi Hortsch's Classes

Bayesian statistics (🌶🌶, Ruthi Hortsch, 3 days)
Statistics is the science of analyzing data in the presence of uncertainty or with incomplete information. Since there is little in the world that is certain, and information is always scarce, we humans can't go a day without doing some kind of statistics – in our routine cognitive functions, in science, in the political arena, etc.

On one hand, good statistics is a way of making our belief about whether something is true rigorous. On the other hand, statistics can also, through negligence or malice, be manipulated to show all kinds of not-particularly-true things, which is how we get the famous quote by Mark Twain: "There are three kinds of lies: lies, damned lies, and statistics."

In this class, we'll talk about some of the pitfalls statistics often leads us to fall into, how to think critically about the data we're given, and how Bayesian statistics can be a clean way of analyzing our intuitions. If you're interested at all in how math can be used in real life, this is the class for you!

Prerequisites: None. Knowing basic probability and/or calculus will be helpful, but is not necessary.

Homework: Recommended

The theorem formerly known as the Mordell conjecture (🌶🌶🌶🌶, Ruthi Hortsch, 1 day)
The premise of arithmetic geometry is this: we can turn questions from algebraic number theory into problems about geometry, and use our geometric knowledge to solve the problem there. The beautiful and surprising way these interplay changed the way people think about number theory and has formed modern research in the field. One of the key theorems is Faltings' Theorem (né the Mordell Conjecture), which states that no curve of genus 2 or higher has infinitely many rational points. In this talk we will discuss what this means, some results we can get from it, and some of the key elements of a proof first presented by Bombieri (perhaps touching upon the tour de force which landed Faltings the Fields Medal in 1986).

Prerequisites: Know what an abelian group is, and some general comfort with algebra.

Homework: None

Topology your friend (🌶🌶🌶, Ruthi Hortsch, 3 days)
The theorems about continuous functions from calculus rely on a notion of what it means for things to be close. If you've studied more advanced calculus, you've been introduced to how this is formalized with the "epsilon-delta" approach. This can sometimes get, how shall we put it, messy? In this class we will discuss how notions of closeness can be generalized to defining a topology on a set, focusing on relating it back to calculus. In particular, we will aim to give alternative (very slick!) approaches to the proofs of the intermediate and extreme value theorems.

Prerequisites: Calculus

Homework: Recommended

Steve's Classes

Further topics in forcing (🌶🌶🌶🌶, Steve, 1–4 days)
In this class we'll continue where Susan left off, with fun, strange, deep, or silly further applications of the forcing method. Possible topics include iterated forcing ("why force once when you could force twice?"), $\mathcal{P}(\omega)/\mathit{fin}$ ("let's solve all Ramsey problems at once forever!"), and using forcing to prove **outright theorems** as opposed to mere independence facts ("... what?").

Prerequisites: The continuum hypothesis

Homework: Recommended

How not to integrate complicated functions (🌶, Steve, 1 day)
One of the cutest applications of polar coordinates is the computation of the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

We'll see that this is an almost-completely-useless trick: in a precise sense, only the Gaussian integral itself (and minor variations thereof) can be solved in this way. Time permitting, we'll consider a generalization of this trick and show that it's *also* useless.

Prerequisites: Multivariable calculus

Homework: None

Universal algebra (🌶🌶🌶–🌶🌶🌶🌶, Steve, 3–4 days)
This class begins with a simple question: why, in multiple different areas of abstract algebra, do the same three constructions (homomorphic images, substructures, and product structures) play such a key role? This is of course a very open-ended question with lots of answers, but we'll start with one particular answer: the HSP theorem. This says that in a precise sense any class of algebraic structures describable by equations can be built just using these three operations from appropriate building blocks.

The HSP theorem isn't the end of the story, however: it motivates a particular perspective on algebraic structures, called **universal algebra**, which both generalizes many aspects of group and ring theory but also has its own distinctive flavor. We'll spend the rest of the class getting a sense of that flavor, looking at as many weird open questions in the subject as we can get to.

