CLASS DESCRIPTIONS—WEEK 1, MATHCAMP 2023

Contents

9:10 Classes 1
10:10 Classes 4
11:10 Classes 6
1:10 Classes 8
Colloquia 11

9:10 Classes

Cubic curves (🌶🌶🌶, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)
A curve in the x, y-plane is called a cubic curve if it is given by a polynomial equation $f(x, y) = 0$ of degree 3. Compared to conic sections (which have degree 2), at first sight cubic curves are unpleasantly diverse and complicated; Newton distinguished more than 70 different types of them, and later Plücker made a more refined classification into over 200 types. However, as we'll see, by using complex numbers and points at infinity we can bring a fair amount of order into the chaos, and cubic curves have many elegant and excellent properties. One of those properties in particular, which is about intersections, will allow us to prove a beautiful theorem of Pascal about hexagons and conic sections, and it will also let us define a group structure on any cubic curve—well, almost. We may have to leave out a singular ("bad") point first, but a cubic curve has at most one such point (which may be well hidden; for example, $y = x^3$ has one!), and most of them don't have any. Cubic curves without singular points are known as *elliptic* curves, and they are important in number theory, for example in the proof of the Fermat–Wiles–Taylor theorem (a.k.a. "Fermat's Last Theorem"). However, in this week's class we probably won't look at that aspect at all, and no knowledge of number theory (or even groups) is required. With any luck, along the way you'll pick up some ideas that extend beyond cubic curves, such as how to deal with points at infinity (using "homogeneous coordinates"), what to expect from intersections, and where to look for singular points and for inflection points.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: A bit of differential calculus, probably including partial derivatives; complex numbers; a bit of experience with determinants.

Fourier series (🌶🌶, *Jonathan Tannenhauser*, $\boxed{\text{TW}\Theta\text{F}}$S)
A Fourier series is an expansion of a periodic function as a sum of sines and cosines. After studying how and why Fourier series work and seeing some basic examples, we'll look at two applications.

(1) The heat equation. Joseph Fourier introduced Fourier series in 1807 to solve the differential equation that models how heat diffuses through a given region.
(2) The isoperimetric problem. Among closed plane curves of fixed perimeter, which curve maximizes the enclosed area? Adolf Hurwitz in 1901 used Fourier series to show that the answer is a circle.

Homework: Recommended

Class format: Interactive lecture
Prerequisites: None.

Hlod onto yoru ahts![1] (🌶🌶, Tim!, TWΘF$\boxed{\text{S}}$)
Here's a game for fifteen campers: Every camper will have a random color of hat placed on their head. Each hat is either black or white with 50% probability, independently chosen. A camper can't see their own hat, but they can see the other fourteen hats. Simultaneously, all fifteen campers make a choice: they can either guess the color of their own hat, or remain silent. They must all make this decision simultaneously, they can't use anyone else's guess (or lack thereof) as part of their decision. The game is cooperative; they all win the game if at least one camper guesses their hat color correctly and nobody guesses incorrectly. They all collectively lose the game if *anybody* guesses wrong, or if everybody remains silent.

The campers are allowed to strategize before they receive their hats. What strategy should the campers use to maximize their probability of winning? What is that probability?

Certainly, they can win 50% of the time: Before the game, they agree on one camper who will make an arbitrary guess, while the other fourteen campers remain silent. That camper has no information whatsoever about their own hat—they know the other campers' hat colors, but those were chosen independently so are of no help, and they can't communicate with the other campers in any way. So their guess is just a shot in the dark; they have a 50% of being right by pure luck.

Perhaps unsurprisingly, they can't win 100% of the time. Again, no camper has any information about their own hat color, and can't communicate with anyone before they guess. So if anyone guesses at all, there is some chance they will be wrong. And if nobody guesses, they just lose.

But perhaps very surprisingly, they *can* win more than 90% of the time. When I first heard about this, it sounded impossible: Every camper has a 50% chance of being wrong if they guess, and if multiple campers guess, the situation can only get worse, right?

Homework: None
Class format: Interactive lecture
Prerequisites: None

[1] P.S. I want to apologize about the title of this class; it was supposed to be "Hold onto your hats!" But I guess you figured that out. I only slightly misspelled my class title; it was *approximately* correct (at least it had all the right letters). So you were able to figure out *approximately* what the title was supposed to be. Wait, actually, you were probably able to figure out *exactly* what the title was supposed to be. You're able to understand exactly what my typos were and how to fix them. That's sort of wild and exciting, isn't it? That I give you something that's only *approximately* right, but you can figure out *exactly* what I mean? What a useful superpower. Think about this in other contexts. Let's say you are sending photos back to earth from a Mars rover, or just sending a text message to a friend. The physical world is an imperfect place, so some of those electronic signals will inevitably get a bit distorted. But even if the *signal* is a little distorted, you don't want the *message* (the photo or the text message) to be corrupted even a little bit. You want to encode the message in such a way that even the signal is distorted, you can (or really, your computer/phone can) correct the error and figure out what the original message was. To allow the recipient to correct errors in your message, you want to encode your message using an **error-correcting code**. That basically means you want to add a little redundancy to the message you send. English is inherently redundant; that's why when I type "yoru" you can guess that I meant "your". English is kind of inconsistently redundant though; if I tell you I moved one letter of a word and got "zeabr" you can guess that the word was "zebra", but if instead I give you "daer", you don't know if the word was "dear" or "dare". So English is slightly inefficient as an error-correcting code in that sense. And you would have a harder time if I were allowed to move multiple letters. Different error correcting codes are able to handle different amounts of data correction, and different error correcting codes are more consistent/efficient than others in terms of how much error correcting they can do in different situations. For example, **Hamming codes** are codes that can only do a bit of error-correction, but they are very consistent about how much error correction they can do; in that sense they make very efficient use of their signal. Sorry for rambling; I could go on about this for a while. Actually, I think I'll stop rambling, and instead of that silly hat game, I'll teach a class about error-correcting codes and Hamming codes in particular.

Inspecting gadgets (🌶🌶🌶, Della, $\boxed{\text{TW}\Theta\text{FS}}$)
Imagine you're designing a level in a video game like Mario, and want to construct something the player can only use in certain ways: maybe a hallway that they can go through in only one direction, or a hallway that they can't go through until they visit some other location. How would you do this? If you build one of them, could you use it to build the other?

A good way to model this kind of problem is with 'gadgets', which are an abstraction for self-contained components which lets us stop thinking about the details of the game the gadgets were built in. Once you have such gadgets, you can sometimes use them to build other, cooler gadgets—but not always. How can you know when this sort of 'simulation' is and isn't possible?

This class will be about a particular genre of gadgets with a lot of fun simulations. You'll spend the first half working together to tryng to find as many simulations as possible, which feels like solving puzzles or designing video game levels. Towards the end, I'll show you general results about when simulations are or aren't possible: we'll see a gadget that can simulate *every* possible gadget!

Gadgets originally come from hardness proofs in computational complexity (e.g. NP-hardness), but this class will study gadgets in their own right without that context. If you've seen and enjoyed NP-hardness proofs in the past, you'll probably have fun simulating gadgets.

I intend to offer a project on gadgets—there's a lot more to learn that what I can cover in a class, including a lot of approachable open problems.

Homework: Recommended

Class format: Days 1-3 will be mostly IBL; days 4-5 will be lectures.

Prerequisites: None.

Introduction to linear algebra (🌶, Narmada, $\boxed{\text{TW}\Theta\text{FS}}$)
Linear algebra is ostensibly the study of linear equations, but that's like calling writing the study of the alphabet. The real beauty of the subject lies in the techniques we'll learn.

This course will take a non-traditional approach by focusing on matrices: the star players in finite dimensions. By the end of this course, you'll understand what it means to say a matrix behaves like a real number, and what that tells us about the geometry of high-dimensional spaces.

Homework: Recommended

Class format: Lecture with some group work

Prerequisites: None.

Required for: Representation theory of the symmetric groups (W2); The transcendence of many numbers (including π and e) (Week 2 of 2) (W2); Polytopes (Week 1 of 2) (W2); Solving equations with origami (W3); How to count rings (W3); Polytopes (Week 2 of 2) (W3); Linear algebra through knots (W3); Finite fields (W4); Vhat are your vectors vorth? or, part of the part of combinatorics and discrete geometry that ve can do easily vith linear algebra (W4); Gaussian magic (W4); Quantum computing (W4)

Khinchin's constant and the ergodic theorem (🌶🌶🌶🌶, Ben, $\boxed{\text{TW}\Theta\text{FS}}$)
Every irrational number can be written as a "continued fraction," that is, as an expression like

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

where $a_1, a_2, \ldots$ are positive natural numbers and a_0 is an integer. Here's a fun fact: if you take the geometric mean of $a_1, a_2, \ldots, a_n$, and let n go to infinity, this will almost always converge to a specific number, which is about 2.685. This is Khinchin's constant.

In this class, we'll investigate how to prove this kind of result. To do so, we'll talk a bit about integration, a bit about limits, and just a little bit about bakers. Why bakers? Why, because one of the things we'll be talking about is this mysterious thing called "ergodicity," and one of the classic examples of an "ergodic map" is called the baker's transformation!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Familiarity with integration and limits—e.g. have taken some calculus class covering both of these topics. No knowledge about continued fractions is needed!

10:10 Classes

Discreet calculus (shh!) (🌶️, Travis, $\boxed{\text{TW}\Theta\text{FS}}$)
There's a parallel world of calculus out there—the one Newton and Leibniz don't want you to know about! Why bother with limits when you can just plug in $\epsilon = 1$?

This class is about a discrete version of calculus: we'll see versions of the derivative, integral, Fundamental Theorem of Calculus, Taylor series, and differential equations. This is a really nice topic, because on the one hand, it's more concrete than calculus (no limits), but it still maintains much the same story. And it has its own rewards, as we'll find out when we evaluate tricky sums simply using "summation by parts" or solve recursions lickety-split.

A note about the chilis: This class is 1 chili because it'll be fairly relaxed. We won't be speeding through a lot of math, but we'll see some interesting things and have a good time along the way.

Homework: Optional

Class format: Interactive lecture

Prerequisites: You don't need to know calculus to take this class. That said, if you have seen the definition of a derivative or the Fundamental Theorem of Calculus, you'll be able to appreciate how discrete calculus compares to "traditional" calculus, and some of the things we do in class might seem more natural.

Homotopy groups of spheres (🌶️🌶️, Kevin, TW$\Theta\boxed{\text{FS}}$)
The spheres are some of the simplest spaces to define, but the mysteries of their topology have stumped mathematicians for nearly a century. In this class, we'll take a look at the open problem of computing the homotopy groups of spheres, which describe the various ways spheres of different dimensions can wrap around one another. We'll talk about how the problem is set up, why it's so difficult, and what we currently know.

I'll introduce all the key objects along the way, so this class will be a short preview of algebraic topology for those of you who think homotopies sound cool. This class will feature a ton of pictures: pictures of spheres, maps, and homotopies on Day 1 and pictures/demos of spectral sequences on Day 2.

Homework: Optional

Class format: Interactive lecture

Prerequisites: Some familiarity with the notion of continuity would be helpful. Group theory could also be helpful but is definitely not required. Don't worry if you haven't seen any topology before!

Introduction to number theory (🌶️🌶️, Mia, $\boxed{\text{TW}\Theta\text{FS}}$)
At some point in your life, you might have wondered about the following *very common* questions:

(1) Given that yaks cost \$23 and xylophones cost \$5, is it possible to spend exactly \$303 on yaks and xylophones?
(2) If Pete the Pirate claims that he split a giant square chocolate bar among 7883 of his minions and had 13 squares left-over, should you believe him? Or is he lying?
(3) If an evil rhinovirus (named Vincent) is planning to take over the world and the only way to stop him is to unlock the VirusVanquisher 2001 which asks you to compute the units digit of 7^{1001} mod 12 with only a pencil and paper, is all lost or is there a way??

Yet underpinning these goofy questions is the elegant structure of the integers. And the goal of this class is to better understand that structure. We'll start with the study of linear Diophantine equations, then transition to the world of modular arithmetic where we'll build from the study of order modulo n to primitive roots to quadratic reciprocity.

Although this class will follow an interactive lecture format, it will be inquiry-focused. We'll start each day with a simple-to-state (yet not necessary easy!) question, build the theory to answer it, and then use that as a springboard to generate our next line of inquiry.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Metric spaces (, Krishan, TWΘFS)
Choose any number between 0 and 1 (non-inclusive) and use a calculator to find its cosine. Then take that number and find its cosine. Repeat this process again and again. Eventually you'll find that the results are approaching a number starting with 0.739085... if they aren't you haven't done it enough times. This number is an irrational number known as the Dottie Number, which is the unique fixed point of the cosine function. Surprisingly, the reason this process works is that —cos(x) - cos(y)— is less than —x-y—, due to a result known as the Banach Fixed Point Theorem. In this class we will work toward proving this theorem in a general setting. We'll discuss metric spaces, which are sets together with a way to measure the distance between points. We'll see how to tell when a metric space has "holes", and we'll see how to generalize the extreme value theorem from calculus. The class will borrow some ideas from calculus, but don't worry if you haven't taken it yet. We will approach these ideas from a different (ε-free) direction, so no knowledge of calculus is required.

Homework: Optional

Class format: Interactive lecture

Prerequisites: None.

Multivariable calculus crash course (, Mark, TWΘFS)
In real life, interesting quantities usually depend on several variables (such as the coordinates of a point, the time, the temperature, the number of campers in the room, the real and imaginary parts of a complex number, ...). Because of this, "ordinary" (single-variable) calculus often isn't enough to solve practical problems. In this class, we'll quickly go through the basics of calculus for functions of several variables. As time permits, we'll look at some nice applications, such as: If you're in the desert and you want to cool off as quickly as possible, how do you decide what direction to go in? What is the total area under a bell curve? What force fields are consistent with conservation of energy? One reason, and maybe the best reason, to take this crash course right now rather than waiting until you encounter the material naturally after BC calculus and/or in college, is to be able to take the course on functions of a complex variable (which have many amazing features) that starts in week 3.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic knowledge of single-variable calculus (both differentiation and integration)

Required for: Functions of a complex variable (Week 1 of 2) (W3); Calculus of variations (W3); Functions of a complex variable (Week 2 of 2) (W4); Gaussian magic (W4)

Reverse mathematics (, Steve, $\boxed{\text{TW}\Theta\text{FS}}$)
Usually in mathematics we proceed from *axioms* to *theorems*. But axioms are really heavy and bulky, and RyanAir doesn't allow carry-on bags. In order to figure out what axioms we really need, we're going to have to go backwards: start with the theorems we care about and figure out what axioms are needed to prove them.

This general line of thought has a long history; it relates, for example, to early investigations in the foundations of geometry (what can you prove without the parallel postulate?), abstract algebra (what algebraic properties of the integers are behind unique factorization?), and set theory (where does the axiom of choice show up?). **Reverse mathematics** is a more recent aspect of this. In reverse mathematics, we focus on theorems which are about natural numbers or sets of natural numbers. Some examples of such statements include Ramsey's theorem, weak Konig's lemma ("every infinite binary tree has an infinite path"), and determinacy principles (such as "Every finite-length two-player perfect-information game is determined"). We measure "axiom strength" along two axes:

- How much **induction** do we need?
- What sort of **set formation** (= **comprehension**) do we need?

In the development of this subject, connections with computability theory and (very) theoretical computer science emerge. Reverse mathematics has recently been described as "the playground of logic"—let's have fun!

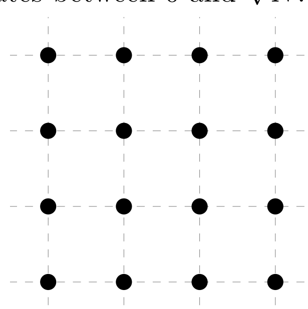
Homework: Recommended

Class format: Lecture

Prerequisites: None

11:10 Classes

Erdős's distinct distance problem (, Neeraja, $\boxed{\text{TW}\Theta\text{FS}}$)
If P is a set of N distinct points in the plane, the set of distances between points in P is called the distance set $\Delta(P)$. The size of the distance set is at most $\binom{N}{2}$ and we can find examples where this upper bound is realized (for instance, choose the endpoints of a scalene triangle). A much more difficult question is to ask how small the distance set can be. For example, $P = \{(1,0),(2,0),\ldots,(N,0)\}$ gives $|\Delta(P)| = N$. Paul Erdős discovered a better example by taking his points in a square lattice, that is, taking all points with integer coordinates between 0 and $\sqrt{N}$:



For this set, $|\Delta(P)|$ works out to be about $N/\sqrt{\log(N)}$. Based on the lattice example, Paul Erdős conjectured in 1946 that $|\Delta(P)| \geq N/\sqrt{\log N}$. This conjecture was proved by Guth and Katz in 2015 (or rather almost proved, as they showed a lower bound of $N/\log N$). In this course we will look at their proof, which uses topological tools such as the polynomial ham sandwich theorem, algebraic geometry tools and incidence geometry arguments.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: vector geometry (vectors in $\mathbb{R}^3$, rotations in $\mathbb{R}^3$, what it means for vectors to be orthogonal).

Geometry, under construction (🌶, Arya, $\boxed{\text{TW}\Theta\text{FS}}$)
From the age of dinosaurs to the age of Gauss, straight-edge and compass constructions have been peak enjoyment activities. But like, what stuff can you really construct? How much information do you really need to construct, say, a triangle? Other polygons? How much algebra can you build geometrically? Get your hard hats and your thinking caps, as we don appropriate headgear and explore the wonders of constructive geometry!

Homework: Recommended

Class format: Interactive lectures and group work.

