CLASS DESCRIPTIONS—WEEK 4, MATHCAMP 2023

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Back to basi(c)s (🌶🌶🌶, Travis, $\boxed{\text{TW}\Theta\text{FS}}$)
One of my favorite things in mathematics is using one kind of math to do another kind of math, and one of my favorite maths to apply to other maths is linear algebra. Why choose randomly when you can elegantly extract a simple solution through the thrilling theory of vector spaces?

Thus, this class consists of ample applications of linear algebra to combinatorics, discrete geometry, and convex geometry. We'll see how fundamental notions in linear algebra, such as dimension and the inner product, can be used to prove results in combinatorics. Then, we'll continue using these techniques to prove things about possible patterns of points and lines in Euclidean space. (For example, how many points can you place in $\mathbb{R}^n$ such that the distance between every pair of points is either 1 or 2?) We'll see how to use eigenvalues to bound the size of any collection of "equiangular lines" through the origin in $\mathbb{R}^n$, and we'll prove some foundational results in convex geometry. And if we have enough time, we'll use vectors to get a surprisingly good approximation for the famous MAX-CUT problem on graphs.

(Previously titled *What are your vectors vorth? or, part of the part of combinatorics and discrete geometry that ve can do easily vith linear algebra.*)

Homework: Optional

Class format: Interactive lecture

Prerequisites: Intro to linear algebra. (You should be comfortable with linear independence and dimension and have seen the dot product before. Comfort with eigenvalues will help in a small part of the class but isn't necessary for the rest.)

Finite fields (🌶🌶🌶, *Aaron Landesman*, $\boxed{\text{TW}\Theta\text{FS}}$)
What do the rational numbers, complex numbers, and real numbers have in common, but not share with the integers? They are all fields; we can add, subtract, multiply, and divide elements in them. But which finite sets also have these properties? What possible sizes can such a finite set have? What are the possible subfields? These questions all have simple, beautiful answers which we will present in this course. Finite fields are crucially used throughout number theory, algebraic geometry, cryptography, and coding theory. After classifying finite fields, we will solve a number of combinatorial counting problems over finite fields, such as computing the average number of roots of a polynomial with coefficients in a finite field.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra, group theory

Functions of a complex variable (Week 2 of 2) (🌶🌶🌶, Mark, TWΘFS)
This is a continuation of last week's class. If you didn't take the class last week and want to join, you are encouraged to check with me first.

Homework: Recommended

Class format: Interactive lecture.

Prerequisites: None.

High-school algebraic geometry (🌶🌶, Neeraja, TWΘFS)
This is a class about real root counting, i.e. finding how many real roots a polynomial $p(x)$ with real coefficients has in a given interval $[a, b]$. We will prove Sturm's theorem which answers this question precisely, and along the way we'll prove some easier results like Descartes' law of signs. At the end of the class, you will be able to answer questions like the following: what conditions should we impose on a, b, c so that the polynomial $ax^3 + bx + c$ has exactly one real root?

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Derivatives (we will need to take derivatives of rational functions)

Kuratowski's game (🌶🌶🌶, Ian, TWΘFS)
Let X denote a subset of real numbers. You may take closure and complement indefinite number of times on this set X. What is the maximal number of distinct sets you can get from applying these operations, and which set X distinguishes these operations? Now, what if we introduce a frontier (or boundary) operator? What if we bring in two topologies? How about n topologies?

In 1922 K. Kuratowski answered the first question through his thesis—the answer was 14. Now the latter questions get more complicated as we increase the number of operators and the number of topologies. In 2021, myself and co. proved that if we have n saturated topological spaces and closure, complement, frontier operators, then the number of distinct sets is finite. Even better, a quartic polynomial $p(n) = \frac{5}{24}n^4 + \frac{37}{12}n^3 + \frac{79}{24}n^2 + \frac{101}{12}n + 2$.

On the first day we establish some fundamentals of point-set topology. Then on the second day, we discuss different variations of the Kuratowski game, specifically the variation I investigated, and see how to obtain the polynomial expression for the number of distinct sets of the closure-complement-frontier problem of the saturated polytopological space.