Prerequisites: Comfort with at least one (ideally two) of the following structures: group, ring, Boolean algebra. (Basically, the notions of homomorphism, substructure, and product structure should make sense.)

Homework: None

"I can't believe it's not matrix algebra!" (🌶🌶🌶🌶, Steve, 2–4 days)
Consider the following two mathematical structures: the **quaternions** (with the defining equalities $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$, and $i^2 = j^2 = k^2 = ijk = -1$) and the **2-by-2 matrices with real coefficients**. These structures, denoted "$\mathbb{H}$" and "$M_2(\mathbb{R})$" respectively, are broadly similar—in each structure we can add and multiply elements in a reasonable way, and each structure is a vector space over $\mathbb{R}$—but clearly non-isomorphic: in particular, in $\mathbb{H}$ we can divide by nonzero elements but in $M_2(\mathbb{R})$ we generally cannot.

However, it turns out that $\mathbb{H}$ and $M_2(\mathbb{R})$ are "potentially isomorphic:" there is a natural way to expand each into a vector space over the complex numbers, and the "$\mathbb{C}$-versions" of these algebras are isomorphic!

In this class we'll study this notion of potential isomorphism. We'll focus specifically on the algebras which are potentially isomorphic to matrix algebras; we will show that these have a very nice description, and in fact form a group (once we mod out by an appropriate equivalence relation, called **Morita equivalence**)! Tensor products and group cohomology will make appearances.

Prerequisites: Ring theory and group theory

Homework: None

Travis's Classes

Bang! (🌶🌶🌶, Travis, 1 day)
In the 1930's, Alfred Tarski asked just how easy it is to cover a table with planks of wood, and it took until 1950 for Thøger Bang to answer the question. We'll see how this is formulated as a problem in convex geometry and solve it using the now-celebrated and eponymous Bang's lemma. This is a really nice problem with a slick solution that provides a good introduction to convex geometry. And it will give us an opportunity to shout Bang! in class!

Prerequisites: Linear algebra (familiarity working in $\mathbb{R}^d$)

Homework: Optional

Counting with linear algebra (🌶🌶–🌶🌶🌶, Travis, 1–2 days)
In
this

class, we'll talk about how linear algebra can solve combinatorial problems.

Prerequisites: Linear algebra (linear independence and the rank-nullity theorem; the week 1 intro class is more than enough)

Homework: Optional

Derangements (🌶🌶, Travis, 1 day)
The principle of inclusion-exclusion is a general counting principle known for reliably returning correct but complicated formulas. In this class, we'll state and prove the principle, and then apply it to the *derangement problem*, which more or less asks this question: Suppose there are n campers, each with a cookie that they hand to a squirrel. The squirrels run off and then return, returning one cookie to each camper. What's the probability that no camper receives their original cookie? Turns out that it's almost exactly $1/e$—who'da thunk?

Prerequisites: None

Homework: Optional

Greed (🌶🌶🌶, Travis, 1 day)
Who's gonna stop me? is the central question when applying the so-called greedy algorithm to discrete optimization problems. Sometimes, the answer is "Nobody." But sometimes, you're not so lucky. In this class, we'll see some examples of the greedy algorithm (both how it can be spectacularly effective and spectacularly defective) and describe exactly when the greedy algorithm will work using *matroids*.

Prerequisites: None

Homework: Optional

Max flow is max fun (🌶🌶, Travis, 1 day)
The Max-Flow Min-Cut theorem is a well-known theorem, especially in algorithmic graph theory. We'll see how it makes several key results in graph theory seriously easy to prove.

Prerequisites: Know what a graph is

Homework: Optional

VIV'S CLASSES

Applications of finite fields (🌶🌶–🌶🌶🌶, Viv, 1–3 days)
Now that we know, from Aaron's class this week or otherwise, that finite fields exist, let's use them! We will weaponize the multiplicative structure of finite fields against their additive structure in order to construct all kinds of interesting, useful things: finite projective planes; Sidon sets, where the sums of any two elements are all distinct; error-correcting codes; a magic trick; and more! The amount of applications we see will depend on timing.