Prerequisites: None.

Information theory and the redundancy of English (🌶🌶, *Mira Bernstein*, $\boxed{\text{TW}\Theta\text{FS}}$)
NWSFLSH: NGLSH S RDNDNT!! (BT DN'T TLL YR NGLSH TCHR SD THT)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting or to have a conversation in a crowded room. The redundancy works as a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

How redundant is English? There are two ways to interpret this question:

- How much information is conveyed by a single letter of English text, relative to how much could theoretically be conveyed? (But what is information? How do you measure it?)
- How much can we compress English text? If we encode it using a really clever scheme, can we reduce the length of the message by a factor of 2? 10? 100? (But how will we ever know if our encoding is the cleverest possible one?)

Fortunately, the two interpretations are related. In this class, we will first derive a mathematical definition of information, based on our intuitive notions of what this word "should" mean. Then we'll prove the Noiseless Coding Theorem: the degree to which a piece of text (or any other data stream) can be compressed is governed by the actual amount of information that it contains. We'll also talk about Huffman codes: the optimal way of compressing data if you know enough about its source. Finally, we'll answer our original question—how redundant is English?—in the way that Claude Shannon, the father of information theory, originally answered it: by playing a game I call *Shannon's Hangman* and using it as a way of communicating with our imaginary identical clones.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Introduction to group theory (🌶🌶🌶, Eric, $\boxed{\text{TW}\Theta\text{FS}}$)
Groups are a super common type of mathematical object. Whenever you've got a way of sticking two mathematical things together to make a third in a reasonable way there's very likely a group in the picture! Familiar processes like addition and multiplication will live in groups, as well as many more wild and strange ones (shuffling cards! performing music on bells! twisting strands of yarn!).

This particular course on group theory will aim to emphasize two things:

- group theory itself is an introduction to the subject of *abstract algebra* so we'll devote time to learning the style of axiomatic arguments used in this subject,

- one of the things that makes group theory interesting is the absolutely insane variety of groups in existence, and so we'll get hands on experience with both some traditional and less traditional examples, and prove or state some results which hint towards the plethora of groups you'll encounter after this class.

Homework: Recommended

Class format: Interactive lectures! We'll move at a fairly brisk pace as there's a lot of exciting material to cover, but there will be activities interspersed throughout helping us drill down into some core concepts.

Prerequisites: None!

Required for: Representation theory of the symmetric groups (W2); Introduction to model theory (W2); Take it to the limit (W2); Coxeter groups (W3); The outer life of inner automorphisms (W4); Braid groups (W4)

Knot invariants (🌶🌶🌶, Raj, TWΘFS)
In this course, we will learn all about how to distinguish knots. Knot invariants are tools we use to distinguish knots, and some major examples include tricolorability and the Jones polynomial. Hopefully by the end of the course, you will be able to tell your friends why the Kinoshita- Teresaka mutant knots are different!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

1:10 Classes

Bhargava's cube (🌶🌶🌶, Kevin, TWΘFS)
This class is about a mathematical story more than a thousand years in the making. In the 7th century, Brahmagupta discovered his identity

$$(x^2 - Ny^2)(z^2 - Nw^2) = (xz + Nyw)^2 - N(xw + yz)^2.$$

In 1801, Gauss generalized Brahmagupta's identity to polynomials of the form $ax^2 + bxy + cy^2$ (a.k.a. binary quadratic forms) by figuring out how to compose different binary quadratic forms. In 2014, Manjul Bhargava won the Fields Medal, partly for figuring out how to generalize Gauss composition to higher dimensional spaces like $2 \times 2 \times 2$ cubes.

In this two day class, we'll learn about the history and theory of binary quadratic forms and the elegant mathematics that goes into understanding them.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Number theory (e.g. you should be comfortable with modular arithmetic)

Infinite arithmetic (🌶🌶🌶, Susan, TWΘFS)
Infinity is a funny concept. Many infinite sets that look like they ought to be different sizes actually turn out to be the same, like the natural numbers, the integers, and the rational numbers. But there are definitely different sizes of infinity—there are more real numbers than rational numbers, and we can prove that any set is smaller than the set of its subsets.

This class deals with different sizes of infinity and explores how they're organized and what happens when we attempt to add, multiply, and exponentiate these infinite numbers. We'll see that cardinal addition and multiplication are beautifully simple (though for complicated reasons), and that cardinal exponentiation is beastly (but can be tamed given the right assumptions). Our journey will take us through another way of categorizing infinities—the ordinal numbers—and a discussion of the generalized continuum hypothesis.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Required for: Infinite Ramsey theory (W2)

Is it possible to gamble successfully? (🌶🌶, Tanya, $\boxed{\text{TW}\Theta\text{FS}}$)
The subject of this course is incredibly dangerous - it almost got banned from Mathcamp 2023! Here you will learn the mathematics behind why gambling can leave you in financial ruin or the richest person on Earth. Along the way, you will be introduced to the foundational concepts of probability theory, which is an area of study that seeks to discover patterns in the inherently unpredictable behavior of random phenomena. Some of the initial attempts to develop a robust formal theory of probability were made in 17th century France thanks to a gambler who was interested in improving his chances of winning and happened to be friendly with Blaise Pascal. In this course, we take inspiration from probability theory's historical roots and investigate a simple gambling scheme that can be mathematically represented by a random walk. Methods of analysis we will be learning will be applicable to the study of a much broader class of stochastic processes known as Markov Chains (unfortunately beyond the scope of the curriculum, but I hope you will be inspired to learn about these objects in the future). This class is inquiry-based, so you will be discovering all these exciting concepts and techniques on your own, with gentle nudging and occasional hints.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic set theory

Mathcamp crash course (🌶, Charlotte, $\boxed{\text{TW}\Theta\text{FS}}$)
Math is useless unless it is properly communicated. Most of math communication happens through a toolbox of terminology and proof techniques that provide us with a backbone to understand and talk about mathematics. These proof techniques are often taken for granted in textbooks, math classes (even at Mathcamp!) and lectures. This class is designed to introduce fundamental proof techniques and writing skills in order to make the rest of the wonderful world of mathematics more accessible. This class will cover direct proofs from axioms, proofs using negation, proofs with complicated logical structure, induction proofs, and proofs using cardinality. If you are unfamiliar with these proof techniques, then this class is highly recommended for you. If you have heard of these techniques, but would like to practice using them, this class is also right for you. Here are some problems that can assess your knowledge of proof writing:

- Negate the following sentence without using any negative words ("no", "not", etc.): "If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."
- Given two sets of real numbers A and B, we say that A dominates B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two disjoint, nonempty sets A and B such that A dominates B and B dominates A.
- Prove that there are infinitely many prime numbers.
- Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
- Define rigorously what it means for a function to be increasing.

- What is wrong with the following argument (aside from the fact that the claim is false)? On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

  Proof. We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. □

If you would not be comfortable writing down proofs or presenting your solutions to these problems, then you can probably benefit from this crash course. If you found this list of questions intimidating or didn't know how to begin thinking about some of them, then you should definitely take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Homework: Required

Class format: Mix of lecture and group work.

Prerequisites: None.

Problem solving: geometry galore (, Ian, TWΘFS)

Welcome to the Geometry Galore! Here we befriend circles, lines, and angles to complete several quests. The quests involve empowering points, searching for radical axes, and saving our majesty Ceva, Menelaus, Desargue, and Pascal!...

Since the Galore involves problem solving, it is strongly encouraged that the participants work on problems outside of the 50-minute time limit, and complete prerequisite quests before each class to make the most of the Galore...

Once you are equipped with the enthusiasm to complete all the quests, the Galore is yours - good luck...

==Start Quest==>

Homework: Required

Class format: The first few minutes will have a lecture format, where we go through theorems and solve problems together. Then the rest of the class will be working on problems individually or in groups.

Prerequisites: A good understanding of basic geometry (some experience with Math Olympiad is recommended)

The transcendence of many numbers (including π and e) (Week 1 of 2) (, *Dave Savitt*, TWΘFS)

We'll prove a result called the Schneider–Lang theorem, which will allow us to show that a great many numbers are transcendental. For example, we will be able to show that e^{α} is transcendental whenever α is algebraic. (Exercise: why does this show that π is transcendental?) The overarching idea is that since the functions z and e^z are algebraically independent (there's no polynomial relationship between them), they shouldn't be simultaneously algebraic very often. During the three days of week 1, we'll prove they cannot be simultaneously rational very often, proving in particular that e and π are irrational. Then in week 2 we'll explain how to upgrade this to the statement about transcendence; this part will additionally need some linear algebra as a prerequisite.

Homework: None

Class format: lecture
Prerequisites: multivariable calculus (week 1), linear algebra (week 2)

Colloquia

Voting Theory, Burlington, VT, and the Gibbard–Satterthwaite Theorem (*Mira Bernstein*, Tuesday)
Happy 4th of July! To celebrate the birthday of the world's oldest democracy, let's talk about voting. It's much more complicated than the US Founding Fathers thought: whenever you have more than two candidates, the outcome of an election depends on the voting system you use, and all voting systems are flawed. There's a whole branch of mathematics called voting theory that deals with these issues.

In this colloquium, I'll give an example of the kinds of problems that can arise in practice by discussing a famous example of a mayoral election held in Burlington, VT in 2009. At the time, Burlington was using a voting method called Instant Runoff Voting (IRV). Over the past few decades, many election reform advocates have been lobbying for IRV (which they inaccurately call "RCV") as a solution to all our voting problems. But in this example, we'll see that while IRV is certainly better than plurality (the most common voting system in the US, in which whoever gets the most votes wins), it is hardly a silver bullet. We will then prove the Gibbard–Satterthwaite (GS) theorem, which says that any reasonable voting system is vulnerable to strategic voting. (When people want to argue that all voting systems are flawed, they usually cite the much more famous Arrow's Theorem; but the GS Theorem is just as interesting and has much more real-world relevance.)

Mediants, circles, and Stern–Brocot patterns (*Assaf Bar-Natan*, Wednesday)
An ant is at the corner of a tetrahedron, and starts walking along a face. When she hits an edge, she goes over it, continuing in a "straight" path, and when she hits a vertex, she explodes. Can the ant return to her starting vertex without exploding? Can you explicitly construct a countable collection of circles in the plane whose centers converge to any real number? How would you find the continued fraction expansion of π? All of these questions, and more, can be answered using "kindergarten addition" of fractions: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. In this talk, we'll see how!

Hacking heads off hydras (Susan, Thursday)
A hydra is a mythical being with many heads. Whenever a head gets cut off two grow back in its place! A mathematical hydra is a game that we play on trees. On the n^{th} turn, we cut off a "head," causing n new "heads" to grow back in this place. In this colloquium we'll discuss the rules of the Hydra game, and explore some questions that arise. The rules of the game make it seem hopeless, but on small enough graphs the game is clearly winnable. For which graphs is there a winning strategy? And what do the ordinal numbers have to do with it?

The only formula it can be! (*Noah Snyder*, Friday)
You can often guess the right formula for a mathematical problem, not by deriving the formula directly, but instead by showing it's the only formula that has the right properties. In this talk I'll explain one of the simplest examples of this phenomenon: Heron's formula for the area of a triangle. This formula says that if you have a triangle whose sides have length a, b, and c and if you let $s = (a+b+c)/2$ be the semiperimeter, then the area is $\sqrt{s(s-a)(s-b)(s-c)}$. We will guess (and eventually prove!) Heron's formula by explaining why it's the only formula that can be right. In particular, we will use much less geometry but a bit more algebra than you might expect.

CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2023

Contents

9:10 Classes 1
10:10 Classes 2
11:10 Classes 4
1:10 Classes 7
Colloquia 10

9:10 Classes

Beyond inclusion/exclusion (🌶🌶🌶, *John Mackey*, $\boxed{\text{TW}\Theta\text{FS}}$)
Inclusion/Exclusion is a useful counting method wherein one successively corrects overapproximations and underapproximations. We'll spend one day reviewing Inclusion/Exclusion and then consider Möbius Inversion on Posets and Sign Reversing Involutions.

Möbius Inversion will provide a useful look inside the dual nature of accruing and sieving objects, and cast an algebraic context onto Inclusion/Exclusion. Sign Reversing Involutions will lend a matching perspective to the calculation of alternating sums.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Experience with elementary counting and matrix algebra recommended.

Epsilons and deltas (🌶🌶, Ben & Charlotte, $\boxed{\text{TW}\Theta\text{FS}}$)
This class is a rigourous introduction to limits and related concepts in calculus. Consider the following questions:

(1) Every calculus student knows that $\frac{d}{dx}(f+g) = f' + g'$. Is it also true that $\frac{d}{dx}\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n'$?
(2) Every calculus student knows that $a + b = b + a$. Is it also true that you can rearrange terms in an infinite series without changing its sum?

Sometimes, things are not as they seem. For example, the answer to the second question is a resounding "no." The Riemann rearrangement theorem, which we will see, states that we can rearrange the terms in infinite series such as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ so that the sum converges to π, e, or whatever we want!

To help us study the questions above and many other ones, the key tool we'll use is the "epsilon-delta definition" of a limit. This concept can be hard to work with at first, so we will study many examples and look at related notions, such as uniform convergence. Being comfortable reasoning with limits is central to the field of mathematical analysis, and will open the door to some very exciting mathematics.

Homework: Required

Class format: A mix of lecture and group work.

Prerequisites: A calculus class of some kind

Required for: Non-standard analysis (W3); Why do we need measure theory? (W3); aspacefillingcurve (W4)

Infinite Ramsey theory (🌶🌶🌶, Susan, $\boxed{\text{TW}\Theta\text{FS}}$)
Suppose you throw a party and invite six people, and some of those people know each other and some don't. A well known result from graph theory tells us that we can guarantee a group of three mutual friends, or three mutual strangers. But suppose you wanted to throw a much, much bigger party... like an infinite party? Can you guarantee an infinite group of friends or strangers?

More generally, suppose we want to color the edges of a graph with κ colors, where κ is an infinite cardinal number. How many vertices do we need in order to guarantee a monochromatic clique? In this class, we'll find a bound for the answer to this question, and prove that our bound is sharp.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Representation theory of the symmetric groups (🌶🌶🌶🌶, Raj, $\boxed{\text{TW}\Theta\text{FS}}$)
Representation theory is a field designed to understand how groups act on vector spaces. In this course, we will focus specifically on the symmetric groups and understand how their representation theory is connected with partitions of natural numbers and combinatorial gadgets known as Young tableaux. Importantly, you will learn how to use combinatorics to tackle abstract problems.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra. Group theory.

What are your numbers worth? (🌶, Eric, $\boxed{\text{TW}\Theta\text{FS}}$)
In this class we will figure out what numbers are worth. Some numbers will be worth a lot, other numbers will be worth negative amounts, yet still others will be worth fractional amounts. We will learn the difference between knowing a number (very extremely mind-bogglingly hard) and knowing what a number is worth (surprisingly incredibly magically easy). We will gain the power of being able to figure out the worth of numbers (and many cool corollaries of this power) by making line doodles. In actuality this is a course about the local part of algebraic number theory, but those words won't show up much until the last day. (If you want more technical words: we'll be learning about the p-adic valuation(s) on $\mathbb{Q}$ and finite extensions thereof, through the lens of Newton polygons. But don't worry if you don't know any of these words yet!)

Homework: Recommended

Class format: Mixture of interactive lecture and worksheets. Tentatively: day 1 will be lecture, days 2 through 4 will be mostly worksheets, day 5 will be back to lecture.

Prerequisites: You'll need to be comfortable with modular arithmetic, at the level of knowing that you can translate between statements in modular arithmetic and statements in divisibility of integers; Mia's week 1 introduction to number theory class will prepare you super well. Towards the end we will do some linear algebra with coefficients in $\mathbb{Z}/p\mathbb{Z}$, but at a 🌶 pace so don't be scared. There will be a few very optional homework problems that use background in ring theory and linear algebra.

10:10 Classes

Green's Theorem (🌶🌶🌶, Mark, $\boxed{\text{T}}$WΘFS)
How can you measure the area of a lake without ever venturing into the water? If you have the right

mechanical device (called a *planimeter*), you can do this by simply moving the device all around the lakeshore. Green's Theorem, which gives a fundamental connection between line integrals over closed curves in the plane and double integrals, explains how this is possible. The theorem (to be stated in class) has other useful applications, as well as interesting generalizations. It will be used (without proof) in the weeks 3-4 class on functions of a complex variable. Meanwhile, in this one-day class we'll quickly review the necessary concepts, state and sketch a proof of the theorem, give a paradoxical proof that $2\pi = 0$, and if time permits, mention a quite general result that has Green's Theorem as a special case. If you've taken the multivariable calculus crash course in week 1, you'll definitely be ready for this class, because we'll start where that course left off.

Homework: None

Class format: Interactive lecture

Prerequisites: Multivariable calculus (if possible, having encountered line integrals as well as double integrals)

Introduction to cryptography (🌶🌶, Ian, T$\boxed{\mathrm{W\Theta FS}}$)
CIPHERTEXT: LIBRX FDQQR WGHFL SKHUW KLVWH AWWKH QBRXV KRXOG WDNHW KLVFO DVVYZ

plaintext: if you cannot decipher this text then you should take this class

Contents:

- Art of encryption & decryption
- Danger of attacks & security

Homework: Recommended

Class format: Mostly lecture with some activities

Prerequisites: Basic probability

Introduction to model theory (🌶🌶🌶, Krishan, T$\boxed{\mathrm{W\Theta FS}}$)
A large part of math boils down to studying "sets with structure." These could be groups, fields, vector spaces, any of the different varieties of graphs, or even more exotic examples like valuation rings or ordered groups. In this class we'll see how all of these are manifestations of the same idea and we'll study them all simultaneously at a meta level. We'll learn how to view mathematical objects like the real number line and the field of complex numbers in a new light. We'll encounter strange objects like infinite groups which behave as if they are finite. We'll see why large random graphs all tend to look the same, and we'll even talk about how you can build models of set theory where the reals are countable!

