CLASS DESCRIPTIONS—WEEK 5, MATHCAMP 2023

About the weird symbols appearing instead of chilis

We are trying an experiment in week 5 this summer! Instead of labeling every class with 🌶, 🌶🌶, 🌶🌶🌶, or 🌶🌶🌶🌶, we are using a two-symbol code to represent two aspects of the class. A one-word summary of the two axes could be “abstraction” and “pacing”, but they are described in more detail below.

As always, feel free to talk to us to learn more about what our classes will be like! Also, no matter which symbols a class has next to it, we are always happy to spend more time outside of class to help you understand the material, resolve lingering doubts, or indulge your curiosity.

The first dimension: ⚑, 🏁, and ⚐

- ⚑: Just like the ⚑ flag is fully colored in, the class is “fully filled in” with examples. The instructor makes an effort to make the ideas easier to grasp immediately, even at the cost of presenting the material less generally.
- 🏁: A class with 🏁 next to it is halfway between ⚑ and ⚐. For example, you might see how familiar objects are special cases of general, less intuitive ideas.
- ⚐: In a class with ⚐ next to it, you will often have to grapple with ideas that are hard to reduce to concrete examples. You might have to wait to see how the new abstract ideas relate to things you already know.

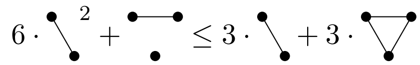
The second dimension: 🚲, 🚗, and ✈

- 🚲: A class with 🚲 next to it will feel relaxed and stress-free. Do not worry at all about slowing the class down if you need to ask more questions to understand; the class plan is built to accommodate this.
- 🚗: The default pace of a Mathcamp class is 🚗: comfortable, yet brisk. Questions are welcome in all Mathcamp classes—but some questions may be postponed until TAU.
- ✈: A class with ✈ next to it will feel like an exhilarating sprint. The class will need to move from topic to topic quickly to get to the finish line; you might feel like you have to review your notes and/or talk to the instructor to fully grasp the material.

Contents

Superclasses

Flag algebra marathon (Misha, $\boxed{\text{T}}$WΘF, 10:10–12:00 and 1:10–3:00)
Do you want to write inequalities where our variables are graphs?



Do you want to see the most roundabout proof that $R(3,3) = 6$?

Do you want to prove graph theory results from 1907 by methods developed in 2007?

Do you want to have gigabytes of multiplication tables to throw at supercomputers?

And do you want to do this **all in one day**?

Then this is the class for you.

This class is being taught in **marathon** format. We'll go 10:10am–12pm, take a break for lunch, and reconvene 1:10pm–3pm.

Homework: None

Class format: Marathon! (It will not all be lecture; you will spend some time in class solving problems.)

Prerequisites: You should be comfortable with matrix multiplication, linear transformations, and an assortment of basic concepts from graph theory.

Not the math we need, but the math we deserve (Ben, Ian, Kevin, Krishan, Narmada, Neeraja, Raj, Tanya, and Travis, $\boxed{\text{T}}$WΘF, 11:10–12:00 and 1:10–3:00)
Do you ever feel like math is the most beautiful, transcendent, magical discipline that connects disparate ideas in a seemingly seamless manner? This class is here to shatter that illusion—unfortunately, math is filled with unintuitive, unwieldy, even monstrous counterexamples that will make you question if anything would ever make sense again. If this somehow doesn't scare you off, come join us on a tour of the ugly side of math the teaching staff has been trying to shield you from, sampled from across different fields according to the specialties of the instructors.

Homework: None

Class format: Lecture

Prerequisites: None

9:10 Classes

Computing trig functions by hand (Misha, TWΘ$\boxed{\text{F}}$)
When you learn about trig functions, you typically memorize a few of their values (for 30° or 45°, say) and if you want to know any of the other values, you get pointed to a calculator.