Prerequisites: Finite fields; specifically knowing for what n there exists a finite field of order n, and understanding the additive and multiplicative groups

Homework: Recommended

How not to prove a group isn't sofic (🌶🌶🌶🌶, Viv, 2 days)
Cayley's Theorem tells us that finite groups are all subgroups of finite permutation groups. A *sofic* group is a possibly-infinite group that we sort of maybe want to have the same property roughly speaking. The group can't always be a subgroup of a finite permutation group, so instead we just require that all finite subsets of our sofic group act kind of like finite subsets of permutation groups. This ends up being a super-useful definition, but we're left with a burning question: do non-sofic groups exist? We don't actually know the answer to this one. We'll spend the class talking about a hopeful candidate for a non-sofic group and one great way not to prove that it isn't sofic.

Prerequisites: Group theory, Bonus group theory part 2 (or knowledge of group actions)

Homework: Recommended

How to juggle (🌶–🌶🌶, Viv, 1–2 days)
In this class, you will learn to juggle. . . in theory.

We'll discuss juggling sequences, a mathematical model for juggling that revolutionized the juggling world!

(Note: when I taught this class four years ago, I did not know how to juggle well enough to do several of the demonstrations that I wanted to do. Since then, I have not practiced juggling at all. This class will feature me attempting and failing to juggle.)

Prerequisites: None

Homework: None

Semi-direct products (🌶🌶–🌶🌶🌶, Viv, 2–4 days)
The most straightforward way to take a product of two groups is the direct product, where for two groups G and H, $G \times H$ is the set of pairs (g, h) for $g \in G$ and $h \in H$, with pointwise operations.

But then. . . there are wilder, more alien group products, which behave in mysterious ways. These are the semi-direct products, where the operations are only semi-pointwise, and where dragons be.

In this class, we'll define semi-direct products and give many examples, including some of my *favorite* examples of groups, like wreath products, and the lamplighter group. We'll also talk about when you can decompose a group as a semi-direct product (hint: often).

Prerequisites: Group theory; Bonus group theory part 2 is helpful but not necessary

Homework: Recommended

The multiplication table problem (🌶🌶🌶, Viv, 1–3 days)
When I was in grade school, we learned our multiplication tables, which looked basically like this:

$\cdot$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

In typing out that table, I typed out one hundred different numbers between 1 and 100 (I literally did this, ugh). But there were a lot of repeats—I only had to type forty-two *distinct* numbers. In particular, I wrote a little less than half of the total numbers from 1 to 100. But what happens to the proportion of numbers between 1 and N^2 that appear in the $N \times N$ multiplication table as N goes to infinity? Does it stay close to half? Does it go to zero? Does it fluctuate?

In this class, we'll answer that question. Along the way, we'll explore several related concepts, like how $\ln \ln x$ is basically constant, and we'll travel through the looking-glass to see other… stranger… multiplication tables in other… stranger… contexts.

Prerequisites: None

Homework: Recommended

Will Dana's Classes

Combinatorial reciprocity: counting through the looking-glass (🌶🌶🌶, Will Dana, 2 days)
Many of the classic counting problems in combinatorics—like counting subsets of a set, colorings of a graph, or integer points in a polyhedron—can be answered with polynomial formulas. Given such a polynomial, we can try plugging negative numbers into it, but it's not immediately clear what this means. What are colorings of a graph with -1 colors, or subsets of a set of -5 elements, or integer points in a cube scaled by a factor of -2?