*Thanks to Aaron Anderson for the blurb inspiration!

Homework: Recommended

Class format: Interactive Lecture

Prerequisites: Group theory (a week at mathcamp would be enough)

Mechanics of fluid flow (🌶🌶🌶, Neeraja, T$\boxed{\mathrm{W\Theta FS}}$)
The Euler and Navier–Stokes equations are partial differential equations that describe the motion of fluid particles (gases and liquids) in $\mathbb{R}^3$. The equations are derived from Newton's second law of motion, with the force equal to a sum of contributions by pressure, external forces and in the Navier–Stokes case, viscous stress. Though the Euler equations were first written down in 1757, and the Navier–Stokes equations were formulated as a generalization of the Euler equations in the 19^{th} century, theoretical understanding of the solutions to these equations remains incomplete. Mathematicians have neither proved that smooth solutions to the equations always exist, nor have they found any counterexamples.

The goal of this class is to develop enough of the theory of fluid mechanics to derive the Euler and Navier–Stokes equations and to appreciate (to whatever extent possible) the difficulty of these open problems. If time permits, we will also discuss the existence of solutions to Euler's equation.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: single-variable calculus

Polygons, friezes, and snakes — oh my! (, Kayla, $\text{T}\boxed{\text{W}\Theta\text{FS}}$)
What do polygons, friezes and snakes all have in common? It turns out that they can all be related using algebraic combinatorics! We will begin the class by exploring the connection between triangulations of polygons and finite arrays of positive integers called frieze patterns. Over the course of the week, we will ramp up this construction to involve not just arrays of positive integers, but eventually, rational functions. When we get into these weeds, we will see snakes! The types of snakes we will see are called snake graphs—a tool for understanding which rational functions can appear in our frieze patterns and are constructed from our triangulations of polygons. Come see how all these cute objects are all connected! Do NOT beware of the snakes, they are friendly and don't bite!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Problem solving: triangle geometry (, *Zach Abel*, $\text{T}\boxed{\text{W}\Theta\text{FS}}$)
Come explore the rich, diverse, and endlessly surprising world of triangle geometry! Triangles have loads of named "center" points, and we'll venture well beyond the classical centroid and orthocenter into some lesser-known yet unreasonably beautiful ones. Why has the symmedian point been called "one of the crown jewels of modern geometry"? Why is the existence of Feuerbach's point even reasonable (I'm still not convinced…), and how might we approach its construction synthetically (i.e., without inversion)? What are the (literally!) more than $10,000$ triangle centers listed in the Encyclopedia of Triangle Centers, and how can this encyclopedia be interpreted?

Homework: Required

Class format: This class is largely problem based: there will be some lecturing, but much of the time you will present your solutions to the previous day's olympiad-style homework problems.

Prerequisites: Some familiarity with synthetic geometry (similar triangles, cyclic quadrilaterals, etc.).

11:10 Classes

Elliptic curves (, *Ruthi Hortsch*, $\text{TW}\boxed{\Theta\text{FS}}$)
Let n be a positive integer. We call n a congruent number provided that there is a right triangle that has all rational sides and area n. Can we find an easy way of determining whether n is a congruent number? Tunnell's theorem gives us a way, but its methods of proof use techniques more advanced and modern than the statement of the question itself: the theory of elliptic curves and modular forms, which are fundamental objects of study in advanced number theory. It is also dependent on a weak form of an unsolved conjecture called the Birch and Swinnerton-Dyer Conjecture. (A proof—or counterexample—of this conjecture is worth a million dollars!)

In this class we will: discuss congruent numbers and Tunnell's theorem, define elliptic curves and their basic properties, explain why answering questions about elliptic curves will tell us about congruent numbers, and get some idea about how this leads us to Tunnell's result.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic modular arithmetic. Group theory is not required but is useful context.

Gödel's incompleteness theorems (, Steve, $\boxed{\text{TW}\Theta\text{FS}}$)

Consider the following very silly argument:

"Let f_i be the i^{th} function $\mathbb{N} \to \mathbb{N}$ which we can prove is well-defined. There are obviously infinitely many such functions. Now consider the function

$$g : \mathbb{N} \to \mathbb{N} : x \mapsto f_x(x) + 1.$$

Obviously we can prove that this function g is well-defined. But g can't be any of the functions that we can prove are well-defined. So, [explosion]."

You may be tempted to reject this immediately as "too informal," or "hiding some subtleties," or "really frikkin' silly." However, with work it can in fact be made perfectly precise! When we do so what we get is a proof that no "reasonable" system of mathematics is able to resolve every question. This is (a version of) **Gödel's first incompleteness theorem**, and in this class we'll explore it, its strengthenings, and its corollaries (note the word "first" there...). This will involve looking under the hood of mathematics, so to speak, and getting really precise about what exactly a definition/theorem/proof in mathematics is; think about the difference between an algorithm and a computer program in some specific language. It will also involve breaking reality a wee bit, and learning how to write sentences that refer to themselves.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Required for: Consistency of arithmetic (W3)

Introduction to ring theory (, Kevin, $\boxed{\text{TW}\Theta\text{FS}}$)

Welcome to the world of rings! First discovered by Dedekind, Hilbert, and Noether in the late 19^{th} century, rings are algebraic structures that capture the abstract ideas of addition and multiplication. It's no exaggeration to say that ring theory touches all of mathematics. The number theorists are still here, but nowadays we have lots of geometers, topologists, and combinatorialists. Even analysts from faraway lands will make the occasional trip.

In this week-long tour, we'll get acquainted with their basic axioms and properties and see some of the sights (the integers, fields, polynomial rings, some weirder stuff near the end). A major theme of the class will be ideals in rings. As one of the ring theorists' key innovations, ideals vastly generalize the notion of divisibility and provide new perspectives on modular arithmetic and prime factorization.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Some familiarity with basic number theory (e.g. modular arithmetic) would be helpful but is not required. Group theory would also be helpful but is definitely not required.

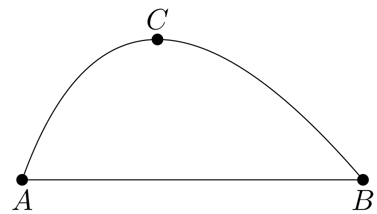
Required for: How to count rings (W3); Finite fields (W4)

Parabolic curves (, Misha, $\boxed{\text{TW}}\Theta\text{FS}$)

First of all, this class has nothing to do with Ruthi's class later in the week. Elliptic curves are algebraic curves with hugely important applications in number theory and cryptography. Parabolic curves are a silly joke—my class is just about parabolas.

You may be familiar with parabolas as the graphs of functions like $y = x^2$. In this class, we'll look at parabolas the way mathematicians thought about them before coordinate geometry was invented.

On the first day of class, we'll just take a tour of some properties of parabolas. The second day of class is a theorem of Archimedes: the "quadrature of the parabola". We will find out how to compute the area of a shape bounded between a line segment and a parabola:



Homework: Optional

Class format: Interactive lecture with lots of diagrams.

Prerequisites: Some Euclidean geometry, but nothing fancy: if you're comfortable with parallel lines, congruent and similar triangles, and the Pythagorean theorem, you should be fine.

The Wythoff array (, Della, $\boxed{\text{TW}\Theta\text{FS}}$)
Here's a simple game! There's a single Chess queen on a large board. Two players take turns moving it any distance down, left, or diagonally down and left. Whoever moves the queen to the bottom left square wins. What's the optimal strategy?

You've probably written numbers in base ten before, and you might have even used base two. But you can also write numbers in base Fibonacci, where digits represent Fibonacci numbers! Under what conditions do numbers have unique representations of this form? What operation does adding a 0 to the end correspond to?

Draw a ray from the origin with slope x, and write down the sequence of horizontal and vertical grid lines it passes. What can we say about this sequence? How is it related to the sequence you'd get for other numbers, such as $x + 1$?

Imagine I beat a drum every x seconds (x isn't an integer). You watch a timer, and record which 1-second intervals have drum beats in them. How does the sequence of numbers you get depend on x? What's interesting about the intervals which *don't* have drum beats in them?

In this class, I'll answer all of these questions! Then we'll see how all of these problems are intrinsically related, and use their relationships to construct a beautiful grid of numbers called the Wythoff array, and learn about some of the magical properties it has: for instance, every positive integer appears exactly once, and each row satisfies the Fibonacci recurrence.

The first two or three days will cover several disjoint topics, but by the end of the class you'll see how it all fits together.

Homework: Recommended

Class format: Lecture

Prerequisites: None.

Wedderburn's Theorem (, Mark, $\boxed{\text{T}}$WΘFS)
Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1,\ \ i\,j = k,\ \ j\,i = -k,\ \ j\,k = i,\ \ k\,j = -i,\ \ k\,i = j,\ \ i\,k = -j.$$

Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Homework: None

Class format: Interactive lecture

Prerequisites: Some group theory; the idea of vector spaces and bases from linear algebra; familiarity with complex roots of unity would be helpful.

When will this end??? (🌶🌶, Arya, TWΘFS)
You can run from your fate, but can you really escape it? On the Earth, sadly not. On the real line, sure - just run off to infinity on either side! How many ways can you escape fate on an infinite plane? In a group? On an arbitrary surface?

In this class, we shall study ends of spaces, which are in some sense, different ways to go off to infinity. A lot of information can be obtained about a space or a group given how its ends behave. At some point, I shall be drawing the Loch Ness monster, and possibly the Eiffel Tower. What's not to love? :)

Homework: Recommended

Class format: Lectures.

Prerequisites: You should know what a group is, and possibly what a graph is. If you don't, chat with me before class!

1:10 Classes

First, choose randomly (🌶🌶🌶, Travis, TWΘFS)
How do you prove that something exists without knowing what it is? In math, there are many ways to do this, and the probabilistic method is, at least in combinatorics, one of the most popular. Counterintuitively, even though it doesn't provide any specific example, it often can still be used to generate examples in practice fairly quickly.

In this class, we'll use probability to prove things about combinatorics and discrete geometry. If you've never seen it before, it's very surprising that probability is an effective tool in combinatorics; and if you have seen it before, even many times, it's surprising how effective it is: The probabilistic method can often turn otherwise very difficult problems into child's play.

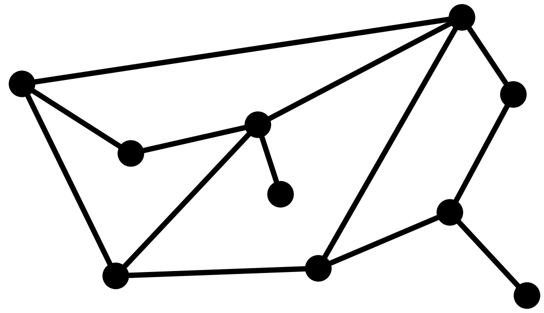
In this class, we'll start by seeing how simple methods in probability can already prove important (and sometimes still cutting-edge) results in combinatorics. Then we'll build up more advanced and refined techniques throughout the week to prove even more. Applications will be far and wide, from graphs, set families, and permutations; to number theory; to discrete geometry. The probabilistic method touches on most areas of combinatorics, and the goal of this class is to give rapid tour of these connections and applications.

Homework: Optional

Class format: Interactive lecture

Prerequisites: familiarity with basic probability is helpful

Introduction to graph theory (🌶🌶, Tim!, TWΘFS)
A graph consists of some *vertices*, which we will draw as dots on our paper, and some *edges*, which we will draw as segments connecting pairs of vertices, like this:



Because the definition of a graph is so lightweight, graphs can represent many different things. For example, each vertex could represent a person at camp, and we could draw an edge between two people if they are close contacts for the purposes of contact tracing. Or, each vertex could represent a path on a manifold, with two vertices being joined by an edge if the corresponding paths don't intersect. Therefore, many problems (both in the real world and in other areas of math) can be transformed into questions about graphs! If you can prove things about graphs, you can solve problems in many different areas!

With the understanding that graphs have lots of important applications, we can feel even more comfortable studying graphs for their own sake. Again because graphs have such a simple definition, there are many different lines of investigation we can take, and we will look at different topic each day. We'll spend one day thinking about *planarity*. One question we will ask: Draw three dots on a piece of paper, representing an electric utility, a water utility, and a gas utility. Draw three more dots, representing three houses. Can you draw a path connecting each utility to each house, without any of those paths crossing? This is a famous puzzle; it's even been printed on merchandise! Here it is on a coffee mug:

From `https://mathsgear.co.uk/`

The more general question is: Which graphs can we draw in the plane without edges crossing? Is there a nice way to describe which ones can be and which ones can't be?

Topics that may appear on other days of class include: trees, connectivity, matchings, and graph colorings.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None

Required for: Graph colorings (W3); Matroids and the chromatic polynomial (W4)

Packing permutation patterns (, Misha, TWΘ$\boxed{\text{FS}}$)
Prepare by picking a permutation π and a pattern P. Probabilistically pick $|P|$ pieces of π: perhaps putting them together produces P? Let $\rho_P(\pi)$ be the probability of producing P.

To pack P in π, puff up this probability, making P as plentiful as possible. We will ponder the packing problem for $P = 132$ (and plenty of its pals) using a progression of powerful problem-solving procedures.

Homework: Recommended

Class format: Fill-in-the-blank lecture: to help you along, I'll provide lecture notes at the beginning of class, but some details will be left out for you to supply as we go. Homework is sparse, but I highly recommend taking the time to make sure you are happy with all the examples if you want to get the most out of this class.

Prerequisites: Nothing we'll do will require more background than knowing what $n!$ and $\binom{n}{k}$ count. Ask me after the end of the week if you're curious about the heavy-prerequisite version of this topic (which can involve graph theory, linear and polynomial algebra, logic, and semidefinite programming).

Polytopes (Week 1 of 2) (, Susan, $\boxed{\text{TW}\Theta\text{FS}}$)
A polygon is a two-dimensional shape bounded by line segments. A polyhedron is a three-dimensional shape bounded by polyhedra. A polytope is a d-dimensional shape, bounded by $(d-1)$-dimensional polytopes. In this class we'll explore these high-dimensional objects.

Our first task is to define them—what is a polytope? One possible definition is that a polytope is the convex hull of its vertices. Another is that a polytope is the bounded intersection of a collection of half-spaces. These two definitions seem to produce the same collection of objects, but it's not at all obvious why this should be the case!

This week we'll build a small library of example polytopes, see what types of theorems can be proved with each of our two definitions, and show that the definitions are equivalent.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Randomized vs deterministic computation (, Tanya, $\boxed{\text{TW}\Theta\text{FS}}$)
If you're using a computational method to test whether your submersible vehicle can withstand high pressure it would experience several miles underwater, you would like to use simulation techniques as precise and detailed as possible to avoid a potential catastrophe. However, if you are doing so on your personal laptop which doesn't have very much processing power, you may be inclined to take some shortcuts and perhaps run a simpler, less computationally expensive, but potentially error prone algorithm. Ideally, we would like to have algorithms that are efficient in runtime and memory usage, solve our desired question exactly and are easy to analyze theoretically so the risks posed by the aforementioned scenario never have to be taken. However, in practice, there are often trade-offs between all of these desired qualities. In this course we will explore what happens if we introduce some small error to our solution by allowing random operations in the example of the classical problem of finding a minimum cut of a graph. We will start with a deterministic approach which yields an exact solution after some non-trivial analysis and compare it to an almost painfully simple to state and analyze, however error-prone, randomized solution.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

The transcendence of many numbers (including π and e) (Week 2 of 2) (🌶🌶🌶🌶, *Dave Savitt*, TWΘFS)
This is a continuation of my week 1 class. Linear algebra is an additional prerequisite for the second week.

Homework: None

Class format: Lecture

Prerequisites: linear algebra, multivariable calculus

Colloquia

One-half factorial from scratch (*Zach Abel*, Tuesday)
Even though factorials are all about counting orderings of people, and the idea of permuting just half of a person makes you queasy, you may nevertheless have some vague suspicion that the factorial of 1/2 can somehow be made sense of. You just can't shake this hazy impression that if you squint real hard and pray to Gauss to please please please let ϵ be greater than zero, somehow a value for (1/2)! pops out in some reasonable manner. You may even hear voices on the wind that this value is somehow connected to the square root of π, which is kinda hard to believe, tbh.

Armed only with this hunch, and without looking anything up, and without calculus in any form, how might you rediscover what value should be assigned to (1/2)! on your own, and convince yourself that it's the right one?

Some stories about squares (mod p) (*Dave Savitt*, Wednesday)
Which tend to be larger, on average: the squares mod p, or the non-squares mod p? And by how much? This simple question turns out to have surprising depth.

Computer-aided mathematics and satisfiability (*John Mackey*, Thursday)
During this talk we'll review some of the successes over the last 40+ years in computer-aided mathematics, and then focus on recent applications of SAT solvers to problems in discrete math.

A SAT solver considers a logical formula in conjunctive normal form, for example (A or -B) and (B or C) and (-A or -B or C) and (-C), and attempts to determine whether it is possible to assign truth values to the variables such that the formula is true.

We'll show how to encode some discrete math problems in conjunctive normal form and see how modern SAT solvers process them.

Project selection fair (Staff, Friday)
A special event scheduled instead of a Friday colloquium: at the Project Selection Fair, you'll get a chance to ask staff about the projects they've proposed (project blurbs will be available on Thursday at sign-in). If you're interested in participating in a staff-supervised project at camp, whether it's already been proposed, or you have an idea of your own that you're excited about, make sure to come to the project selection fair and fill out a preference form.

CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2023

Contents

9:10 Classes 1
10:10 Classes 3
11:10 Classes 6
1:10 Classes 7
Colloquia 11

9:10 Classes

Consistency of arithmetic by killing hydras (🌶🌶🌶🌶, Della, $\boxed{\text{TW}\Theta\text{FS}}$)
Steve proved last week—and Gödel proved in 1931—that if the basic axioms of number theory (formalized as Peano Arithmetic) are consistent, then they can't prove that they're consistent. I'm going to ignore this inconvenient fact, and prove that PA is consistent from as simple axioms as I can (no set theory!).

To do this, we'll turn proofs into trees and play something like the hydra game on them. Once the hydra is defeated, we'll have an extra simple proof that definitely isn't a proof of a contradiction. Because of Gödel, we'll need a pinch of axioms beyond PA: as with hydras, defeating our proofs will require ordinal numbers.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: You should be familiar with Peano Arithmetic, Gödel's second incompleteness theorem, and ordinal numbers up to ε_0. If you took Steve's week 2 class and remember Susan's colloquium about killing hydras, you're prepared.

Functions of a complex variable (Week 1 of 2) (🌶🌶🌶, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)
Spectacular (and unexpected) things happen in calculus when you allow the variable (now to be called $z = x + iy$ instead of x) to take on complex values. For example, functions that are "differentiable" in a region of the complex plane now automatically have power series expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both in- and outside math. For example, functions of a complex variable were used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if a and b are positive integers with $\gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, \ldots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann Hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet's theorem is certainly beyond the scope of this class and heat conduction probably is too, but we'll prove an important theorem due to Liouville that 1) leads to a proof of the so-called "Fundamental Theorem of Algebra", which states that any nonconstant polynomial (with real or even complex coefficients)

has a root in the complex numbers and 2) is vital for the study of "elliptic functions", which have two independent complex periods, and which may be the topic of a week 5 class. Meanwhile, we should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and venturing boldly forth into the complex plane! This class runs for two weeks, but it should be worth it. (If you can take only the first week, you'll still get to see a good bit of interesting material, including one or two of the things mentioned above.)

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Multivariable calculus (the week 1 crash course will be enough). Familiarity with Green's Theorem won't hurt, but the theorem will be stated, so it's fine if you didn't take the one-day class about it in week 2.

Music: the number theory of sound (🌶🌶, *J-Lo*, $\boxed{\text{TW}\Theta\text{FS}}$)

A "major scale" is obtained by dividing an "octave" into seven steps, where the third and seventh of these are "half" and the rest are "whole," for a total of twelve half steps. To form a "major triad," a particularly pleasant combination of notes, you can combine a "third" (four half steps) and a "fifth" (seven half steps).

Regardless of your music theory background, the description above may seem like a jumble of numbers with more idiosyncracies than the imperial system of units. Even if we ignore the naming conventions, why divide the octave into 12 equal pieces? Why do $\frac{4}{12}$ and $\frac{7}{12}$ of an octave sound so natural, while $\frac{6}{12}$ of an octave sounds so unsettling? What even *is* an octave?

In this class, we begin with the basic (but vague) question underlying all of the above: *which notes "work" together?* For most of the class you will explore the question yourselves: you will run auditory and graphical experiments, come up with hypotheses, and discuss theoretical models to understand the situation. Along the way you will find that this deceptively simple question has a very rich and complex structure (SPOILERS[1]). I will share some mathematical tools that can be used to better understand this structure (continued fractions and Diophantine approximation, to name a few), but the goal is for you to develop a theory describing these observations yourselves. As a result, this course will be somewhat open-ended, and its direction may change depending on where our exploration leads.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Problem solving: olympiad inequalities (🌶🌶, Ian, $\boxed{\text{TW}\Theta\text{FS}}$)

One day, your teacher decides to change the way to calculate the average of your exam scores. If you have n scores, instead of adding the scores and dividing by n, the teacher decides to multiply all your scores and take the n^{th} root. Will you form a mob and complain to the teacher? Do you have grounds for such action? In this class, we will be discussing when one thing is always greater than (or equal to) another: inequalities!

In this five day course, we will be discussing rearrangement inequalities, AM-GM, Cauchy, Jensen, and a mix of useful inequalities for Math Olympiad. This class is more for campers who have not seen Olympiad inequalities before.

Homework: Recommended

Class format: Lecture + IBL

Prerequisites: Basic inequality (e.g. what happens to the inequality when we multiply -1?)

[1] it's number theory

Solving equations with origami (🌶🌶, Eric, TWΘFS)
Put a piece of paper in front of you. Mark two points on it. Pretend that your paper is a piece of the complex plane, and that your two marked points are 0 and 1. Which other points in the plane can you construct by folding your paper and marking where your existing points fold to? If you allow arbitrary folds you can hit anything, but what if we restrict ourselves to folds we can "line up" using our existing points and lines?

We'll be able to answer this question precisely by developing a system of axioms for one-fold origami and analyzing their algebraic potential. Along our journey to understanding the limits of the algebraic power of origami we'll travel the world (Japan to India to Austria to Italy and back), gently encounter some flavours of math from abstract algebra and algebraic geometry, and employ a truly wonderful piece of 19^{th} century mathematics to solve equations by shining lasers on turtles. If that's not enough for you, one of the historical characters we'll encounter has possibly the greatest name in mathematics: Margherita Piazzola Beloch (arguably the first person to really understand the algebraic power of origami).

Homework: Recommended

Class format: This will be a very active class! There will be some lectures, we'll do lots of paperfolding activities in class (I will supply the paper), and you'll spend a fair bit of time working through constructions on your own or in small groups.

Prerequisites: Comfort with the idea of dimension of a vector space. You do not need any prior knowledge of origami to follow this course.

10:10 Classes

A very chill intro to measure theory + dimension (🌶, Charlotte, TWΘFS)
Suppose I gave you a set in $\mathbb{R}^2$, and asked you how large it is. If the shape were simple enough, you'd probably compute its area. But what if the shape were not so simple? What if I asked you to provide some other sense of size? In this class we will take an exploratory approach to different ways that we can measure the size of sets in $\mathbb{R}$ and $\mathbb{R}^2$, with a particular focus on measure theory and dimension.

You can think of a measure in

$$\mathbb{R}$$

and

$$\mathbb{R}^2$$

as a generalization of length and area, respectively. We'll discuss what properties we want our definition of measure to satisfy, try out a couple of definitions, and see what kinds of sets our definition of measure works well with. Most measure theory classes involve a lot of detailed proofs and a lot of epsilons, which can make the ideas seem much more complicated than they actually are. We'll focus less on rigour and more on the ideas of measure theory.

We'll also spend some time looking at different types of fractals sets, which loosely, are sets that no matter how far you zoom in on, exhibit fine structure and possibly, self-similarity (i.e., a fractal may be a union of multiple smaller copies of itself!). There's a notion of size, called dimension, which attempts to quantify the fine structure of a fractal. We'll look at multiple ways that we can define a set's dimension, and discuss how it agrees with the intuitive sense of dimension that we already have.

These explorations will lead us to some intriguing or even counterintuitive examples, like a set with largest possible dimension but smallest possible measure, and a function that is basically flat everywhere, yet increases from zero to one.

You may have noticed that Tanya's also teaching a class about measure theory this week: you should take my class if you'd prefer a slower-paced, worksheet-based, geometric and example focused intro to measure theory.

Homework: Optional

Class format: Mostly group work and worksheet based.

Prerequisites: Familiar with the idea that $\mathbb{Q}$ is countable while $\mathbb{R}$ is not, and an intuitive idea of what limits are. Familiarity with derivatives and integrals could be useful, but not required.

Graph colorings (🌶🌶, Mia, T$\boxed{\text{W}\Theta\text{FS}}$)

A new island has been discovered in the Arctic Ocean! While the geographers are arguing over how to divide the island, the cartographers begin to wonder about the map: how many colors are needed to color the countries so that any two countries that share a border get different colors? The Four Color Theorem says just four. We won't prove this—it took over 100 years and a computer program that checked 1,936 different cases to prove this theorem—but we will use this question as a springboard to others.

Suppose the countries decide that they have *non-negotiable* color preferences. For instance, the country Zudral demands to be cyan or magenta. And the country Scaecia refuses to be anything but light blue, sky blue, or cornflower blue. Given that each country now has a list of allowable colors, how does that change the cartographer's ability to color the map?

Or what if we are allowed countries to be shaded in with several colors? In this case, Zudral could be indecisive and be one half cyan and one half magenta. Or what if we changed up the objective entirely and instead of focusing on the total number of colors used, we tried to minimize the number of colors "seen" on the neighboring countries?

Secretly, the questions above can be changed into questions about graph colorings; specifically, list coloring, fractional coloring, and "local coloring" respectively. With each new coloring, there arises a new chromatic number and we return to our central questions:

(1) What are the bounds for this chromatic number?
(2) Can we construct a family of graphs that forces this chromatic number arbitrarily far from its bounds?

Note: Although maps are an excellent motivating example, we will be focusing on general graphs, not just planar graphs!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic definitions from graph theory

Guess Who? (Week 1 of 2) (🌶 → 🌶🌶, Tim!, T$\boxed{\text{W}\Theta\text{FS}}$)

The children's board game *Guess Who?* is the hardest game that you thought was easy. In case you haven't played it, the rules are simple: you ask yes/no questions to try to determine your opponent's mystery character from among 24 possibilities, and whoever guesses correctly with fewer questions wins.

You may think (like many mathy folks I have talked to) that you know the optimal strategy. In fact, YouTube star (and former NASA engineer) Mark Rober made a whole video about this "BEST strategy." But I realized watching the video that this strategy isn't really the "BEST." Actually, it can be totally destroyed.

I wrote a little program to compute the real optimal strategy, and the answer is bizarre. The solution exhibited a pattern so unexpected that it left me asking "Where did this come from?" even *after* I had written out a proof.[2]

[2] Note: "Where did this come from?" is a different question than "What is the proof?" I had a proof! But being able to prove that something has weird behavior is different question than what is *causing* that behavior.

At this point, I became pretty obsessed with this game. I thought about this game on and off for years trying to understand why this simple children's game exhibited such a surprising pattern. And I finally got to a pretty satisfying answer.

The answer goes through several areas of math; for each topic, we'll see the fundamentals of the topic, then apply what we learned to the *Guess Who?* problem. Over the course of **two weeks**, we'll see decision trees, **information theory** and entropy, **matrix games**, relaxations, **linear programming** and convex optimization, **continuous random variables** and cumulative distribution functions, and **network flows**. By the end of the class, you'll have an introduction to each of these topics, as well as an answer to the *Guess Who?* mystery (and some open questions you can think about).

The class is 🌶️ to 🌶️🌶️🌶️🌶️ over the two weeks as we move through the topics. It's a sampling platter of chili levels — you may get to see some material at a slightly higher chili level than you are used to, and other material at a slightly lower chili level; I hope you will enjoy the variety. Overall, the first week will start at 🌶️ but will then be mostly 🌶️🌶️, while the second week will be mostly 🌶️🌶️🌶️ with a bit of 🌶️🌶️🌶️🌶️.

The class will go through a satisfying story arc in just its first week, so it's possible to take the first week without the second.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None

How to build a donut (🌶️🌶️🌶️, Kayla, T$\boxed{\mathrm{W\Theta FS}}$)
Whether you prefer Krispy Kreme, Dunkin, or something fancy from Portland, donuts are a well-loved treat. People who especially love donuts call themselves topologists. In this class, we will be baking as topologists! In particular, we will be doing things a bit out of order starting with the icing first. Using a chocolate icing function, we will characterize special properties of donuts topologically to construct the donut. Once we devour this example, we will formulate some basic topological notions and develop more general techniques to show that defining special icing functions from a topological space to $\mathbb{R}$ can completely determine any topological pastry up to continuous deformation.

Homework: Optional

Class format: Interactive lecture.

Prerequisites: Calculus, up to knowing what a derivative is.

How to count rings (🌶️🌶️🌶️, Kevin, T$\boxed{\mathrm{W\Theta FS}}$)
How many ways can we make the abelian group $\mathbb{Z}\oplus\mathbb{Z}$ into a ring? What about $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$? $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$? It turns out that we can answer these questions in a simple and satisfying way (but only if there aren't too many $\mathbb{Z}$'s!).

This class will be an introduction to the parametrization of rings of low degree (2, 3, and 4). Along the way, I'll provide an introduction to some basic ideas from algebraic number theory and arithmetic statistics, where the study of low degree rings has some cool applications. For those who enjoyed my class on Bhargava's cube, I'll be talking about the algebraic/number-theoretic side of the story, which relates binary quadratic forms and $2 \times 2 \times 2$ cubes to ideals in quadratic rings.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic ring theory and abstract linear algebra (e.g. abstract vector spaces and linear operators, bases, trace/determinant)

11:10 Classes

All aboard the Möbius (🌶🌶🌶🌶, Narmada, $\boxed{\text{TW}\Theta\text{FS}}$)
BüS SCHEDULE
McMobil from **Möbius function** to **Prime Number Theorem**
Scheduled in week 3
What's it like on board?
Lots of Greek letters · The *primest* numbers · Funky finite sums
Stops

- Arithmetical functions
- Möbius inversion
- Sketchy descriptions of the distribution of primes by a lapsed number theorist

Homework: Optional

Class format: Lecture

Prerequisites: basic modular arithmetic

Calculus of variations (🌶🌶🌶, Ben & Steve, $\boxed{\text{TW}\Theta\text{FS}}$)
The calculus of variations, as a technique, is only a little younger than the calculus as a whole; Newton is the first person known to have used it, in the 1680s. Broadly speaking, where the "normal" calculus finds *points* that optimize *functions*, the calculus of variations finds *functions* that optimize functions-of-functions, or *functionals*.

In this class, we'll aim to study some of the general theory of the subject: this will help draw out some of the similarities with the usual differential calculus. We will also aim to study some examples—both simple and complex—to see the power of the methods, to see where they leave a lot of work to be done, and to explore some of the historical context of the subject.

Homework: Recommended

Class format: Lecture

Prerequisites: Multivariable calculus (familiarity with the multivariable chain rule, in particular)

Generating functions, Catalan numbers, and partitions (🌶🌶, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)
Generating functions provide a powerful technique, used by Euler and many later mathematicians, to analyze sequences of numbers; often, they also provide the pleasure of working with infinite series without having to worry about convergence.

The sequence of *Catalan numbers*, which starts off $1, 2, 5, 14, 42, \dots$, comes up in the solution of many counting problems, involving, among other things, voting, lattice paths, and polygon dissection. We'll use a generating function to come up with an explicit formula for the Catalan numbers.

A *partition* of a positive integer n is a way to write n as a sum of one or more positive integers, say in nonincreasing order; for example, the seven partitions of 5 are

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, \text{ and } 1+1+1+1+1\,.$$

The number of such partitions is given by the partition function $p(n)$; for example, $p(5) = 7$. Although an "explicit" formula for $p(n)$ is known and we may even look at it (in horror?), it's quite complicated. In our class, time permitting, we'll combine generating functions and a famous combinatorial argument due to Franklin to find a beautiful recurrence relation for the (rapidly growing) partition function. This formula was used by MacMahon to make a table of values for $p(n)$ through $p(200) = 3972999029388$, back when "computer" still meant "human being who does computations".

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Summation notation; geometric series. Some experience with more general power series may help, but is not really needed. A bit of calculus may come in handy, but you should be able to get by without.

Polytopes (Week 2 of 2) (🌶🌶🌶, Susan, TWΘFS)
This week in Polytopes, we'll clean up any loose ends from our proof of the equivalence of our two definitions of polytopes and move on to consider a cool class of examples: cyclic polytopes!

If you choose n arbitrary points from the curve parametrized by $(t, t^2, t^3, \ldots, t^d)$ and take the convex hull, you obtain a polytope that has precisely those n points as vertices. That is to say, no point falls within the convex hull of the others. And this is true no matter how many points you choose! Also, these polytopes have way more edges than our three-dimensional intuition would suggest. In four dimensions and above, the line segment between any pair of vertices is an edge of the polytope!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Polytopes Week 1

The sum-product conjecture (🌶🌶, Neeraja, TWΘFS)
Let A be any set of N integers. The *sum set of* A is the set

$$A + A := \{a + b : a, b \in A\}$$

and the *product set of* A is the set

$$A \cdot A := \{a \cdot b : a, b \in A\}.$$

If A is given by N randomly chosen integers, we would expect that both the sum set and the product set have size around N^2. However, each of the two can be made much smaller by choosing A carefully; for example, choosing A to consist of elements of an arithmetic progression, i.e. $A = \{a+d, a+2d, a+3d, \ldots, a+Nd\}$ makes $|A + A| = 2N - 3$. The sum-product conjecture suggests that no set A has both a small sum set and a small product set. Roughly speaking, the conjecture says that if N is large enough, then for any set A,

$$\max\{|A + A|, |A \cdot A|\} \approx N^2.$$

This unsolved conjecture has a number of applications in many different fields, including number theory, incidence geometry, and computer science. In this class we will survey some of the partial progress made towards proving the conjecture.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: A little experience with epsilon-delta proofs may be helpful, but is not required

1:10 Classes

Coxeter groups (🌶, Kayla, TWΘFS)
Coxeter groups are a special class of groups that capture the symmetries of objects. For example, how many symmetries are there of a cube? Icosahedron? Coxeter groups give a combinatorial way for us to think about these types of problems. We will look at examples of of Coxeter groups and state a classification theorem of finite Coxeter groups that is one of my all time favorite theorems!