Homework: None

Class format: Lecture

Prerequisites: Some basic combinatorics (counting principles)

Markov chain Monte Carlo (🌶🌶, *Moon Duchin*, TWΘFS)
Turns out a really good way to study complex systems is through a kind of mostly blind exploration where you make dumb, semi-random decisions and wait a really long time. This is the magic of "MCMC", a sneaky powerful method of harnessing what computers do best: repeat easy calculations a lot. In this course we'll approach this topic from scratch, through the theory of Markov chains (random walks with no memory), and will get as far as the Metropolis method, which the IEEE lists as the #1 algorithm of the 20th century. We'll also see applications to codebreaking, autocomplete, and elections.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

McKelvey's Chaos Theorem (🌶, Ben, TWΘ$\boxed{\text{FS}}$)
In Mira's colloquium, she mentioned—cryptically—that there's a good way to pick a winner if everyone's preferences are "linear" or "single-peaked." This class will begin by finding one way in which there is a "natural winner" in this case (under some modest simplifying assumptions, including the "linear" business—this means that the "issue space" is one-dimensional). That's a nice result, which is a pleasant and happy counterpoint to all of the other results Mira told us about.

IF YOU WANT TO STAY HAPPY PLEASE SKIP TO THE NEXT BLURB NOW.

You might be wondering "What if we had TWO dimensions of issue-space instead of one? How much does that break?"

It breaks everything! There is a natural sense in which, instead of having ONE natural winner, we (probably) enter a world in which everyone is a winner—so no one is.

Homework: Recommended

Class format: Lecture, with a few small activities to explore the topic

Prerequisites: Essentially none

10:10 Classes

Gaussian magic (🌶🌶🌶🌶, Tanya, T$\boxed{\text{W}\Theta\text{FS}}$)
Did your teacher ever say that a class is being graded on "a curve"? The "curve" in this context is referring to the Gaussian distribution, which, for reasons still unbeknownst to me, is thought to be the "natural" distribution that grades must follow. Despite this potentially misguided usage, the Gaussian random variable is ubiquitous throughout probability and statistics for a number of good reasons, which we will explore in this class. Turns out, Gaussians have many wonderful properties that simplify computations and, in certain cases, allow us to relate functions of non-Gaussian variables to those involving Gaussian ones. If time permits, I will briefly discuss some recent advances in understanding of Gaussian random matrices, which is related to the subject of my PhD. Attend this class to learn how Gaussian magic happens!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: some previous exposure to multivariable calculus, analysis (particularly limits), and linear algebra

How to rob your friends (🌶🌶, Arya, T$\boxed{\text{W}\Theta}$FS)
Imagine a world where you're broke and severely in debt, but all your friends are rich and generous. You make your friends stand on the vertices of a graph. At each instance, choose a vertex, and send \$1 from the vertex to each of its neighbours. Can you repeat this move over and over to eventually get out of debt, WITHOUT sending some other friends into debt?

In this class, we shall talk about how greed is necessary to prevail in society, and study some linear algebra with graphs. One of the proofs shall involve starting a wildfire and watching it spread (too soon?). Weirdly enough, this theory builds on to study sand piles and Riemann surfaces.

Homework: None

Class format: Lectures.

Prerequisites: You should know what a graph is.

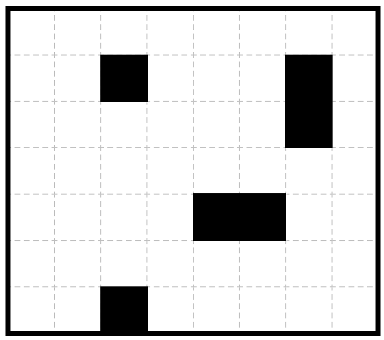
How to rob your friends 2: non-transitive dice boogaloo (🌶️, Eric, TWΘ$\boxed{\text{FS}}$)
You might think that if I give you a set of 3 dice A, B, C and tell you that on average A rolls higher than B and B rolls higher than C, it would have to be the case that A rolls higher than C. But this isn't true! The property of *transitivity* fails for the relation "X rolls higher than Y on average." In this class we'll make sense of that statement and show that we can make transitivity fail as badly as we want!

Homework: Optional

Class format: Interactive lecture

Prerequisites: None.