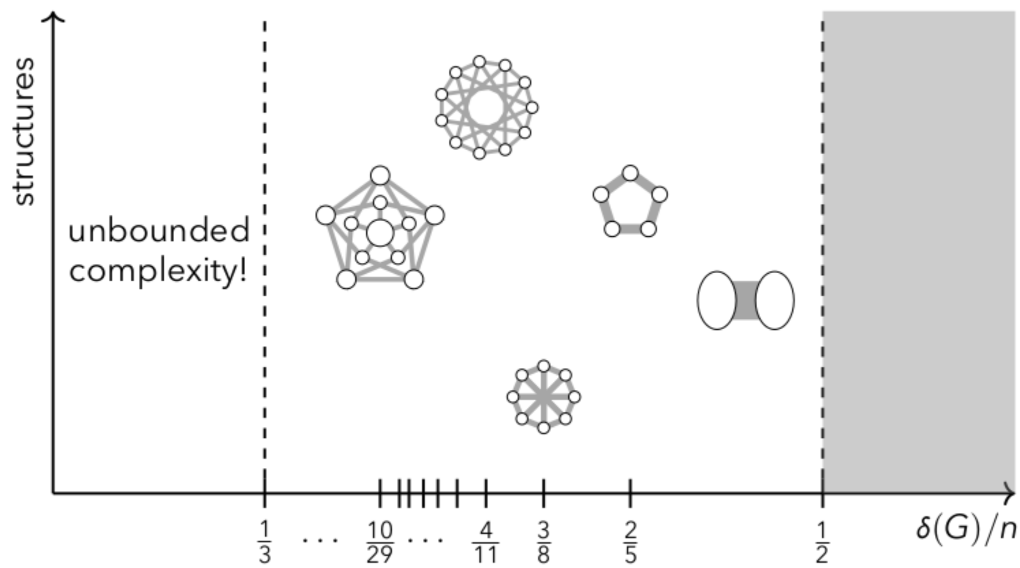
In the first part of this class, we'll go through the suspiciously similar answers to these questions. In the second part, we'll state an elegant theorem of Richard Stanley that generalizes all of them using generating functions, and unpack why it is a generalization.

Prerequisites: Previous exposure to generating functions would be helpful for day 2.

Homework: Recommended

Yuval's Classes

δ_χ and $\delta_{\mathbf{hom}}$ (🌶–🌶🌶🌶, Yuval, 1–2 days)
This is one of my favorite pictures in graph theory:



Come learn what it means!

Prerequisites: Graph theory

Homework: Recommended

Extremaler grapher theoryer (🌶🌶🌶🌶, Yuval, 1–3 days)
More extremal graph theory! There was a ton of stuff that I didn't cover in Week 2 that I'd be happy to cover.

The most likely topic is the Erdős–Simonovits stability theorem, which is an important strengthening of Turán's theorem, and which says that if a graph G is K_r-free and has almost as many edges as the Turán graph $T_{r-1}(n)$, then G "looks like" $T_{r-1}(n)$. Using this, we'll be able to prove some cool facts, such as that $\text{ex}(n, C_5)$ is exactly equal to $\lfloor n^2/4 \rfloor$ for sufficiently large n. But other topics are possible—talk to me if you have requests!

Prerequisites: Extremal graph theory (week 2)

Homework: Recommended

Factoring and packing graphs (🌶–🌶🌶, Yuval, 1–2 days)
Recall that *Bear, Camper, Squirrel* is played like Rock, Paper, Scissors, but modeled after everyday Colby campus interactions. Squirrel chases Bear off campus, Camper chases Squirrel up trees, and Bear eats Camper for breakfast.

I want to organize a Mathcamp-wide Bear, Camper, Squirrel tournament among all 124 campers, so that every pair of campers plays. But that requires running $\binom{124}{2} = 7626$ games, and that'll take a while. So maybe I can run the games in parallel: have all 124 campers playing at once, so that 62 games happen at a time.

Luckily for me, 7626 is divisible by 62, so I'd only need 123 rounds of games. But can I actually organize the tournament so that every camper plays every other camper, while ensuring that 62 games happen simultaneously for 123 rounds?

This is an example of a *graph factorization* problem. In this class, we'll solve this problem, see some other beautiful solutions to related problems, and very quickly get to the absolute limits of human knowledge.