Food for thought: what is the sequence $1, \infty, 3, 5, 3, 4, 4, 4, 3, 3, 3, 3, \ldots$?

Homework: Optional

Class format: Interactive lecture

Prerequisites: Group theory encouraged!

Latin squares (🌶🌶, *Zoe Wellner*, $\boxed{\text{TW}}\Theta\text{FS}$)
In addition to sudokus being a type of Latin square, Latin squares are quite the mysterious object. They have various connections to projective geometry and there are some questions about their properties that have been unknown for over 60 years! In this class, we will learn about Latin squares and why some of the questions about them are hard.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Linear algebra through knots (🌶🌶🌶, Raj, $\boxed{\text{TW}}\Theta\text{FS}$)
In this course, we will learn how to visualize maps between vector spaces using braids. The point is that long products of maps can be manipulated very easily by untwisting and unturning the corresponding braids.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra.

Logic puzzles (🌶, Misha, $\text{TW}\boxed{\Theta\text{FS}}$)
Here are a few logic puzzles, some more well-known than others:

(1) You are shown two closed boxes, each containing either cookies or carrots. (There is no guarantee that one contains cookies and the other contains carrots.) They are labeled as follows:

Box 1: At least one of these boxes contains cookies.
Box 2: The other box contains carrots.

You are told that either both labels are true, or both are false. If you want cookies, which box should you open?

(2) Every mentor at Mathcmap either always tells the truth, or always lies. You run into Tanya and Travis at dinner, and decide to find out whether they tell the truth or lie.

"Are you a truthteller?" you ask Tanya, who responds, "Banana." This is a word in food tongue that either means "Yes" or "No", but you have forgotten which one.

"Does banana mean 'Yes'?" you ask Travis. Travis replies, "It does. You shouldn't believe Tanya, though; she always lies."

What can you conclude about Tanya and Travis?

(3) X and Y are two integers bigger than 1; their sum is 100 or less. Sam is told their sum $X+Y$, and Priya is told their product $X \cdot Y$. Both Sam and Priya always tell the truth, and know all the information in this paragraph.

Sam and Priya then hold the following conversation:

Sam: I know that Priya does not know X and Y.
Priya: Now I know X and Y.
Sam: Now I also know X and Y.

What are X and Y?

In this class, we will explore logic puzzles like these, solve many of them, talk about how we can solve them more easily, and see some ways in which they're connected to other kinds of math.

Homework: Recommended

Class format: All over the place; there will be some lecture, some independent work on logic puzzles, and some discussion.

Prerequisites: None.

Neural codes (, *Zoe Wellner*, TW$\boxed{\Theta\mathrm{FS}}$)
The stimuli of a given neuron can be modeled by a convex set and combinatorial objects known as neural codes can extract information about the space covered by these regions. Although these were initially of interest to determine how the brain stores information, topologists and geometers have expanded on the questions we ask about these objects. We will be looking at how codes behave in different dimensions and get used to working with the spaces that arise from codes and other related objects.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Non-standard analysis (, Krishan, TW$\boxed{\Theta\mathrm{FS}}$)
Intuitively, a function on the reals is continuous if it sends points which are close together to other points which are close together. This idea is relatively simple, but unfortunately, the definition of continuity from real analysis is more complicated. In real analysis, we introduce epsilons and deltas as bookkeeping devices to make the intuition precise, but what if I told you that we could do without them?

In non-standard analysis we have access to actual infinitesimal numbers. So we can say that a function on the reals is continuous when it sends points that are infinitesimally close to other points that are infinitesimally close. Similarly, we have access to infinite numbers, so we can say that a sequence converges to a limit, L, if the "infinite" terms in the sequence are infinitesimally close to L.

This class will mention some ideas from logic, but logic will not be the main focus, and no prior experience with logic is required to follow the class.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Familiarity with epsilon-delta proofs

Predicting the future (, *Rice Neyman*, TW$\boxed{\Theta\mathrm{FS}}$)
You're deciding where to hold Mathcamp in 2026 and want to make sure that the campus is safe for campers. And so you decide you need to know how many bears there will be on the Champlain College campus in 2026. But you're no bear expert, so you decide to consult the local ursinologist (bear expert) for a forecast. You're happy to spend some money to make sure that the forecast is good, so you decide that you will pay the ursinologist according to the accuracy of their prediction. (They make a prediction about the number of bears, you wait until 2026, check how many bears there are, and the more accurate the prediction, the more you pay them.) It turns out that unless you're careful with your payment scheme, you could incentivize the ursinologist to lie to you! How should you set up the payment scheme so that a self-interested ursinologist will tell you the truth?

But then you decide that it's really important to know how many bears there will be, so you decide to ask not one but *three* ursinologists for a forecast. And they all give you different answers: one says that there will be 5 bears on campus, another says 10 bears, and a third predicts a whole 100 bears! What's the best way to aggregate these predictions into a single number? It turns out that the answer depends on how you decided to pay the ursinologists!

On Day 1 we will talk about *forecast elicitation*: how to incentivize forecasters to tell you their true beliefs. On Day 2 we will talk about *forecast aggregation*: combining multiple forecasts into one.

And on Day 3 we will talk about *prediction markets*, which do both of these things at the same time. Come learn about the secrets of predicting the future!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

The Borsuk–Ulam Theorem (🌶️, Arya, $\boxed{\text{TW}}\Theta\text{FS}$)

In this class, we shall discuss the Borsuk–Ulam theorem, which states that at any point of time, there are two diametrically points on Earth that have the same temperature and air pressure. One can proceed to make deep statements about the topology of spheres, or one can proceed to make combinatorial statements about triangulations and colourings.

This shall be an introduction to topological combinatorics, relating results in topology (such that continuous functions on the disk have a fixed point) to combinatorial results (the game of Hex cannot end in a draw). Time permitting, we shall talk a bit about algebra and the secret ingredient - projective spaces.

Homework: Optional

Class format: Interactive lecture

Prerequisites: None.

Ultrafilters and voting (🌶️🌶️, Krishan, $\boxed{\text{TW}}\Theta\text{FS}$)

Imagine you and your friends are trying to decide where to go for dinner. You all have your own personal ranking of the options but somehow you need to combine your individual rankings into a group ranking. If you were hoping that math could help you with this problem then you're out of luck! It turns out that there is no "fair" way to solve this type of problem. This result is known as Arrow's Impossibility Theorem. In this class we will formulate the theorem precisely and will see how this concrete result can be proven using ultrafilters (a type of object most commonly seen in logic).

Homework: Optional

Class format: Interactive lecture

Prerequisites: Familiarity with basic set theory (infinite unions, infinite intersections, etc)

Why do we need measure theory? (🌶️🌶️🌶️, Tanya, $\text{TW}\boxed{\Theta\text{FS}}$)

Suppose you're trying to compute the volume of your aquarium—one way to do so is by multiplying the dimensions of the container. However, what if what you really care about is the number of fish that the aquarium can support. Or, perhaps, instead you're interested in the amount of oxygen in the water. Whilst the number of fish and oxygen levels are not conventional notions of "volume", they are very reasonable measures of size of the aquarium. In this class, we will investigate what it means for a set of real numbers to be measurable and why we need to develop the theory of measures carefully. In particular, we will construct the infamous Vitali set, study the Caratheodory criterion for measurability and compute the volume of the Cantor set.

You may have noticed that Charlotte's also teaching a class about measure theory this week: you should take this class if you already have some familiarity with analysis and are curious to dive deeper into technical details behind foundations of measure theory.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: some basic analysis

Colloquia

Teaching Math to Computers (*Apurva Nakade*, Tuesday)
Will computers ever be able to understand mathematics? Will computers ever be able to prove theorems? How does one even communicate proofs to a computer? What is the meaning of life?

In this talk, I will discuss the project of math formalization using Lean theorem prover. Lean is an open-source programming language which can be used to encode mathematical proofs. There is a vast and rapidly growing library of mathematical proofs in Lean maintained by mathematicians and computer scientists around the world. In the first half of the talk, I will give an overview of the library and show some example proofs in Lean. In the second half, I will discuss type theory (an alternative to set theory) which forms the basis of the Lean programming language.

The talk only requires a basic understanding of mathematical logic and a mild interest in programming. A good sense of humor is recommended but not required.

Antinomy: meditations on Gödel's undecidable sentences (*Ari Nieh*, Wednesday)
The word "paradox" gets thrown around a lot in mathematics, but what does it actually mean? The celebrated logician Quine classified them into three types:

- Veridical: using correct premises and reasoning to reach a true (but surprising) conclusion
- Falsidical: using faulty premises or reasoning to reach a false conclusion
- Antinomy: using "correct" premises and reasoning to nonetheless reach a contradiction

Antinomies are the most disturbing kind of paradox. They indicate something shaky in the foundations upon which your logic is built. In fact, resolving them may require rewriting the fundamental rules of mathematics!

In this talk, we'll explore several entertaining examples of Quine's three types of paradoxes. We'll see that, in some sense, they're secretly all the same type. We'll discuss the influence of paradoxes on the development of logic and set theory in the 20$^{\text{th}}$ century. Despite the spooky abstract language, this talk will mostly be fun and accessible, with just a teensy bit of brain-melting.

An introduction to cryptography (*Jess Wernig*, Thursday)

Future of Mathcamp (Staff, Friday)
This is an event we host every year where we ask YOU the campers for feedback on how camp is going. We care about the things you have to say and many times, campers come up with brilliant ideas to improve camp! Please come and help brainstorm how to make Mathcamp better!

CLASS DESCRIPTIONS—WEEK 4, MATHCAMP 2023

Contents

9:10 Classes	1
10:10 Classes	3
11:10 Classes	5
1:10 Classes	9
Colloquia	11

9:10 Classes

Back to basi(c)s (, Travis, $\boxed{\text{TW}\Theta\text{FS}}$)
One of my favorite things in mathematics is using one kind of math to do another kind of math, and one of my favorite maths to apply to other maths is linear algebra. Why choose randomly when you can elegantly extract a simple solution through the thrilling theory of vector spaces?

Thus, this class consists of ample applications of linear algebra to combinatorics, discrete geometry, and convex geometry. We'll see how fundamental notions in linear algebra, such as dimension and the inner product, can be used to prove results in combinatorics. Then, we'll continue using these techniques to prove things about possible patterns of points and lines in Euclidean space. (For example, how many points can you place in $\mathbb{R}^n$ such that the distance between every pair of points is either 1 or 2?) We'll see how to use eigenvalues to bound the size of any collection of "equiangular lines" through the origin in $\mathbb{R}^n$, and we'll prove some foundational results in convex geometry. And if we have enough time, we'll use vectors to get a surprisingly good approximation for the famous MAX-CUT problem on graphs.

(Previously titled *What are your vectors vorth? or, part of the part of combinatorics and discrete geometry that ve can do easily vith linear algebra.*)

Homework: Optional

Class format: Interactive lecture

Prerequisites: Intro to linear algebra. (You should be comfortable with linear independence and dimension and have seen the dot product before. Comfort with eigenvalues will help in a small part of the class but isn't necessary for the rest.)

Finite fields (, *Aaron Landesman*, $\boxed{\text{TW}\Theta\text{FS}}$)
What do the rational numbers, complex numbers, and real numbers have in common, but not share with the integers? They are all fields; we can add, subtract, multiply, and divide elements in them. But which finite sets also have these properties? What possible sizes can such a finite set have? What are the possible subfields? These questions all have simple, beautiful answers which we will present in this course. Finite fields are crucially used throughout number theory, algebraic geometry, cryptography, and coding theory. After classifying finite fields, we will solve a number of combinatorial counting problems over finite fields, such as computing the average number of roots of a polynomial with coefficients in a finite field.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra, group theory

Functions of a complex variable (Week 2 of 2) (🌶🌶🌶, Mark, TWΘFS)
This is a continuation of last week's class. If you didn't take the class last week and want to join, you are encouraged to check with me first.

Homework: Recommended

Class format: Interactive lecture.

Prerequisites: None.

High-school algebraic geometry (🌶🌶, Neeraja, TWΘFS)
This is a class about real root counting, i.e. finding how many real roots a polynomial $p(x)$ with real coefficients has in a given interval $[a, b]$. We will prove Sturm's theorem which answers this question precisely, and along the way we'll prove some easier results like Descartes' law of signs. At the end of the class, you will be able to answer questions like the following: what conditions should we impose on a, b, c so that the polynomial $ax^3 + bx + c$ has exactly one real root?

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Derivatives (we will need to take derivatives of rational functions)

Kuratowski's game (🌶🌶🌶, Ian, TWΘFS)
Let X denote a subset of real numbers. You may take closure and complement indefinite number of times on this set X. What is the maximal number of distinct sets you can get from applying these operations, and which set X distinguishes these operations? Now, what if we introduce a frontier (or boundary) operator? What if we bring in two topologies? How about n topologies?

In 1922 K. Kuratowski answered the first question through his thesis—the answer was 14. Now the latter questions get more complicated as we increase the number of operators and the number of topologies. In 2021, myself and co. proved that if we have n saturated topological spaces and closure, complement, frontier operators, then the number of distinct sets is finite. Even better, a quartic polynomial $p(n) = \frac{5}{24}n^4 + \frac{37}{12}n^3 + \frac{79}{24}n^2 + \frac{101}{12}n + 2$.

On the first day we establish some fundamentals of point-set topology. Then on the second day, we discuss different variations of the Kuratowski game, specifically the variation I investigated, and see how to obtain the polynomial expression for the number of distinct sets of the closure-complement-frontier problem of the saturated polytopological space.

Homework: None

Class format: Lecture

Prerequisites: Some basic combinatorics (counting principles)

Markov chain Monte Carlo (🌶🌶, *Moon Duchin*, TWΘFS)
Turns out a really good way to study complex systems is through a kind of mostly blind exploration where you make dumb, semi-random decisions and wait a really long time. This is the magic of "MCMC", a sneaky powerful method of harnessing what computers do best: repeat easy calculations a lot. In this course we'll approach this topic from scratch, through the theory of Markov chains (random walks with no memory), and will get as far as the Metropolis method, which the IEEE lists as the #1 algorithm of the 20th century. We'll also see applications to codebreaking, autocomplete, and elections.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

McKelvey's Chaos Theorem (🌶, Ben, TWΘFS)
In Mira's colloquium, she mentioned—cryptically—that there's a good way to pick a winner if everyone's preferences are "linear" or "single-peaked." This class will begin by finding one way in which there is a "natural winner" in this case (under some modest simplifying assumptions, including the "linear" business—this means that the "issue space" is one-dimensional). That's a nice result, which is a pleasant and happy counterpoint to all of the other results Mira told us about.

IF YOU WANT TO STAY HAPPY PLEASE SKIP TO THE NEXT BLURB NOW.

You might be wondering "What if we had TWO dimensions of issue-space instead of one? How much does that break?"

It breaks everything! There is a natural sense in which, instead of having ONE natural winner, we (probably) enter a world in which everyone is a winner—so no one is.

Homework: Recommended

Class format: Lecture, with a few small activities to explore the topic

Prerequisites: Essentially none

10:10 Classes

Gaussian magic (🌶🌶🌶🌶, Tanya, TWΘFS)
Did your teacher ever say that a class is being graded on "a curve"? The "curve" in this context is referring to the Gaussian distribution, which, for reasons still unbeknownst to me, is thought to be the "natural" distribution that grades must follow. Despite this potentially misguided usage, the Gaussian random variable is ubiquitous throughout probability and statistics for a number of good reasons, which we will explore in this class. Turns out, Gaussians have many wonderful properties that simplify computations and, in certain cases, allow us to relate functions of non-Gaussian variables to those involving Gaussian ones. If time permits, I will briefly discuss some recent advances in understanding of Gaussian random matrices, which is related to the subject of my PhD. Attend this class to learn how Gaussian magic happens!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: some previous exposure to multivariable calculus, analysis (particularly limits), and linear algebra

How to rob your friends (🌶🌶, Arya, TWΘFS)
Imagine a world where you're broke and severely in debt, but all your friends are rich and generous. You make your friends stand on the vertices of a graph. At each instance, choose a vertex, and send $1 from the vertex to each of its neighbours. Can you repeat this move over and over to eventually get out of debt, WITHOUT sending some other friends into debt?

In this class, we shall talk about how greed is necessary to prevail in society, and study some linear algebra with graphs. One of the proofs shall involve starting a wildfire and watching it spread (too soon?). Weirdly enough, this theory builds on to study sand piles and Riemann surfaces.

Homework: None

Class format: Lectures.

Prerequisites: You should know what a graph is.

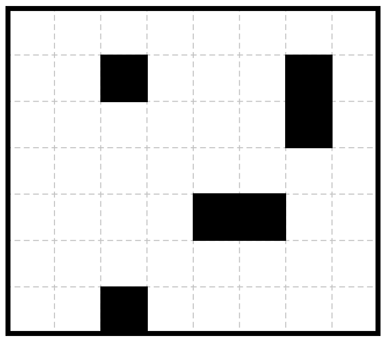
How to rob your friends 2: non-transitive dice boogaloo (🌶️, Eric, TWΘ$\boxed{\text{FS}}$)
You might think that if I give you a set of 3 dice A, B, C and tell you that on average A rolls higher than B and B rolls higher than C, it would have to be the case that A rolls higher than C. But this isn't true! The property of *transitivity* fails for the relation "X rolls higher than Y on average." In this class we'll make sense of that statement and show that we can make transitivity fail as badly as we want!

Homework: Optional

Class format: Interactive lecture

Prerequisites: None.