Mathematical Concepts for Solving Puzzles (🌶️🌶️ → 🌶️🌶️🌶️🌶️, Della, T$\boxed{\text{W}\Theta\text{FS}}$)
I have a puzzle for you! Draw a loop that goes horizontally and vertically between adjacent white squares, which visits every white square exactly once.[1]



In this class, you'll solve lots of puzzles along these lines! Some of them will require some math: to make progress, you'll have to discover an interesting lemma about the puzzle.

Actually, this is three independent classes, and you can come to any subset of them! They'll each focus on some concept from math and some types of puzzles it can be applied to:

- MCSP: Planarity (🌶️🌶️, T$\boxed{\text{W}}$ΘFS)
- MCSP: Parity (🌶️🌶️🌶️, TW$\boxed{\Theta}$FS)
- MCSP: Penalty (🌶️🌶️🌶️🌶️, TWΘ$\boxed{\text{FS}}$)

Homework: Optional

Class format: Working on puzzles, with some discussion of the math at the end.

Prerequisites: Basic graph theory, such as the definitions of 'planar' and 'cycle'.

Polynomial methods in combinatorics (🌶️, Narmada, T$\boxed{\text{W}\Theta\text{FS}}$)
Have you ever wondered why there are so many algebraic combinatorialists out there? Come to my class to answer this burning question! We'll define finite fields as the right setting to frame combinatorial problems, and then we'll see how algebra over finite fields helps us solve these problems. Our two big results will be the Schwartz–Zippel lemma and the Combinatorial Nullstellensatz, which we'll use to provide short proofs of famous results like the Kakeya conjecture and the Cauchy–Davenport theorem.

Homework: Optional

Class format: A mix of lecture and group work

Prerequisites: None.

[1]Fancy graph theorists would call this a Hamiltonian cycle.

Problem solving: induction (🌶🌶, Misha, T$\boxed{\text{W}\Theta\text{FS}}$)
You probably first saw induction in the context of proving a result like

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}.$$

This is not easy for everyone, but once you've solved one problem like this, you can solve every problem like this one; all it takes is a tiny bit of algebra.[2] That's not what this class is about!

We will see how these proofs can get much more complicated. Our induction will start out strong, and on each day of class it will get stronger than all the previous days combined. You'll see examples of crazy induction in algebra, game theory, number theory, and other theories. You'll learn how to use induction (and how *not* to use it) to solve problems of your own, olympiad and otherwise.

In class, we will spend time solving problems together; I will focus less on answering the question "why is this claim true?" and more on answering the question "why would we think of solving a problem this way?" There will be plenty of problems left for homework, and you will not get much out of a problem-solving class unless you spend time solving those problems.

Homework: Recommended

Class format: A bit of lecture, and a lot of what I call "problem-solving lecture": you will tell me how to solve the problems I gave you.

Prerequisites: None.

11:10 Classes

Guess Who? (Week 2 of 2) (🌶🌶🌶 → 🌶🌶🌶🌶, Tim!, $\boxed{\text{TW}\Theta\text{FS}}$)
This is the second week of *Guess Who?*! If you came to the first week, I highly recommend you to come back for the second to see the full story come together! The chili level is changing from 🌶–🌶🌶 last week to 🌶🌶🌶–🌶🌶🌶🌶 (mostly 🌶🌶🌶) this week, but regardless of chilis, if you liked last week, I encourage you to come back for this week.

If you didn't take the first week of the class but want to take the second, come talk to me; there are things you would need to catch up on.

Here's what's coming up in the second week:

In the first week, we saw that the divide-in-half strategy was not optimal, and by expressing *Guess Who?* as a zero sum matrix game, we were able to find the actual optimal strategy.

While the divide-in-half strategy is not the "BEST" strategy that Mark Rober claims it is, it does have a compelling story: that broad questions are better than narrow questions. On the other hand, the actual optimal strategy doesn't come with a clear story attached—and the pattern is so nice and beautiful, and so different from divide-in-half, that really *ought* to have a story. There should be some good reason why the numbers are what they are other than just "we can do a calculation that shows it's true."

There *is* a good reason and we will find it. To get there, we will have to temporarily set *Guess Who?* aside and consider some strange variations on it. In regular *Guess Who?*, each question that you ask applies to some integer number of characters. In *Super Guess Who*, each question applies to some number of characters that doesn't have to be an integer. And *Ultra Guess Who* takes this a step further.... Once we've solved the very odd game of *Ultra Guess Who*, we'll be able to return to regular *Guess Who?* and understand what is going on. We'll discuss continuous random variables and cumulative distribution functions (this will involve a little calculus).