Prerequisites: Graph theory

Homework: Optional

Nilpotent matrices (🌶🌶🌶, Yuval, 1–2 days)
If you were in Moon's colloquium last week, you learned about nilpotent groups, such as the Heisenberg group. This class has nothing to do with that.

A nilpotent matrix is one with a power that is the zero matrix. For example, the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}$$

satisfies

$$A^2 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ -4 & 0 & -2 \end{pmatrix}$$

and

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There's a lot to say about such matrices! We'll see some such things, including a surprising connection to graph theory.

Prerequisites: Linear algebra, graph theory

Homework: Optional

Singularity of random matrices (🌶🌶🌶–🌶🌶🌶🌶, Yuval, 2 days)
Suppose I flip a coin n^2 times, and record the results in an $n \times n$ grid: I write a 0 if the coin came up tails, and a 1 if it came up heads. In this way, I end up with a random $n \times n$ array of zeros and ones.

But now I can interpret this as a matrix, and start asking questions about it! For example, what is the probability that this random $n \times n$ matrix is singular (i.e. not invertible)?

This turns out to be a surprisingly difficult and subtle question, and there have been major breakthroughs on it during each of the last three years. It also turns out to be closely related to *anticoncentration*, an important concept in probability theory, which itself turns out to be closely related to the structure of partially ordered sets. In this class, we'll see many of these connections, and prove that the probability that the random matrix is singular tends to 0 as $n \to \infty$.

Prerequisites: Linear algebra, probability

Homework: Optional

The combinatorial Nullstellensatz (🌶🌶🌶–🌶🌶🌶🌶, Yuval, 2–3 days)
The combinatorial Nullstellensatz is a very powerful technique for solving intractable-looking combinatorial problems. The core of the technique is:

(1) Write down a magic polynomial that encodes your problem.
(2) Prove that the polynomial does not vanish everywhere.
(3) ???????
(4) Profit.

In this class, I'll discuss a variety of applications of the combinatorial Nullstellensatz, including topics from high-dimensional geometry, additive number theory, and graph theory.

Note: The combinatorial Nullstellensatz is, apparently, very useful for solving certain olympiad-style problems (including an infamous problem from the 2007 IMO). This class will not be about problem solving.

Prerequisites: Know what it means to say that the integers mod a prime p form a field. Also, it would be helpful to have some familiarity with graph theory, but I will define all relevant concepts as they come up.

Homework: Optional

The happy ending problem (🌶–🌶🌶, Yuval, 1–2 days)
Here's a nice exercise. Prove that among any five points in the plane (no three collinear), there are four of them forming the vertices of a convex quadrilateral.

If that was too easy, try proving that among any 33 points in the plane (no three collinear), there are seven of them forming the vertices of a convex heptagon.

OK, that sounds harder (and, in fact, it's an open problem). But a great deal is known about this beautiful problem, which lies at the intersection of Euclidean geometry and combinatorics, and which we'll cover in this class.

Prerequisites: None

Homework: Optional

The uncertainty principle (🌶🌶🌶–🌶🌶🌶🌶, Yuval, 1–2 days)
If you talk to a physicist, they'll tell you that Heisenberg's uncertainty principle says that

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

If you talk to a Fourier analyst, they'll tell you that Heisenberg's uncertainty principle says that

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)| \, \mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \xi^2 |\widehat{f}(\xi)| \, \mathrm{d}\xi\right) \geq \frac{\|f\|_2^2 \|\widehat{f}\|_2^2}{16\pi^2}.$$

If you talk to me, I'll tell you that the *real* uncertainty principle is

$$(*) \qquad \frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|\widehat{f}\|_1}{\|\widehat{f}\|_\infty} \geq 1.$$

Come learn what uncertainty principles are, and why $(*)$ is the *real* one.