Mathematical Concepts for Solving Puzzles (🌶️🌶️ → 🌶️🌶️🌶️🌶️, Della, T$\boxed{\text{W}\Theta\text{FS}}$)
I have a puzzle for you! Draw a loop that goes horizontally and vertically between adjacent white squares, which visits every white square exactly once.[1]



In this class, you'll solve lots of puzzles along these lines! Some of them will require some math: to make progress, you'll have to discover an interesting lemma about the puzzle.

Actually, this is three independent classes, and you can come to any subset of them! They'll each focus on some concept from math and some types of puzzles it can be applied to:

- MCSP: Planarity (🌶️🌶️, T$\boxed{\text{W}}$ΘFS)
- MCSP: Parity (🌶️🌶️🌶️, TW$\boxed{\Theta}$FS)
- MCSP: Penalty (🌶️🌶️🌶️🌶️, TWΘ$\boxed{\text{FS}}$)

Homework: Optional

Class format: Working on puzzles, with some discussion of the math at the end.

Prerequisites: Basic graph theory, such as the definitions of 'planar' and 'cycle'.

Polynomial methods in combinatorics (🌶️, Narmada, T$\boxed{\text{W}\Theta\text{FS}}$)
Have you ever wondered why there are so many algebraic combinatorialists out there? Come to my class to answer this burning question! We'll define finite fields as the right setting to frame combinatorial problems, and then we'll see how algebra over finite fields helps us solve these problems. Our two big results will be the Schwartz–Zippel lemma and the Combinatorial Nullstellensatz, which we'll use to provide short proofs of famous results like the Kakeya conjecture and the Cauchy–Davenport theorem.

Homework: Optional

Class format: A mix of lecture and group work

Prerequisites: None.

[1]Fancy graph theorists would call this a Hamiltonian cycle.

Problem solving: induction (🌶🌶, Misha, T$\boxed{\text{W}\Theta\text{FS}}$)

You probably first saw induction in the context of proving a result like

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}.$$

This is not easy for everyone, but once you've solved one problem like this, you can solve every problem like this one; all it takes is a tiny bit of algebra.[2] That's not what this class is about!

We will see how these proofs can get much more complicated. Our induction will start out strong, and on each day of class it will get stronger than all the previous days combined. You'll see examples of crazy induction in algebra, game theory, number theory, and other theories. You'll learn how to use induction (and how *not* to use it) to solve problems of your own, olympiad and otherwise.

In class, we will spend time solving problems together; I will focus less on answering the question "why is this claim true?" and more on answering the question "why would we think of solving a problem this way?" There will be plenty of problems left for homework, and you will not get much out of a problem-solving class unless you spend time solving those problems.

Homework: Recommended

Class format: A bit of lecture, and a lot of what I call "problem-solving lecture": you will tell me how to solve the problems I gave you.

Prerequisites: None.

11:10 Classes

Guess Who? (Week 2 of 2) (🌶🌶🌶 → 🌶🌶🌶🌶, Tim!, $\boxed{\text{TW}\Theta\text{FS}}$)

This is the second week of *Guess Who?*! If you came to the first week, I highly recommend you to come back for the second to see the full story come together! The chili level is changing from 🌶–🌶🌶 last week to 🌶🌶🌶–🌶🌶🌶🌶 (mostly 🌶🌶🌶) this week, but regardless of chilis, if you liked last week, I encourage you to come back for this week.

If you didn't take the first week of the class but want to take the second, come talk to me; there are things you would need to catch up on.

Here's what's coming up in the second week:

In the first week, we saw that the divide-in-half strategy was not optimal, and by expressing *Guess Who?* as a zero sum matrix game, we were able to find the actual optimal strategy.

While the divide-in-half strategy is not the "BEST" strategy that Mark Rober claims it is, it does have a compelling story: that broad questions are better than narrow questions. On the other hand, the actual optimal strategy doesn't come with a clear story attached—and the pattern is so nice and beautiful, and so different from divide-in-half, that really *ought* to have a story. There should be some good reason why the numbers are what they are other than just "we can do a calculation that shows it's true."

There *is* a good reason and we will find it. To get there, we will have to temporarily set *Guess Who?* aside and consider some strange variations on it. In regular *Guess Who?*, each question that you ask applies to some integer number of characters. In *Super Guess Who*, each question applies to some number of characters that doesn't have to be an integer. And *Ultra Guess Who* takes this a step further.... Once we've solved the very odd game of *Ultra Guess Who*, we'll be able to return to regular *Guess Who?* and understand what is going on. We'll discuss continuous random variables and cumulative distribution functions (this will involve a little calculus).

We'll also look at the calculation side of things. We were able to construct a matrix game for *Guess Who?* that a computer could solve reasonably quickly to give us the optimal strategy, but we were

[2] If you went to Apurva's colloquium, you know that to write a proof for such a problem in Lean, you use the exact same code, no matter what the actual sum is. That's not otherwise related to this class, but it's indicative of how boring the proof turns out to be.

lucky that the number of possible mystery characters, $n = 24$, was relatively small. This technique does not give reasonably sized matrix games when n is larger than 24. But, with another technique, we'll be able to solve *Guess Who?* with n characters efficiently for larger n (in time polynomial in n). To do this, we'll need to discuss linear programming and convex optimization, as well as network flows in graphs. The key will be to lift the problem to a higher dimensional space.

Also, sometimes everything becomes clear when you figure out the right picture to draw. I promise a variety of interesting and illuminating diagrams. Now let's finish giving this children's game the mathematical analysis it deserves!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None

Perron trees (everyone loves analysis, part 1) (, Charlotte, TWΘFS)
As everyone knows, this is the year that everyone loves analysis!!!! You've probably heard the analysis staff talk about the existence of pathological, terrifyingly counterintuitive examples that seem to break everything we believe about the natural order of things—and how much fun this is! I am very happy to tell you that we will talk about one of my favourite examples of this in this class!

Specifically, we'll talk about the "Kakeya needle problem," which is about the following: imagine you have a needle of width zero and length one lying on a table, and you would like to rotate it continuously 180 degrees. You could achieve this feat just by rotating it within a circle of diameter 1. But what if I challenged you to rotate it within a region of smaller area? Let's say you managed to achieve this. What if I challenged you to rotate it within a region of even smaller area? And so on...

Eventually, this game would have to end, because surely the smallest region for which I could do this is not *that* small, right?? Right????? WRONG.

Homework: Optional

Class format: Mix of lecture and group work.

Prerequisites: none

Quiver representations part I (, Kayla, TWΘFS)
Don't *quiver* in fear: path algebras are here! We as humans aim to understand the world *linearly*—we approximate curvy functions with a tangent line, areas bounded by functions with rectangles in calculus. Algebraists care about *linearizing* finite-dimensional algebras! To achieve this linearization, people began the study of representation theory. In this class, we will be studying the representation theory of finite dimensional algebras! It sounds quite spicy, but don't *quiver* in fear, there are nice concrete ways that people study these abstract algebraic objects. In particular, we will be studying quiver representations—a linearization of directed graphs. Come see the basics of quiver representations, a classification theorem that connects quiver representations to many different areas of math and about path algebras that allow us to study any finite dimensional algebra in terms of quivers!

Homework: Optional

Class format: Interactive lecture.

Prerequisites: Linear algebra, group or ring theory: comfortability with quotient groups or rings.

Quiver representations part II (, Raj, TWΘFS)
This is the second installation of quiver representations. As you will see in Kayla's part of the class, quivers are directed graphs, and a quiver representation is an assignment of a vector space to every vertex of the quiver and a linear transformation to every edge. The theory of quiver representations is broad and touches many different fields of mathematics, including the theory of Dynkin diagrams, the

study of finite subgroups of $SU(2)$, and algebraic geometry. We will see many of these connections in this course, and hopefully by the end of this course, you will be convinced of the power of quivers!

Homework: Recommended

Class format: Interactive lecture.

Prerequisites: Quiver representations part I

The outer life of inner automorphisms (🌶🌶🌶🌶, Steve, $\boxed{\text{TW}\Theta\text{FS}}$)

Suppose G is a group, $g \in G$, and $f : G \to H$ is a group homomorphism. Via conjugation, we know that g induces an automorphism of G, namely

$$\alpha_g^G : h \mapsto ghg^{-1}.$$

Such automorphisms are called **inner automorphisms**, and are very important.

But something even cooler happens: this automorphism *makes sense after we apply* f, in the sense that

$$\alpha_{f(g)}^H : H \to H : h \mapsto f(g)hf(g)^{-1}$$

is an automorphism of H which is "built from" the original conjugation automorphism in a natural way. This is neat, because generally automorphism *don't* "travel along" homomorphisms in this way.

A natural question at this point is whether there are any *other* ways to define an automorphism (besides conjugation, that is) so that it will "travel along" homomorphisms in a good way. It turns out that the answer is negative, and so inner-ness can be detected from the outside. This (and more) is a theorem due to Bergman only 11 years ago. In this class we'll follow Bergman's paper and get a rather unusual introduction to the abstract subject of **category theory**. (No experience with category theory is expected!)

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Group theory (specifically, being comfortable with the statement "Conjugation by a fixed element is an automorphism"; if you can follow the first paragraph of this blurb, you're fine)

Trail mix (🌶 → 🌶🌶🌶, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)

Is your mathematical hike getting a little too strenuous? Would you like to relax a bit with a class that offers an unrelated topic every day, so you can pick and choose which days to attend, and that does not expect you to do homework? If so, some Trail Mix may be just what you need to regain energy. Individual descriptions of the five topics follow.

Trail Mix Day 1: The Prüfer Correspondence (🌶–🌶🌶).

Suppose you have n points around a circle, with every pair of points connected by a line segment. (If you like, you have the complete graph K_n.) Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments, but in only one way. (This tree will be a spanning tree for K_n.) How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll find a combinatorial proof that is especially cool.

Prerequisites: none

Trail Mix Day 2: Cyclotomic Polynomials and Migotti's Theorem (🌶🌶🌶).

The cyclotomic polynomials form an interesting family of polynomials with integer coefficients, whose roots are complex roots of unity. Looking at the first few of these polynomials leads to a natural conjecture about their coefficients. However, after the first hundred or so cases keep confirming the conjectured pattern, eventually it breaks down. In this class we'll see, and if time permits prove, a theorem due to Migotti, which sheds some light on what is going on, and in particular on why the conjecture finally fails just when it does.

Prerequisites: Some experience with complex numbers, preferably including complex roots of unity; some experience with polynomials.

Trail Mix Day 3: Integration by Parts and the Wallis Product (🌶🌶🌶).

Integration by parts is one of the only two truly general techniques for finding antiderivatives that are known (the other is integration by substitution). In this class you'll see (or review) this method, and encounter two of its applications: How to extend the factorial function, so that $(1/2)!$ ends up making sense (although the standard notation used for it is a bit different), and how to derive the famous product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots ,$$

which was first stated by John Wallis in 1655.

Prerequisites: Basic single-variable calculus.

Trail Mix Day 4: Perfect Numbers (🌶).

Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called perfect, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes — a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Prerequisites: None

Trail Mix Day 5: The Jacobian Determinant and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (🌶🌶🌶).

How do you change variables in a multiple integral? In the "crash course" in week 1 we saw that when you change to polar coordinates, a somewhat mysterious factor r is needed. This is a special case of an important general fact involving a determinant of partial derivatives. We'll see how and roughly why this works; then we'll use it to evaluate the famous sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} .$$

(You may well know the answer, but do you know a proof? If so, do you know a proof that doesn't require Fourier series or complex analysis?)

Prerequisites: Multivariable calculus (the crash course is plenty); some experience with determinants.

Homework: None

Class format: Interactive lecture

Prerequisites: Varies by day—see blurb(s) above

aspacefillingcurve (everyone loves analysis, part 2) (🌶🌶, Charlotte, TW$\boxed{\Theta\mathrm{FS}}$)

As everyone knows, because I just told you, this is the year that everyone loves analysis!!!! You've probably heard the analysis staff talk about the existence of pathological, terrifyingly counterintuitive examples that seem to break everything we believe about the natural order of things—and how much fun this is! I am very happy to tell you that we will talk about another one of my favourite examples of this in this class!

We have intuition about what continuous functions are supposed to look like—they are nice and friendly! Things can't get too crazy with them! Surely, if you give me a continuous function from $[0,1]$ to $\mathbb{R}$, I can draw its graph with ease! It's just a nice looking curve in the plane, right????

Surely, a continuous function must map the one-dimensional interval $[0,1]$ to a one-dimensional curve, right????? WRONG.

We will see just how emphatically WRONG this is by constructing a horrible creature called aspacefillingcurve. Using compactness and completeness (which are concepts we will discuss in class), we will define a continuous function on $[0,1]$ that fills up the entire TWO-dimensional square $[0,1]^2$!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Have taken the epsilons and deltas class, or something equivalent. In particular, you should be comfortable with the epsilon-N definition of a limit and proofs involving epsilons. You should be comfortable with sequences of functions and uniform convergence. You should have seen the definition of a Cauchy sequence before. You should definitely be familiar with metric spaces. If you do not have one of these pre-reqs, then add one to the chili level.

1:10 Classes

Braid groups (, Arya & Kevin, $\boxed{\text{TW}\Theta\text{FS}}$)
Take n strings. Tie one end of each string to a metal bar, twist the strings around a bunch, and tie the free ends of the strings to another bar. You've got yourself a braid!

This class will be an introduction to the topology and algebra of braids. We'll learn about several different perspectives on braids (the Artin group presentation, configuration spaces, mapping class groups) before moving on to applications of braids to modern research in cryptography and 3-dimensional topology.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic group theory

Continued fractions (, Ben, $\boxed{\text{TW}\Theta\text{FS}}$)
Did you take Ben's class on the ergodic theorem and wonder what the deal was with all of the continued fraction stuff at the end? Have you seen a lot of relay problems that look like

$$5+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\dots}}}}$$

or something like it? Do you want to learn about lost[3] and secret[4] knowledge that mathematicians of the past had?

This class will do all of those things![5] In particular, we'll develop the theory of continued fraction expansions, prove their (very nearly almost) uniqueness, talk about why they're better than decimal expansions, and, time permitting, prove a lovely result of Lagrange characterizing the quadratic irrationals (those of the form $a+b\sqrt{c}$ for $a,b\in\mathbb{Q}$ and $c\in\mathbb{N}$ in terms of their continued fraction expansions.

The most important piece of secret knowledge I will impart is how to write these things without filling half a board.

Homework: Recommended

Class format: Lecture

Prerequisites: None!

[3]Not that lost, since I'm teaching it.
[4]Not that secret, since I'm teaching it.
[5]Except, again, the knowledge is neither lost nor secret.

Intersections of algebraic plane curves (🌶🌶🌶🌶, *Nic Ford*, $\boxed{\mathrm{TW\Theta FS}}$)

An *algebraic plane curve* is the set of points in the plane where some polynomial is equal to zero. For example, a circle is an algebraic plane curve, because it's the set of points (x, y) where $x^2 + y^2 - 1 = 0$. One of my favorite facts about these objects is Bézout's Theorem, which says that if you have two algebraic plane curves, one cut out by a polynomial of degree d and one by a polynomial of degree e, then they intersect in exactly de points. This theorem is simple, beautiful, and fun to prove, but—as I've stated it here—unfortunately quite false. (See how many counterexamples you can come up with!)

But we're not the sort of people to let something as trivial as falsehood get in the way of proving such a pretty result. It will turn out that, with enough extra adjectives and caveats, we will be able to modify this version of Bézout's Theorem into something we *can* prove, and that's the task we'll take up in this class. Along the way, we'll also learn how to define and compute the *resultant* of two polynomials, a method for solving systems of polynomial equations that isn't taught much anymore but which is both critical to our proof of Bézout's Theorem and a fun object in its own right.

Homework: Required

Class format: Interactive lecture

Prerequisites: You should have comfort with the complex numbers and with manipulating polynomials in several variables. It will also be very helpful if you know how to compute the determinant of a matrix and know what it tells you about solutions of systems of linear equations, but we'll briefly review this in class.

Quantum computing (🌶🌶🌶, Krishan, $\boxed{\mathrm{TW\Theta FS}}$)

In a normal computer the fundamental unit of information is a bit, a value that can either be 0 or 1. In quantum computing, we treat **0** and **1** as vectors, so that the data of a single quantum bit (or qubit) is a linear combination (or superposition) of **0** and **1**. These quantum bits can also be entangled meaning that their states are correlated in a way that has no classical analogue.

In this class, we'll study the math underlying quantum computation (it's a lot of linear algebra). Then we'll prove some theorems about what quantum computers can and cannot do and will examine some quantum algorithms. Don't worry if you haven't seen much physics before, I'll explain all of the physics we need during class.

Homework: Required

Class format: Interactive lecture

Prerequisites: Linear algebra is required, and some familiarity with big O notation and pseudocode is suggested (+🌶 if missing)

{Game, graph} theory against the world (🌶🌶, Ania, $\boxed{\mathrm{TW\Theta FS}}$)

Consider the following scenarios and questions. Imagine that . . .

- You have access to information about links between all the Internet pages. Which is the most important one?
- You have some information about connections within a terrorist network, but you have resources to tap the phone of only one of the terrorists. Which one should you choose?
- Company A sells cats for \$10, and company B sells hats for \$5. They decide to merge and sell cats wearing hats for \$20. How should they split the income?

What do these questions have in common? They are all related to {game, graph} theory, and we will discuss them during this class.

We will start by talking about different centrality measures, which are the ways of finding the most "central" vertex of a graph. In particular, we'll define and understand PageRank, a centrality which was originally used in the Google's search engine. Then we'll dive into cooperative game theory, discuss concepts like Shapley and Myerson value, and see how we could break terrorist networks using game-theoretic centrality measures.