We'll also look at the calculation side of things. We were able to construct a matrix game for *Guess Who?* that a computer could solve reasonably quickly to give us the optimal strategy, but we were

[2] If you went to Apurva's colloquium, you know that to write a proof for such a problem in Lean, you use the exact same code, no matter what the actual sum is. That's not otherwise related to this class, but it's indicative of how boring the proof turns out to be.

lucky that the number of possible mystery characters, $n = 24$, was relatively small. This technique does not give reasonably sized matrix games when n is larger than 24. But, with another technique, we'll be able to solve *Guess Who?* with n characters efficiently for larger n (in time polynomial in n). To do this, we'll need to discuss linear programming and convex optimization, as well as network flows in graphs. The key will be to lift the problem to a higher dimensional space.

Also, sometimes everything becomes clear when you figure out the right picture to draw. I promise a variety of interesting and illuminating diagrams. Now let's finish giving this children's game the mathematical analysis it deserves!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None

Perron trees (everyone loves analysis, part 1) (🌶, Charlotte, $\boxed{\text{TW}}\Theta\text{FS}$)
As everyone knows, this is the year that everyone loves analysis!!!! You've probably heard the analysis staff talk about the existence of pathological, terrifyingly counterintuitive examples that seem to break everything we believe about the natural order of things—and how much fun this is! I am very happy to tell you that we will talk about one of my favourite examples of this in this class!

Specifically, we'll talk about the "Kakeya needle problem," which is about the following: imagine you have a needle of width zero and length one lying on a table, and you would like to rotate it continuously 180 degrees. You could achieve this feat just by rotating it within a circle of diameter 1. But what if I challenged you to rotate it within a region of smaller area? Let's say you managed to achieve this. What if I challenged you to rotate it within a region of even smaller area? And so on...

Eventually, this game would have to end, because surely the smallest region for which I could do this is not *that* small, right?? Right????? WRONG.

Homework: Optional

Class format: Mix of lecture and group work.

Prerequisites: none

Quiver representations part I (🌶🌶🌶, Kayla, $\boxed{\text{TW}\Theta}\text{FS}$)
Don't *quiver* in fear: path algebras are here! We as humans aim to understand the world *linearly*—we approximate curvy functions with a tangent line, areas bounded by functions with rectangles in calculus. Algebraists care about *linearizing* finite-dimensional algebras! To achieve this linearization, people began the study of representation theory. In this class, we will be studying the representation theory of finite dimensional algebras! It sounds quite spicy, but don't *quiver* in fear, there are nice concrete ways that people study these abstract algebraic objects. In particular, we will be studying quiver representations—a linearization of directed graphs. Come see the basics of quiver representations, a classification theorem that connects quiver representations to many different areas of math and about path algebras that allow us to study any finite dimensional algebra in terms of quivers!

Homework: Optional

Class format: Interactive lecture.

Prerequisites: Linear algebra, group or ring theory: comfortability with quotient groups or rings.

Quiver representations part II (🌶🌶🌶, Raj, $\text{TW}\Theta\boxed{\text{FS}}$)
This is the second installation of quiver representations. As you will see in Kayla's part of the class, quivers are directed graphs, and a quiver representation is an assignment of a vector space to every vertex of the quiver and a linear transformation to every edge. The theory of quiver representations is broad and touches many different fields of mathematics, including the theory of Dynkin diagrams, the

study of finite subgroups of $SU(2)$, and algebraic geometry. We will see many of these connections in this course, and hopefully by the end of this course, you will be convinced of the power of quivers!

Homework: Recommended

Class format: Interactive lecture.

Prerequisites: Quiver representations part I

The outer life of inner automorphisms (🌶🌶🌶🌶, Steve, $\boxed{\text{TW}\Theta\text{FS}}$)

Suppose G is a group, $g \in G$, and $f : G \to H$ is a group homomorphism. Via conjugation, we know that g induces an automorphism of G, namely

$$\alpha_g^G : h \mapsto ghg^{-1}.$$

Such automorphisms are called **inner automorphisms**, and are very important.