Prerequisites: You should know what the Fourier transform is (e.g. from Ben's Overly convoluted plans class)

Homework: Optional

Why June Huh won the Fields medal (🌶🌶–🌶🌶🌶, Yuval, 1 day)
If you were in David Roe's colloquium about hyperspheres, you heard about why Maryna Viazovska won the Fields medal a few weeks ago: she found the optimal way to pack spheres in dimensions 8 and 24.

If you're in Viv's class about the distribution of primes, you will probably hear about why James Maynard won the Fields medal a few weeks ago: he found new and powerful techniques for understanding the gaps between consecutive prime numbers.

If you come to this class, you'll hear why June Huh won the Fields medal a few weeks ago: he proved a variety of powerful theorems connecting Euclidean geometry, combinatorics, and algebraic geometry. I'll also tell you a bit about his personal history, which I find very interesting and inspiring.

Prerequisites: None

Homework: None

Zack's Classes

Borsuk's conjecture (, Zack, 2 days)
In 1932, Karl Borsuk published a paper entitled "Drei Sätze über die n-dimensionale euklidische Sphäre," which became famous for the first proof of the celebrated Borsuk–Ulam theorem. Elsewhere in the paper, though, Borsuk showed that it's always possible to decompose an n-dimensional ball into $n+1$ sets of diameter[4] strictly less than 1, but that in general n sets is not sufficient. This motivated him to ask the following question:

Can every bounded subset E of $\mathbb{R}^n$ be partitioned into $n+1$ subsets, each of diameter smaller than that of E?

Over the next 60 years the conjecture was resolved positively for many special cases: in dimensions 2 and 3, as well as whenever E is smooth or centrally symmetric.

Finally, in 1993 Kahn and Kalai proved that the answer in general to Borsuk's question is *no*, at least for n at least 2000 or so. Their construction is one of the gems of modern combinatorics: it's clean, beautiful, and so unmotivated it must have been divine inspiration.

We'll explore Kahn and Kalai's proof, and be amazed by how little it has to do with geometry and how much it has to do with a TPS problem from week 2.

Prerequisites: Linear algebra

Homework: Optional

Superpatterns (, Zack, 2–3 days)
Suppose Assaf, Ben, Charloutte, Dan, and Emily stand in a row, in that order, and suppose Ben is taller than Emily, who is taller than Charloutte, who is taller than Assaf, who is taller than Dan. Then, something terrible happens to two of them. We have a few cases:

- If Ben and Dan disappear, the remaining staff are in increasing order of height. We call this a 123-pattern.
- If Dan and Emily suddenly vanish, we have the remaining pattern $A > C > B$. Call this a 132-pattern.
- If Assaf and Ben are pulled into the void, Charloutte is now in the middle—taller than Dan, but shorter than Emily. Call this a 213-pattern.
- If Charloutte and Emily are nowhere to be found, we similarly a 231-pattern.
- If Assaf and Dan are "at lunch," the remaining staff form a 312-pattern.
- Finally, if Assaf and Emily are being devoured by bears, during that process the remaining staff are in decreasing order—a 321-pattern.

So it turns out that we can get any "sub-order" from this initial permutation. This tells us that the ordering we started with, 25314, is a **superpattern**. One question is: could we see this phenomenon with fewer than 5 staff? Well, if Ben wasn't invited to the staff party, we could only have 4 ways of choosing 3 staff, so we could never get all 6 patterns. But what about larger values of 3? Let's let $f(k)$ denote the minimum number of staff we'd need to see all $k!$ subpatterns of size k.

[4] The **diameter** of a set is the maximum distance between any two points

Arratia, who introduced this problem, showed the (pretty straightforward) bounds

$$k^2 \geq f(k) \geq k^2/e^2,$$

and conjectured that the lower bound was asymptotically tight. The upper bound saw successive improvements over time, but the lower bound proved trickier, until 2020 when it was improved very slightly to

$$f(k) \geq k^2/e^2 \cdot 1.00007.$$

We'll take a tour through this and related problems, including a result whose only known proof was posted anonymously on 4chan, and spend some time going through as much of the proof of the superpattern lower bound as we can, which is from a paper of mine.