Lastly, we'll talk about the algorithm which helps maximise the numbers of patients getting kidney transplants ("the kidney-exchange problem"), and possibly about other topics from the social choice theory.

Homework: Optional

Class format: Interactive lecture and some problem-solving on your own or in small groups

Prerequisites: Very basic definitions from graph theory

Colloquia

The geometry of fractal sets (Neeraja, Tuesday)
Many spatial patterns that occur in nature are fragmented and irregular to such an extreme degree that classical geometry is of little help in describing them. For a long time, these fragmented patterns and shapes were regarded by mathematicians as "pathological" and unworthy of study. In the last fifty or so years, it has become evident that certain irregular shapes give a much better description of nature than the figures of classical geometry. A framework has been developed to study the geometry of such sets and this general theory is called fractal geometry. In this colloquium talk, we'll see examples of fractal sets occurring in many different contexts, including self-similar sets like the Cantor set, fractal sets in the study of dynamical systems like the Julia sets and the Mandelbrot set, "fractal curves" like the Weierstrass function, paths of Brownian motion, and fractals arising in the theory of Diophantine approximation. We'll also see some of the interesting geometric properties of fractal sets.

The hat-xiom of choice (Travis, Wednesday)
The JCs, after several weeks under the thrall of their pink sunglasses, have been corrupted by the constant flow of pink power and attempted a coup of Mathcamp. Unfortunately, due to their preternaturally powerful logistical abilities, the JCs have succeeded and captured all the mentors. But with their goal accomplished, they have no good reason to keep the mentors locked up, so they decide to play a game. They tell the mentors they will place either a duck or a dino on each mentor's head, and any mentor who guesses correctly what animal they've been behatted with will be allowed to escape. But what they don't tell the mentors is that during their reign, they've secretly hired infinitely many other mentors who will also be forced to play this game. . .

In this ~~colloquium~~ tell-all exposé, I'll give an insider's view of the mentors' *secret strategy* and how we used a practical application of the Axiom of Choice to outwit the JCs and break the real numbers. Come hear the mentors' secret knowledge to help us prevent the JCs from overthrowing camp yet again!

(Mira's secret surprise class at the beginning of Week 3 overlaps with this colloquium, but we'll also see several other versions of the hat game and how this relates to measure theory!)

The evolution of proofs in computer science (*Yael Tauman Kalai*, Thursday)
In this talk we will learn about the evolution of proofs in computer science. We will start by introducing the magical notion of zero-knowledge proofs and get a glimpse into how to construct them. Then we will talk about how these proof systems changed the way we think about mathematical proofs, and mention interesting applications.

I'd like some geometry with my topology (*Moon Duchin*, Friday)
In this colloquium I'll introduce some concepts of geometric topology, or all the ways to make stretchy shapes rigid. Tori and triangles are good basic examples, and I'll also let things get a little weirder.

CLASS DESCRIPTIONS—WEEK 5, MATHCAMP 2023

About the weird symbols appearing instead of chilis

We are trying an experiment in week 5 this summer! Instead of labeling every class with 🌶, 🌶🌶, 🌶🌶🌶, or 🌶🌶🌶🌶, we are using a two-symbol code to represent two aspects of the class. A one-word summary of the two axes could be "abstraction" and "pacing", but they are described in more detail below.

As always, feel free to talk to us to learn more about what our classes will be like! Also, no matter which symbols a class has next to it, we are always happy to spend more time outside of class to help you understand the material, resolve lingering doubts, or indulge your curiosity.

The first dimension: ⚑, 🏁, and ⚐

- ⚑: Just like the ⚑ flag is fully colored in, the class is "fully filled in" with examples. The instructor makes an effort to make the ideas easier to grasp immediately, even at the cost of presenting the material less generally.
- 🏁: A class with 🏁 next to it is halfway between ⚑ and ⚐. For example, you might see how familiar objects are special cases of general, less intuitive ideas.
- ⚐: In a class with ⚐ next to it, you will often have to grapple with ideas that are hard to reduce to concrete examples. You might have to wait to see how the new abstract ideas relate to things you already know.

The second dimension: 🚲, 🚗, and ✈

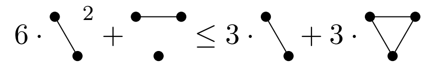
- 🚲: A class with 🚲 next to it will feel relaxed and stress-free. Do not worry at all about slowing the class down if you need to ask more questions to understand; the class plan is built to accommodate this.
- 🚗: The default pace of a Mathcamp class is 🚗: comfortable, yet brisk. Questions are welcome in all Mathcamp classes—but some questions may be postponed until TAU.
- ✈: A class with ✈ next to it will feel like an exhilarating sprint. The class will need to move from topic to topic quickly to get to the finish line; you might feel like you have to review your notes and/or talk to the instructor to fully grasp the material.

Contents

Superclasses 2
9:10 Classes 2
10:10 Classes 4
11:10 Classes 6
1:10 Classes 9
2:10 Classes 12
Colloquia 14

Superclasses

Flag algebra marathon (, Misha, $\boxed{\text{T}}$WΘF, 10:10–12:00 and 1:10–3:00)
Do you want to write inequalities where our variables are graphs?



Do you want to see the most roundabout proof that $R(3,3) = 6$?

Do you want to prove graph theory results from 1907 by methods developed in 2007?

Do you want to have gigabytes of multiplication tables to throw at supercomputers?

And do you want to do this **all in one day**?

Then this is the class for you.

This class is being taught in **marathon** format. We'll go 10:10am–12pm, take a break for lunch, and reconvene 1:10pm–3pm.

Homework: None

Class format: Marathon! (It will not all be lecture; you will spend some time in class solving problems.)

Prerequisites: You should be comfortable with matrix multiplication, linear transformations, and an assortment of basic concepts from graph theory.

Not the math we need, but the math we deserve (, Ben, Ian, Kevin, Krishan, Narmada, Neeraja, Raj, Tanya, and Travis, $\boxed{\text{T}}$WΘF, 11:10–12:00 and 1:10–3:00)
Do you ever feel like math is the most beautiful, transcendent, magical discipline that connects disparate ideas in a seemingly seamless manner? This class is here to shatter that illusion—unfortunately, math is filled with unintuitive, unwieldy, even monstrous counterexamples that will make you question if anything would ever make sense again. If this somehow doesn't scare you off, come join us on a tour of the ugly side of math the teaching staff has been trying to shield you from, sampled from across different fields according to the specialties of the instructors.

Homework: None

Class format: Lecture

Prerequisites: None

9:10 Classes

Computing trig functions by hand (, Misha, TWΘ$\boxed{\text{F}}$)
When you learn about trig functions, you typically memorize a few of their values (for $30°$ or $45°$, say) and if you want to know any of the other values, you get pointed to a calculator.

Has that ever seemed unsatisfying to you? If so, take this class, in which we'll see that finding some of these values is as easy as solving polynomials, and approximating all of them is as easy as multiplication. If time allows, we'll learn how to compute inverse trig functions, and also how to quickly find lots of digits of π.

Homework: None

Class format: Interactive lecture

Prerequisites: Be familiar with the formula $e^{ix} = \cos x + i \sin x$.

Dimers and webs (, Kayla, TW$\boxed{\Theta}$F)
In chemistry, a *dimer* is a polymer with only two atoms. A *dimer covering* of a graph G is a collection

of edges that covers all the vertices exactly once. One can think of vertices of G as univalent atoms that bond to exactly one neighbor. This is more commonly known as a perfect matching! A dimer model for a graph G is the set of all perfect matchings or dimer coverings of G.

In this class, we will be generalizing the notion of a dimer model to double and triple dimer models that satisfy some "web connectivity". What this boils down to is superimposing single dimer models such that their underlying graphs reduce to objects called non-elliptic webs.

If you like graphs, coloring edges of graphs and a lot of math with picture, this is the class for you!

Homework: None

Class format: Interactive lecture with activities!

Prerequisites: None

Elliptic functions (, Mark, $\boxed{\text{TW}\Theta\text{F}}$)

Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k)\,,$$

where $\sigma_i(k)$ is the sum of the i^{th} powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1],$$

which you are welcome to check if you run out of things to do.)

Homework: Optional

Class format: Interactive lecture

Prerequisites: Functions of a complex variable; in particular, Liouville's theorem

Introduction to Schubert calculus (, Raj, T$\boxed{\text{W}\Theta\text{F}}$)

"How many lines generically meet four given lines in space?" One goal of modern Schubert calculus is to solve such enumerative problems using tools from linear algebra and algebraic geometry. This course will be an introduction to this subject. The main components we will discuss are Schubert polynomials and the Grassmannian of linear subspaces of fixed dimension in space.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra, group theory

The Kakeya problem (, Alan Chang and Neeraja, $\boxed{\text{TW}\Theta\text{F}}$)

In this class, we will introduce the famous Kakeya problem, which asks, roughly: how small can a set be if it contains a line segment in every direction? This question can turn into a variety of math problems, depending on how we choose to interpret "smallness" of a set. For example: what is the smallest possible area of a set in the plane that contains a line segment in every direction? Or alternatively, what is the fractal dimension of such a set? In the class, we will also introduce a variant of the Kakeya problem that asks about the “smallness” of a set which contains a circle of every radius. Note: the construction of Kakeya sets that we’ll do in this class will be different from those done in Charlotte’s Perron trees class in Week 4.

Homework: Optional

Class format: Interactive lecture

Prerequisites: None

The puzzle of the superstitious basketball player (Tim!, T$\boxed{\text{W}}\Theta$F)
Sometimes I encounter a math problem, go though a bunch of work to solve it, then arrive at an answer too simple or elegant for the mess of work I did. At that point, I know that there is something deeper and more interesting going on, and I have to know what it is! If you took my class on Guess Who?, you’ve gone on that journey with me. I also had such an experience with the math puzzle below. It’s from Mike Donner, and it was published on FiveThirtyEight.

> A basketball player is in the gym practicing free throws. He makes his first shot, then misses his second. This player tends to get inside his own head a little bit, so this isn’t good news. Specifically, the probability he hits any subsequent shot is equal to the overall percentage of shots that he’s made thus far. (His neuroses are very exacting.) His coach, who knows his psychological tendency and saw the first two shots, leaves the gym and doesn’t see the next 96 shots. The coach returns, and sees the player make shot No. 99. What is the probability, from the coach’s point of view, that he makes shot No. 100?

I remember solving it. I had to do a bit of tedious calculation to arrive at the final answer. And when I saw the answer, I was surprised. It was so simple. I thought I was done with the puzzle, but really I was just beginning. Such a simple answer had to have a simple explanation, right? There are in fact a few simple explanations, each more satisfying than the previous.

In the end, I will make the following claim: even if we accept the scenario described by the puzzle, the basketball player’s view of the world is totally wrong, and he is probably just superstitious. Perhaps there is a lesson here that we can take back with us to our real lives.

Homework: None

Class format: Interactive lecture

Prerequisites: None

10:10 Classes

A couple things Ben kinda knows about measure zero sets (Ben, T$\boxed{\text{W}\Theta\text{F}}$)
Everyone[1] knows that the *countable* union of measure zero sets has measure zero. It’s not hard to convince yourself that you can take the union of $|\mathbb{R}|$-many measure zero sets and build a set that doesn’t have measure zero.

Continuum Hypothesis fans may now be wondering: what happens if you take the union of a number of sets that is *more* than countable and *less* than the size of the real numbers? This class will explore that topic! This will require us to ponder some apparently unrelated content—filters! dominating functions! posets!—and sit nicely between the worlds of set theory and analysis.

As advertised in the title, this is something I’ve seen once before, but haven’t brushed up on in a while!

Homework: Recommended

[1] Or at least everyone who has taken Tanya/Charlotte’s Week 3 class, or other measure theory.

Class format: Lecture

Prerequisites: Having seen measure theory will help motivate one day of this class, but nothing is really necessary except having seen induction.

From high school arithmetic to group cohomology (, Eric, TWΘF)
Adding two digits numbers is something we can do mostly automatically, even though we might have to do an annoying "carrying" operation. In this class we will think **incredibly hard** about what carrying is. It turns out that carrying is an instance of group cohomology! We'll explore that connection and use it as a pathway into learning about the subject of group cohomology.

Homework: Recommended

Class format: Learning through worksheets!

Prerequisites: Basic group theory, to the point of understanding that the groups $\mathbb{Z}/100\mathbb{Z}$ and $\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ are not isomorphic.

Geometry Gala (, Ian, TWΘF)
Welcome to the Geometry Gala, the epilogue of the Geometry Galore! In the Gala, we will discuss Jacobi's Theorem, an application of Trig Ceva and/or radical axis theorem which were discussed in the Galore. Do not worry if you haven't taken the Galore class, since I will briefly mention Trig Ceva and radical axis theorem in the beginning so that everyone is on the same page. Are you ready for the Grand geometric finale?

Homework: Recommended

Class format: 20 minute lecture + 20 minute problem solving + 10 minute wrap up

Prerequisites: Geometry Galore recommended (though not required)

Symmetric Functions and their Combinatorics (, Ian, TWΘF)
The topic of symmetric functions has a deep connection with combinatorics. In this course, our goal is to describe the symmetric functions with combinatorial objects such as Young tableaux and a set of lattice paths.

The main topics include:

- Monomial symmetric polynomials/functions;
- Elementary, homogeneous, Power sum symmetric functions;
- Young tableaux;
- Schur function;
- Jacobi–Trudi identity.

Homework: Recommended

Class format: Lecture

Prerequisites: A good understanding of basic combinatorics

Unicorns and Poland (, Arya, TWΘF)
"Unicorn paths" were defined by Polish mathematician Piotr Przytycki, because he initially wanted to define "one-cornered paths", and the word for "one-cornered" in Polish is very similar to the word "unicorns", and "unicorns" is way cooler. Polish people are so cool ¡3

But why do we care about these (and what are these)? Associated to every surface, there is a graph called the "curve graph", which very literally is a graph of curves on the surface. For deep reasons, this graph is very cool. Unicorns guide the way to travel along the unyielding terrains of this graph, and tell us a lot about the geometry of this graph.

Come to this class to get a feel about stuff people study in modern geometric topology!

Homework: Optional

Class format: Lectures (hopefully interactive? :P)

Prerequisites: None!

11:10 Classes

Calculus without calculus (, Tim!, $\boxed{\text{T}}$WΘF)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Eleni is 5 cubits tall and Krishan is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Eleni's head to the top of Krishan's head that touches the ground in the middle. What is the shortest length of string you can use?
- Della rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an very straight section of the Lake Champlain shoreline. The dog's person stands 20 meters away along the shoreline, and throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- When Mathcamp rented out the movie theater to see the *Barbie* movie, you had the chance to choose the optimal seat. Which seat should you have chosen so to make the screen take up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: We won't use calculus (that's the point), but it would be good if you've seen it for context.

How not to integrate (, Steve, TW$\boxed{\Theta}$F)
The function e^{-x^2} has no elementary antiderivative, but the doubly-improper "Gaussian integral"

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

can be easily computed: it's just $\sqrt{\pi}$. To see this, square it, rewrite the result as a double integral, and change to polar coordinates—this shifts from "$dx\,dy$" to "$r\,dr\,d\theta$," letting us wrap up with a simple u-substitution.

This is a really clever trick, and it's natural to hope that it can be used to solve other integrals. Unfortunately, it can't: in a precise sense, any integral which can be solved this way is just a trivial modification of the Gaussian integral itself. This is a simple argument due to Robert Dawson, and we'll see it in detail. If time remains, we'll examine a follow-up argument by Denis Bell, that (1) there is actually a more general version of the Gaussian integral trick and (2) it's also useless.

Homework: Recommended

Class format: Lecture

Prerequisites: Multivariable calculus; be comfortable with the solution of $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ discussed in the blurb.

Imperfection (🏁🚗, Mia and Nathan, T$\boxed{\text{W}}$ΘF)
While things might be all sunshine, daisies, and perfection in the world of graph theory, life gets yuckier in the land of analysis. Exhibit A: $0.9999\ldots = 1$. Who allowed this nonsense?!?! Maybe they should have thought about *not* including so many darn digits. Why can't 0 and 1 be enough? Which brings us to Exhibit B: Have you ever successfully represented 2 in base 10 using just the numbers 0 and 1? We didn't think so. (And if you have, do let us know.) In short, there is an inherent tension between representing real numbers uniquely and representing every real number. AKA bases suck. In this class, we'll prove why.

Homework: Optional

Class format: Interactive lecture

Prerequisites: Epsilons and deltas

Lastly, choose randomly (⚑✈, Travis, TWΘ$\boxed{\text{F}}$)
Have you been taking a random walk on the Champlain campus, pining over the loss of random choices in Mathcamp classes since week 2? Have you despaired over the dire dearth of combinatorial chaos? Then rejoice as we revive that which you thought you had lost.

In this class, we talk about yet one more method in the probabilistic arsenal for solving combinatorial problems, one which that dastardly devil Travis has thus far hidden from your prying eyes. We'll see how to use the Lovász Local Lemma to solve problems in graph theory, or to color the real number line to break *every single pattern in the world.*

Homework: Optional

Class format: Interactive lecture

Prerequisites: **You do not need to have taken *First, choose randomly* to take this class.** Prior experience with graphs (the combinatorial ones) will help make one application easier to understand.

Let $\varepsilon_0 > 0$ be sufficiently small (⚐🚗, Della, TW$\boxed{\Theta\text{F}}$)
Ordinal numbers are what you get if you decide to count and never stop, not even at infinity. A lot of introductions to ordinal numbers go through the ordinals up to ε_0, and then skip straight past all the countable ordinals to ω_1. But in between ε_0 and ω_1, there's a lot of stuff happening—in fact, an uncountable amount of stuff!