But something even cooler happens: this automorphism *makes sense after we apply* f, in the sense that

$$\alpha_{f(g)}^H : H \to H : h \mapsto f(g)hf(g)^{-1}$$

is an automorphism of H which is "built from" the original conjugation automorphism in a natural way. This is neat, because generally automorphism *don't* "travel along" homomorphisms in this way.

A natural question at this point is whether there are any *other* ways to define an automorphism (besides conjugation, that is) so that it will "travel along" homomorphisms in a good way. It turns out that the answer is negative, and so inner-ness can be detected from the outside. This (and more) is a theorem due to Bergman only 11 years ago. In this class we'll follow Bergman's paper and get a rather unusual introduction to the abstract subject of **category theory**. (No experience with category theory is expected!)

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Group theory (specifically, being comfortable with the statement "Conjugation by a fixed element is an automorphism"; if you can follow the first paragraph of this blurb, you're fine)

Trail mix (🌶 → 🌶🌶🌶, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)

Is your mathematical hike getting a little too strenuous? Would you like to relax a bit with a class that offers an unrelated topic every day, so you can pick and choose which days to attend, and that does not expect you to do homework? If so, some Trail Mix may be just what you need to regain energy. Individual descriptions of the five topics follow.

Trail Mix Day 1: The Prüfer Correspondence (🌶–🌶🌶).

Suppose you have n points around a circle, with every pair of points connected by a line segment. (If you like, you have the complete graph K_n.) Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments, but in only one way. (This tree will be a spanning tree for K_n.) How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll find a combinatorial proof that is especially cool.

Prerequisites: none

Trail Mix Day 2: Cyclotomic Polynomials and Migotti's Theorem (🌶🌶🌶).

The cyclotomic polynomials form an interesting family of polynomials with integer coefficients, whose roots are complex roots of unity. Looking at the first few of these polynomials leads to a natural conjecture about their coefficients. However, after the first hundred or so cases keep confirming the conjectured pattern, eventually it breaks down. In this class we'll see, and if time permits prove, a theorem due to Migotti, which sheds some light on what is going on, and in particular on why the conjecture finally fails just when it does.

Prerequisites: Some experience with complex numbers, preferably including complex roots of unity; some experience with polynomials.

Trail Mix Day 3: Integration by Parts and the Wallis Product (🌶🌶🌶).

Integration by parts is one of the only two truly general techniques for finding antiderivatives that are known (the other is integration by substitution). In this class you'll see (or review) this method, and encounter two of its applications: How to extend the factorial function, so that $(1/2)!$ ends up making sense (although the standard notation used for it is a bit different), and how to derive the famous product formula

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\ \cdots\ ,$$

which was first stated by John Wallis in 1655.

Prerequisites: Basic single-variable calculus.

Trail Mix Day 4: Perfect Numbers (🌶).

Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called perfect, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes — a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Prerequisites: None

Trail Mix Day 5: The Jacobian Determinant and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ (🌶🌶🌶).

How do you change variables in a multiple integral? In the "crash course" in week 1 we saw that when you change to polar coordinates, a somewhat mysterious factor r is needed. This is a special case of an important general fact involving a determinant of partial derivatives. We'll see how and roughly why this works; then we'll use it to evaluate the famous sum

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\,.$$

(You may well know the answer, but do you know a proof? If so, do you know a proof that doesn't require Fourier series or complex analysis?)

Prerequisites: Multivariable calculus (the crash course is plenty); some experience with determinants.

Homework: None

Class format: Interactive lecture

Prerequisites: Varies by day—see blurb(s) above

aspacefillingcurve (everyone loves analysis, part 2) (🌶🌶, Charlotte, TW$\boxed{\Theta\text{FS}}$)

As everyone knows, because I just told you, this is the year that everyone loves analysis!!!! You've probably heard the analysis staff talk about the existence of pathological, terrifyingly counterintuitive examples that seem to break everything we believe about the natural order of things—and how much fun this is! I am very happy to tell you that we will talk about another one of my favourite examples of this in this class!

We have intuition about what continuous functions are supposed to look like—they are nice and friendly! Things can't get too crazy with them! Surely, if you give me a continuous function from $[0, 1]$ to $\mathbb{R}$, I can draw its graph with ease! It's just a nice looking curve in the plane, right????