Prerequisites: None, but some combinatorics background will be helpful.

Homework: Recommended

The hardest IMO problem (🌶🌶🌶🌶, Zack, 1 day)
2007 was sort of a weird year. Rickrolling was sweeping the internet, the first iPhone was released, Britney Spears shaved her head, and the IMO featured two extraordinarily difficult problems. The latter, problem 6, is considered by some to be the hardest problem ever to appear on the IMO, and received only 5 full solutions (one due to a future Fields medalist!).

This problem, about arranging planes in 3-dimensional space, is largely responsible for the prevalence of a technique known as the Combinatorial Nullstellensatz, originally due to Noga Alon. We'll see a quick statement and proof of the Nullstellensatz before moving on to see how we can use it to knock out some very difficult combinatorics problems, including this one and (time permitting) the hardest TSTST problem.

Prerequisites: None

Homework: Recommended

The most remarkable finite group (🌶🌶🌶🌶, Zack, 3–4 days)
Often in math, we like results and theorems that generalize. But some mathematical objects defy generalization, shining with their own unique beauty and intriguing mathematicians for millennia. These "sporadic" objects often involve numerical coincidences and unexpected symmetry, and are super weird and great.

We'll start with the idea of a perfect error-correcting code and see that, because of what seems like a numerical coincidence, one might exist in 23 dimensions. We'll construct it using the icosahedron, and show that though we "baked in" 2-, 3-, and 5-fold symmetry, miraculously it also has 7-fold, 11-fold, and 23-fold symmetry! We'll see a few more of the crazy properties of this object and its huge symmetry group, which Conway called "the most remarkable of all finite groups." Finally, we'll discuss a few even deeper related objects and their connections to sphere packing and number theory.

Prerequisites: Group theory

Homework: Recommended

Zoe's Classes

Elo ratings (🌶🌶–🌶🌶🌶, Zoe, 1–3 days)
If you have ever played online chess you may have noticed that as you win and lose games you get a number attributed to your account. This is your Elo rating. It is modified differently depending on the skill of the player you play against, and different confidence ratings. This rating system has been further adapted for other online games and even team games. If you are curious how this is modeled and decided mathematically this is a great class for you!

Prerequisites: None
Homework: Optional

Squirrel math (🌶️–🌶️🌶️🌶️, Zoe, 1–2 days)
If we looked at the different states squirrels can be in and tried to figure out how likely they are to change states we could then find out how long it would take for them to reach any state. What if squirrels have to wait in line? They probably wouldn't love that. We will talk about Markov chains and some of their applications to queueing theory using mostly squirrel motivated examples.

Prerequisites: None
Homework: Recommended

Who are my archers? Spectral sequences (🌶️🌶️🌶️–🌶️🌶️🌶️🌶️, Zoe, 1–3 days)
If you have ever been in a class with way too many commutative diagrams, then maybe you will appreciate the fact that spectral sequences reduce a few different commutative diagrams into one sequence! Spectral sequences are a tool in homological algebra and algebraic topology that often come up in some of the more complicated computations.

Prerequisites: Homology would be good to know
Homework: Optional

Zoe and Eric's Classes

Partially Ordered Galois Group Equivalence RelationS (🌶️🌶️🌶️, Zoe and Eric, 2–3 days)
As it turns out, Frobenius is all powerful. If you have been enjoying some of the finite fields and Galois theory classes, this will be a totally poggers follow up. It turns out you can secretly do all of Galois theory for finite fields (even infinite extensions of them!) without knowing any Galois theory at all. That's absolutely poggers, ya know. If you were looking closely, you may have noticed that one of the topology-enthusiastic staff is helping to teach this class and that is no coincidence! From the actions we will look at we will also get some fundamentally poggers topology showing up.

This class is really going to be soooo poggers. Actual math is included.

Prerequisites: Familiarity with fields and field extensions. You don't actually need to know Galois theory!
Homework: None