Has that ever seemed unsatisfying to you? If so, take this class, in which we'll see that finding some of these values is as easy as solving polynomials, and approximating all of them is as easy as multiplication. If time allows, we'll learn how to compute inverse trig functions, and also how to quickly find lots of digits of π.

Homework: None

Class format: Interactive lecture

Prerequisites: Be familiar with the formula $e^{ix} = \cos x + i \sin x$.

Dimers and webs (Kayla, TW$\boxed{\Theta}$F)
In chemistry, a *dimer* is a polymer with only two atoms. A *dimer covering* of a graph G is a collection

of edges that covers all the vertices exactly once. One can think of vertices of G as univalent atoms that bond to exactly one neighbor. This is more commonly known as a perfect matching! A dimer model for a graph G is the set of all perfect matchings or dimer coverings of G.

In this class, we will be generalizing the notion of a dimer model to double and triple dimer models that satisfy some "web connectivity". What this boils down to is superimposing single dimer models such that their underlying graphs reduce to objects called non-elliptic webs.

If you like graphs, coloring edges of graphs and a lot of math with picture, this is the class for you!

Homework: None

Class format: Interactive lecture with activities!

Prerequisites: None

Elliptic functions (, Mark, $\boxed{\text{TW}\Theta\text{F}}$)

Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k)\,,$$

where $\sigma_i(k)$ is the sum of the i^{th} powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1],$$

which you are welcome to check if you run out of things to do.)

Homework: Optional

Class format: Interactive lecture

Prerequisites: Functions of a complex variable; in particular, Liouville's theorem

Introduction to Schubert calculus (, Raj, T$\boxed{\text{W}\Theta\text{F}}$)

"How many lines generically meet four given lines in space?" One goal of modern Schubert calculus is to solve such enumerative problems using tools from linear algebra and algebraic geometry. This course will be an introduction to this subject. The main components we will discuss are Schubert polynomials and the Grassmannian of linear subspaces of fixed dimension in space.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra, group theory

The Kakeya problem (, Alan Chang and Neeraja, $\boxed{\text{TW}\Theta\text{F}}$)

In this class, we will introduce the famous Kakeya problem, which asks, roughly: how small can a set be if it contains a line segment in every direction? This question can turn into a variety of math problems, depending on how we choose to interpret "smallness" of a set. For example: what is the smallest possible area of a set in the plane that contains a line segment in every direction? Or alternatively, what is the fractal dimension of such a set? In the class, we will also introduce a variant of the Kakeya problem that asks about the "smallness" of a set which contains a circle of every radius. Note: the construction of Kakeya sets that we'll do in this class will be different from those done in Charlotte's Perron trees class in Week 4.

Homework: Optional

Class format: Interactive lecture

Prerequisites: None

The puzzle of the superstitious basketball player (🏴🚗, Tim!, T$\boxed{\text{W}}\Theta$F)
Sometimes I encounter a math problem, go though a bunch of work to solve it, then arrive at an answer too simple or elegant for the mess of work I did. At that point, I know that there is something deeper and more interesting going on, and I have to know what it is! If you took my class on Guess Who?, you've gone on that journey with me. I also had such an experience with the math puzzle below. It's from Mike Donner, and it was published on FiveThirtyEight.

> A basketball player is in the gym practicing free throws. He makes his first shot, then misses his second. This player tends to get inside his own head a little bit, so this isn't good news. Specifically, the probability he hits any subsequent shot is equal to the overall percentage of shots that he's made thus far. (His neuroses are very exacting.) His coach, who knows his psychological tendency and saw the first two shots, leaves the gym and doesn't see the next 96 shots. The coach returns, and sees the player make shot No. 99. What is the probability, from the coach's point of view, that he makes shot No. 100?

I remember solving it. I had to do a bit of tedious calculation to arrive at the final answer. And when I saw the answer, I was surprised. It was so simple. I thought I was done with the puzzle, but really I was just beginning. Such a simple answer had to have a simple explanation, right? There are in fact a few simple explanations, each more satisfying than the previous.