Surely, a continuous function must map the one-dimensional interval $[0,1]$ to a one-dimensional curve, right????? WRONG.

We will see just how emphatically WRONG this is by constructing a horrible creature called aspacefillingcurve. Using compactness and completeness (which are concepts we will discuss in class), we will define a continuous function on $[0,1]$ that fills up the entire TWO-dimensional square $[0,1]^2$!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Have taken the epsilons and deltas class, or something equivalent. In particular, you should be comfortable with the epsilon-N definition of a limit and proofs involving epsilons. You should be comfortable with sequences of functions and uniform convergence. You should have seen the definition of a Cauchy sequence before. You should definitely be familiar with metric spaces. If you do not have one of these pre-reqs, then add one to the chili level.

1:10 Classes

Braid groups (🌶️🌶️, Arya & Kevin, TWΘFS)
Take n strings. Tie one end of each string to a metal bar, twist the strings around a bunch, and tie the free ends of the strings to another bar. You've got yourself a braid!

This class will be an introduction to the topology and algebra of braids. We'll learn about several different perspectives on braids (the Artin group presentation, configuration spaces, mapping class groups) before moving on to applications of braids to modern research in cryptography and 3-dimensional topology.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic group theory

Continued fractions (🌶️, Ben, TWΘFS)
Did you take Ben's class on the ergodic theorem and wonder what the deal was with all of the continued fraction stuff at the end? Have you seen a lot of relay problems that look like

$$5+\cfrac{1}{2+\frac{1}{3+\frac{1}{2+\frac{1}{3+\dots}}}}$$

or something like it? Do you want to learn about lost[3] and secret[4] knowledge that mathematicians of the past had?

This class will do all of those things![5] In particular, we'll develop the theory of continued fraction expansions, prove their (very nearly almost) uniqueness, talk about why they're better than decimal expansions, and, time permitting, prove a lovely result of Lagrange characterizing the quadratic irrationals (those of the form $a+b\sqrt{c}$ for $a,b\in\mathbb{Q}$ and $c\in\mathbb{N}$ in terms of their continued fraction expansions.

The most important piece of secret knowledge I will impart is how to write these things without filling half a board.

Homework: Recommended

Class format: Lecture

Prerequisites: None!

[3]Not that lost, since I'm teaching it.
[4]Not that secret, since I'm teaching it.
[5]Except, again, the knowledge is neither lost nor secret.

Intersections of algebraic plane curves (, *Nic Ford*, $\boxed{\text{TW}\Theta\text{FS}}$)
An *algebraic plane curve* is the set of points in the plane where some polynomial is equal to zero. For example, a circle is an algebraic plane curve, because it's the set of points (x, y) where $x^2 + y^2 - 1 = 0$. One of my favorite facts about these objects is Bézout's Theorem, which says that if you have two algebraic plane curves, one cut out by a polynomial of degree d and one by a polynomial of degree e, then they intersect in exactly de points. This theorem is simple, beautiful, and fun to prove, but—as I've stated it here—unfortunately quite false. (See how many counterexamples you can come up with!)

But we're not the sort of people to let something as trivial as falsehood get in the way of proving such a pretty result. It will turn out that, with enough extra adjectives and caveats, we will be able to modify this version of Bézout's Theorem into something we *can* prove, and that's the task we'll take up in this class. Along the way, we'll also learn how to define and compute the *resultant* of two polynomials, a method for solving systems of polynomial equations that isn't taught much anymore but which is both critical to our proof of Bézout's Theorem and a fun object in its own right.

Homework: Required

Class format: Interactive lecture

Prerequisites: You should have comfort with the complex numbers and with manipulating polynomials in several variables. It will also be very helpful if you know how to compute the determinant of a matrix and know what it tells you about solutions of systems of linear equations, but we'll briefly review this in class.

Quantum computing (, Krishan, $\boxed{\text{TW}\Theta\text{FS}}$)
In a normal computer the fundamental unit of information is a bit, a value that can either be 0 or 1. In quantum computing, we treat **0** and **1** as vectors, so that the data of a single quantum bit (or qubit) is a linear combination (or superposition) of **0** and **1**. These quantum bits can also be entangled meaning that their states are correlated in a way that has no classical analogue.