In this class, I'll introduce the ordinal numbers in a somewhat unconventional way, and talk about ways to represent countable ordinals. Then we'll try to construct larger and larger countable ordinals, and get a sense of the flavor of the uncountable amount of stuff that's waiting for us. Of course, we'll only be able to understand a tiny (countable) portion of it, but we can get much further than ε_0.

Homework: Recommended

Class format: Lecture

Prerequisites: 🚗 becomes ✈ if you haven't seen ordinal numbers before.

Not theory (⚐✈, Steve, $\boxed{\text{TW}}$ΘF)
Classical propositional logic is pretty boring; there isn't a lot you can do with just two truth values. At various points more interesting logical systems have been proposed, with varying levels of

success/interestingness. Eventually a coherent "theory of propositional logic**s**" emerged, treating in a very general way the properties that any "reasonable" deductive system can have. For example, in a precise sense it *is* possible to add a "half-negation" operator to classical logic without breaking it, but it is *not* possible to add an "un-negation" operator to intuitionistic logic (= classical logic without double negation cancelling) without breaking it.

(That is, and in contrast to things Raj may have told you, a square not is simpler than an un-not.)

We'll explore this area of logic, with particular focus on negation.

Homework: Recommended

Class format: Lecture

Prerequisites: Comfort with Boolean algebra (basically, you should know what a truth table is, and what the truth table for "If A then B" looks like). You do **not** need to have heard of intuitionistic logic before!

Perfection (Mia, TW$\boxed{\Theta}$F)

Commutative algebraists have their excellent rings and algebraic geometers have their wonderful compactifications, but no one achieves perfection like graph theorists. In this class, we will prove none other than the perfect graph theorem which, in addition to having an excellent[2] name, has an exceedingly clever proof. So, what is perfection? In Graph colorings, we proved that $\omega(G) \leq \chi(G)$ and then asked, can we push those two invariants arbitrarily far apart. An alternative question one might ask is, what graphs achieve equality? Or even better, which graphs achieve equality and have that their subgraphs achieve equality too? The answer, perfect graphs! And what's more, the perfect graph theorem gives us an elegant characterization of these graphs.

Note: Graph colorings is not a prerequisite.

Homework: Optional

Class format: Interactive lecture

Prerequisites: Graph theory

Philosophy of math (Neeraja, TWΘ$\boxed{\text{F}}$)

What is the relationship between a mathematical proof and our knowledge of the theorem that it proves? Do some mathematical proofs go beyond establishing the truth of their theorems and actually explain why the theorems are true? How has our mathematical knowledge grown throughout history? We'll discuss some or all of these questions. Most of the questions and examples motivating the discussion come from the book *Proofs and Refutations* by Imre Lakatos.

Homework: None

Class format: Some lecture, mainly discussion

Prerequisites: None

Seven trees in one (Della, T$\boxed{\text{W}}$ΘF)

A (binary) tree is either a single node, or a root with a subtree on the left and a subtree on the right. Algebraically, if T is the set of all trees, this can be expressed as $T = 1 + T^2$.

If I gave you the equation $x = 1 + x^2$, you would use the quadratic formula to get $x = \frac{1 \pm \sqrt{-3}}{2}$. But that doesn't make any sense in the context of trees, right? And the fact that the solutions satisfy $x = x^7$ can't possibly tell us anything about 7-tuples of trees, can it?

Homework: None

Class format: Lecture

[2] Excellent under the English definition, not the algebraic one.

Prerequisites: None

Sophie Germain primes (, Mia, TWΘ$\boxed{\text{F}}$)
Born in 1776, Sophie Germain was a French mathematician, physicist, and philosopher. Despite having to publish under a male pseudonym (in order to have her work recognized) she spent much of her life at the forefront of mathematics and did ground-breaking work studying the Fermat equation $x^q + y^q = z^q$ for primes q with the property that $p = 2q + 1$ is also prime. Such primes are now known as *Sophie Germain primes.*

In this class, we'll study Sophie Germain primes, prove a surprising fact about primitive roots modulo a Sophie Germain prime, and celebrate a seriously awesome female mathematician!

Homework: Optional

Class format: Interactive lecture

Prerequisites: Introduction to number theory

The Ra(n)do(m) graph (, Travis, T$\boxed{\text{W}}$ΘF)
In *First, choose randomly*, we limited ourselves to randomly producing finite graphs. But it turns out that strange things happen once we start to choose *infinite* random graphs, and in this class, I will tell you the story of these graphs. It is surprisingly short—short enough to fit in 50 minutes—but we'll hit upon some serious mathematics, including the 0–1 law for graphs, which says that every "first-order" graph property is either true for almost every graph or false for almost every graph, with no middle ground.

So settle in, my friends, while I tell you the tale of the infinite random graph. Lean back while I weave for you the disparate threads of its history, from probability theory and model theory to the enduring legacy of popular folk singer-songwriter Arlo Guthrie's most enduring song. Come experience the full range of human emotion, shouting with excitement, gasping with amazement, and weeping over what might have been, as you revel in the dramatic legend of the infinite random graph.

Note: I taught the first half of this class under the same name last year; this version will cover more material.

Homework: Optional

Class format: Storytime

Prerequisites: Be familiar with basic laws of probability. **You do not need to have taken *First, choose randomly* to take this class.**

The transcendence of a single number (including Liouville's constant) (, Travis, TW$\boxed{\Theta}$F)

Proving the transcendence of many numbers is good, but proving the transcendence of one number is good enough. Plus, for this, we can prove transcendence simply and quickly: in one day only!

In this class, we'll take the quick route to finding an explicit transcendental number (Liouville's constant) and then see why the title of this class is actually not telling the truth and consequently find an uncountable set of real numbers and prove that all of them are transcendental.

Homework: None

Class format: Interactive lecture

Prerequisites: You don't need nuthin'!

1:10 Classes

Axiom of choice (, Narmada, $\boxed{\text{TW}}$ΘF)
Why do mathematicians get so fussy about the axiom of choice? We'll talk a little bit about why

the axiom of choice isn't just obviously true. We'll look at some obviously fake statements that are equivalent to the axiom of choice. We'll see why math without the axiom of choice might be sad sometimes. And by the end of this class, *you* get to be a mathematician who's fussy about the axiom of choice!

(If you are a returning camper, this is the same class I ran last year.)

Homework: None

Class format: Lecture

Prerequisites: None

Galois theory crash course (, Mark, $\boxed{\text{TW}\Theta\text{F}}$)
In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully. Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later).

If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we might conceivably be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class.) Even if we don't get that far, the so-called Galois correspondence (which we should be able to get to, and probably prove) is well worth seeing!

Homework: Optional

Class format: Interactive lecture

Prerequisites: Group theory; linear algebra; some familiarity with fields and with polynomial rings

Honey, I shrunk the vectors (, Tanya, $\text{TW}\boxed{\Theta}\text{F}$)
Imagine you have a high dimensional dataset (e.g. how much each camper likes every possible LN2 ice cream flavor) that you're trying to analyze, however, every algorithm that you attempt to use ends up being much too slow. There are several possible ways to address this issue—use a cleverer, faster algorithm, get a more powerful computer to run your code on, or find a way to shrink your data without losing too much of the original structure. The focus of this class will be the latter technique. We will see a simple randomized procedure for dimensionality reduction while (almost) preserving pairwise distances between the points in your dataset with high probability. If you enjoyed the second half of my Week 2 class comparing randomized and deterministic computation, you will most likely have fun here as well!

Homework: None

Class format: Interactive lecture

Prerequisites: Knowing how matrices act on vectors

How the compactness theorem got its name (, Krishan, $\text{TW}\Theta\boxed{\text{F}}$)
The compactness theorem is one of the most important theorems in the field of model theory. Surprisingly, it gets its name from the topological notion of compactness. In this class we'll see why this

is the case. We'll define a certain topological space associated with a theory, and we'll prove that the compactness theorem is equivalent to the fact that this space is compact.

Homework: None

Class format: Interactive lecture

Prerequisites: Model theory

Percolating through percolation theory (⚑🚗, Tanya, T$\boxed{\text{W}}$ΘF)
Percolation theory is an area of probability theory that studies the structure of infinitely large graphs after edges get randomly removed. We will see that certain properties for such graphs will either hold with probability 0 or 1, however, perhaps surprisingly, we may not be able to easily distinguish between the two situations. We will also learn about some open questions in this field, as they are often rather straightforward to state, yet incredibly challenging to resolve.

Homework: None

Class format: Interactive lecture

Prerequisites: Having seen a bit of probability before might be helpful.

The Chevalley–Warning theorem (⚑🚲, Kevin, TWΘ$\boxed{\text{F}}$)
The Chevalley–Warning theorem is a nice little result that says that the number of solutions to a system of equations mod p must be divisible by p, given that the number of variables is sufficiently high relative to the degrees of the polynomials. We'll see a slick proof of the theorem using Fermat's little theorem, which we'll also introduce in the class.

Homework: None

Class format: Interactive lecture

Prerequisites: Modular arithmetic, polynomials

Why 0 is the biggest prime (⚐🚗, Kevin and Krishan, T$\boxed{\text{W}\Theta}$F)
This class will be about a neat application of model theory to algebra called the Lefschetz principle.

In the first day we'll use model theory to prove the Lefschetz principle, which states that a first order property is true in $\mathbb{C}$ if and only if its true in algebraically closed fields of with arbitrarily large characteristic. We'll use a bunch of concepts from the model theory class in the second week, but a camper who didn't take the class will be able to understand the big ideas.

In the second day, Kevin will talk about how the Lefschetz principle can be used to be prove some cool results about algebra and algebraic geometry, such as (1) the fact that any injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is surjective (2) Hilbert's Nullstellensatz, which describes all maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$.

Homework: None

Class format: Interactive lecture

Prerequisites: Ring theory (required). 🚗 becomes ✈ if you haven't taken model theory.

Zeroes of recurrence sequences through *p*-adics (⚐🚗, Eric, TW$\boxed{\Theta\text{F}}$)
In this class we'll aim to prove (a simple case of) the Skolem–Mahler–Lech theorem: for an integer recurrence sequence a_n, the set of indices n where $a_n = 0$ can be at worst the union of a finite set and some arithmetic progressions. This theorem is really cool because the only known proofs rely on p-adic analysis in a crucial way. I'll introduce some minimal amount of necessary facts about the p-adic numbers and we'll prove the Skolem–Mahler–Lech theorem using p-adic infinite series.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra to the point of being comfortable with diagonalizing 2×2 matrices. Comfort with the idea of defining functions by infinite series, having seen series expansions of e^x and $\log(x)$ before is useful. Modular arithmetic, at the level of a is invertible mod n if and only if $\gcd(a, n) = 1$.

2:10 Classes

Ben teaches Susan's class (Ben, T$\boxed{\text{W}}$ΘF)
Five minutes before class time, Susan will send Ben a slide deck. Good luck, Ben!

Homework: None

Class format: Slideshow Telephone

Prerequisites: None. Prerequisites would be too big a hint.

Everything Ben knows about nonmeasurable sets (Ben, TW$\boxed{\Theta\text{F}}$)
Here are a few questions. Some are easier and some are harder! Here, $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable sets of $\mathbb{R}$. Also, $\mathcal{N}$ denotes the set of Lebesgue non-measurable sets of $\mathbb{R}$. For convenience.

(1) How large is $\mathcal{L}$?
(2) Suppose we declare two sets A, B equivalent if their symmetric difference $(A \setminus B) \cup (B \setminus A)$ has measure zero. How many equivalence classes does $\mathcal{L}$ have?
(3) Same as the previous one. How many equivalence classes does $\mathcal{N}$ have?

This class will answer all of those in pretty abrupt fashion, by breaking out some comically overpowered tools, mostly from logic. Some other topics might include the fact that we can use nonprincipal ultrafilters to build nonmeasurable sets and Bernstein sets, the sets which don't contain or miss any perfect sets.

Homework: Recommended

Class format: Lecture

Prerequisites: Having seen measure (e.g. in Week 3) is probably necessary for this to make sense. It will help to have seen transfinite induction before.

Fair division using topology (Jane Wang, $\boxed{\text{TW}}$ΘF)
How can we fairly divide a cake among multiple people when each person values frosting, edges, etc. differently? We can answer this question using tools from topology, the study of continuous functions and properties that are preserved under continuous deformation. It turns out that topology has many surprising applications to fields ranging from economics to combinatorics to data science. In this short course, we will survey some applications to problems of fair division (of cakes, necklaces, rent, and more). No prior knowledge of topology will be assumed.

Homework: Optional

Class format: Interactive lecture

Prerequisites: None

From the Sato–Tate conjecture to murmurations (David Roe, T$\boxed{\text{W}\Theta\text{F}}$)
The Sato–Tate conjecture was made in the 1960s based on numerical experimentation with elliptic curves, and proven in 2011 (though generalizations to higher genus curves are still open). It focuses on counting solutions to equations like $y^2 + y = x^3 - x$ modulo different primes p. We prove Hasse's

theorem, which tells us that the number of solutions is about $p+1$, with an error of at most $2\sqrt{p}$. The Sato–Tate conjecture focuses on the distribution of the error term as p varies, providing two possible specific limiting distributions which you can see numerically with cool histograms.

Last year, a related phenomenon was observed by several authors, including an undergraduate at the University of Connecticut. Instead of fixing an elliptic curve and allowing p to vary, they ordered elliptic curves with rational coefficients in a specific way and looked at how a certain average value of the error term varied. The resulting plots showed an unexpected oscillation, and behave differently based on whether the elliptic curve has finitely many or infinitely many rational points (rather than points modulo p).

Homework: Optional

Class format: Interactive lecture

Prerequisites: Elliptic curves and Introduction to group theory. Finite fields and Functions of a complex variable will be helpful for a few parts of the course, but not required.

Spherical geometry (, Kira Lewis, TW$\boxed{\Theta}$F)
What if I told you that:

- Parallel lines do not exist?
- The sum of angles in a triangle is strictly $> 180°$?
- The area of any polygon can be calculated in terms of its angles?

This may all sound ridiculous in normal geometry, but in the spooky world of spherical geometry, it's all true. Spherical geometry is where points, lines, and circles live on the sphere instead of in the plane. Come join this class for a tour through some of the craziest theorems in spherical geometry and to see proofs of the statements above.

Homework: None

Class format: Lecture

Prerequisites: None

Taming the grouchy Grassmannian (, Kayla, TW$\boxed{\Theta\mathrm{F}}$)
The Grassmannian is a mathematical object that is natural place to work in. As a set, it is comprised of k-dimensional vector space in some n-dimensional space. However, even with its innocent definition, this space is notoriously grouchy for its complicated associated structures.

Namely, the Grassmannian has topological and geometric structure we can endow it with. For instance, we can view it as a *projective variety* using a set of coordinates called Plücker relations or as a compact smooth manifold by looking at something called its Schubert decomposition.

We will be exploring the dark sides of the Grassmannian and taming its grouchiness with *combinatorics*. Come see some tableaux, plabic graphs and perfect matchings tame this grouchy Grassmannian!

Homework: None

Class format: Interactive lecture

Prerequisites: None (some linear algebra exposure is good e.g. reduced row echelon form, determinants)

(THICCC) Triangles, Hyperbolas, Isogonal Conjugates, and Certain Circles (, Anthony Wang and Nathan Cho, $\boxed{\mathrm{TW}}\Theta$F)
Come explore the rich, diverse, and endlessly surprising world of triangle conic geometry! In this class, we'll develop some theory relating circumconics of triangles with whatever the heck an isogonal conjugate is, and we'll use that theory to prove Feuerbach's theorem, a theorem purely about circles. To do this, we will push plane geometry to its limit, and we'll encounter many cool and relatively unknown tidbits of conic and classical geometry along the way.

Homework: Recommended

Class format: Lecture

Prerequisites: Experience with Euclidean geometry, some familiarity with conics

Van Roomen's problem (, Philip Yao, TWΘ $\boxed{\text{F}}$)
Find the smallest positive x that satisfies:

$$x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} - 14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{25} + 483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} - 34512075x^{11} + 7811375x^{9} - 1138500x^{7} + 95634x^{5} - 3795x^{3} + 45x = \sqrt{\frac{7}{4} - \sqrt{\frac{5}{16}} - \sqrt{\frac{15}{8} - \sqrt{\frac{45}{64}}}}$$

Homework: None

Class format: Lecture

Prerequisites: Trig identities (triple angle formula and angle sum/difference should be fine)

Colloquia

Think different (*Po-Shen Loh*, Tuesday)
One of the most fascinating things in math is when a problem is solved via an apparently-unrelated idea. The talk will start off with two examples within math itself, which were a source of inspiration to the speaker early in his career, when he used the technique on math research, and throughout his teaching.

The second part of the talk will go into the speaker's current work (https://live.poshenloh.com), which focuses on a large-scale real-world problem: teaching people how to generate creative math ideas. This part of the talk is designed specifically for math people who might have interest in doing real-world projects someday. It will highlight places where the math background (particularly out-of-the-box math thinking) ended up being instrumental in inventing new real world solutions. Along the way, the philosophy of game theory and graph theory will make cameo appearances.

A magic show (*Tadashi Tokieda*, Wednesday)
"There cannot be any abstract," Tadashi writes. We will find out what happens on Wednesday!

Project fair (Campers, Friday)
Project fair is an occasion for campers to present the results of their project. Sometimes campers make a poster about what they've done, sometimes the output of the project is more complicated than that.

If you are interested in presenting at project fair, talk to the staff member(s) supervising your project by Tuesday!