In this class, we'll study the math underlying quantum computation (it's a lot of linear algebra). Then we'll prove some theorems about what quantum computers can and cannot do and will examine some quantum algorithms. Don't worry if you haven't seen much physics before, I'll explain all of the physics we need during class.

Homework: Required

Class format: Interactive lecture

Prerequisites: Linear algebra is required, and some familiarity with big O notation and pseudocode is suggested (+ if missing)

{Game, graph} theory against the world (, Ania, $\boxed{\text{TW}\Theta\text{FS}}$)
Consider the following scenarios and questions. Imagine that . . .

- You have access to information about links between all the Internet pages. Which is the most important one?
- You have some information about connections within a terrorist network, but you have resources to tap the phone of only one of the terrorists. Which one should you choose?
- Company A sells cats for \$10, and company B sells hats for \$5. They decide to merge and sell cats wearing hats for \$20. How should they split the income?

What do these questions have in common? They are all related to {game, graph} theory, and we will discuss them during this class.

We will start by talking about different centrality measures, which are the ways of finding the most "central" vertex of a graph. In particular, we'll define and understand PageRank, a centrality which was originally used in the Google's search engine. Then we'll dive into cooperative game theory, discuss concepts like Shapley and Myerson value, and see how we could break terrorist networks using game-theoretic centrality measures.

Lastly, we'll talk about the algorithm which helps maximise the numbers of patients getting kidney transplants ("the kidney-exchange problem"), and possibly about other topics from the social choice theory.

Homework: Optional

Class format: Interactive lecture and some problem-solving on your own or in small groups

Prerequisites: Very basic definitions from graph theory

Colloquia

The geometry of fractal sets (Neeraja, Tuesday)
Many spatial patterns that occur in nature are fragmented and irregular to such an extreme degree that classical geometry is of little help in describing them. For a long time, these fragmented patterns and shapes were regarded by mathematicians as "pathological" and unworthy of study. In the last fifty or so years, it has become evident that certain irregular shapes give a much better description of nature than the figures of classical geometry. A framework has been developed to study the geometry of such sets and this general theory is called fractal geometry. In this colloquium talk, we'll see examples of fractal sets occurring in many different contexts, including self-similar sets like the Cantor set, fractal sets in the study of dynamical systems like the Julia sets and the Mandelbrot set, "fractal curves" like the Weierstrass function, paths of Brownian motion, and fractals arising in the theory of Diophantine approximation. We'll also see some of the interesting geometric properties of fractal sets.

The hat-xiom of choice (Travis, Wednesday)
The JCs, after several weeks under the thrall of their pink sunglasses, have been corrupted by the constant flow of pink power and attempted a coup of Mathcamp. Unfortunately, due to their preternaturally powerful logistical abilities, the JCs have succeeded and captured all the mentors. But with their goal accomplished, they have no good reason to keep the mentors locked up, so they decide to play a game. They tell the mentors they will place either a duck or a dino on each mentor's head, and any mentor who guesses correctly what animal they've been behatted with will be allowed to escape. But what they don't tell the mentors is that during their reign, they've secretly hired infinitely many other mentors who will also be forced to play this game…

In this ~~colloquium~~ tell-all exposé, I'll give an insider's view of the mentors' *secret strategy* and how we used a practical application of the Axiom of Choice to outwit the JCs and break the real numbers. Come hear the mentors' secret knowledge to help us prevent the JCs from overthrowing camp yet again!

(Mira's secret surprise class at the beginning of Week 3 overlaps with this colloquium, but we'll also see several other versions of the hat game and how this relates to measure theory!)

The evolution of proofs in computer science (*Yael Tauman Kalai*, Thursday)
In this talk we will learn about the evolution of proofs in computer science. We will start by introducing the magical notion of zero-knowledge proofs and get a glimpse into how to construct them. Then we will talk about how these proof systems changed the way we think about mathematical proofs, and mention interesting applications.

I'd like some geometry with my topology (*Moon Duchin*, Friday)
In this colloquium I'll introduce some concepts of geometric topology, or all the ways to make stretchy shapes rigid. Tori and triangles are good basic examples, and I'll also let things get a little weirder.