In the end, I will make the following claim: even if we accept the scenario described by the puzzle, the basketball player's view of the world is totally wrong, and he is probably just superstitious. Perhaps there is a lesson here that we can take back with us to our real lives.

Homework: None

Class format: Interactive lecture

Prerequisites: None

10:10 Classes

A couple things Ben kinda knows about measure zero sets (🏳🚗, Ben, T$\boxed{\text{W}\Theta\text{F}}$)
Everyone[1] knows that the *countable* union of measure zero sets has measure zero. It's not hard to convince yourself that you can take the union of $|\mathbb{R}|$-many measure zero sets and build a set that doesn't have measure zero.

Continuum Hypothesis fans may now be wondering: what happens if you take the union of a number of sets that is *more* than countable and *less* than the size of the real numbers? This class will explore that topic! This will require us to ponder some apparently unrelated content—filters! dominating functions! posets!—and sit nicely between the worlds of set theory and analysis.

As advertised in the title, this is something I've seen once before, but haven't brushed up on in a while!

Homework: Recommended

[1] Or at least everyone who has taken Tanya/Charlotte's Week 3 class, or other measure theory.

Class format: Lecture

Prerequisites: Having seen measure theory will help motivate one day of this class, but nothing is really necessary except having seen induction.

From high school arithmetic to group cohomology (, Eric, T$\boxed{\text{W}\Theta\text{F}}$)
Adding two digits numbers is something we can do mostly automatically, even though we might have to do an annoying "carrying" operation. In this class we will think **incredibly hard** about what carrying is. It turns out that carrying is an instance of group cohomology! We'll explore that connection and use it as a pathway into learning about the subject of group cohomology.

Homework: Recommended

Class format: Learning through worksheets!

Prerequisites: Basic group theory, to the point of understanding that the groups $\mathbb{Z}/100\mathbb{Z}$ and $\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ are not isomorphic.

Geometry Gala (, Ian, $\boxed{\text{T}}$WΘF)
Welcome to the Geometry Gala, the epilogue of the Geometry Galore! In the Gala, we will discuss Jacobi's Theorem, an application of Trig Ceva and/or radical axis theorem which were discussed in the Galore. Do not worry if you haven't taken the Galore class, since I will briefly mention Trig Ceva and radical axis theorem in the beginning so that everyone is on the same page. Are you ready for the Grand geometric finale?

Homework: Recommended

Class format: 20 minute lecture + 20 minute problem solving + 10 minute wrap up

Prerequisites: Geometry Galore recommended (though not required)

Symmetric Functions and their Combinatorics (, Ian, T$\boxed{\text{W}\Theta\text{F}}$)
The topic of symmetric functions has a deep connection with combinatorics. In this course, our goal is to describe the symmetric functions with combinatorial objects such as Young tableaux and a set of lattice paths.

The main topics include:

- Monomial symmetric polynomials/functions;
- Elementary, homogeneous, Power sum symmetric functions;
- Young tableaux;
- Schur function;
- Jacobi–Trudi identity.

Homework: Recommended

Class format: Lecture

Prerequisites: A good understanding of basic combinatorics

Unicorns and Poland (, Arya, T$\boxed{\text{W}\Theta\text{F}}$)
"Unicorn paths" were defined by Polish mathematician Piotr Przytycki, because he initially wanted to define "one-cornered paths", and the word for "one-cornered" in Polish is very similar to the word "unicorns", and "unicorns" is way cooler. Polish people are so cool ¡3

But why do we care about these (and what are these)? Associated to every surface, there is a graph called the "curve graph", which very literally is a graph of curves on the surface. For deep reasons, this graph is very cool. Unicorns guide the way to travel along the unyielding terrains of this graph, and tell us a lot about the geometry of this graph.

Come to this class to get a feel about stuff people study in modern geometric topology!

Homework: Optional

Class format: Lectures (hopefully interactive? :P)

Prerequisites: None!

11:10 Classes

Calculus without calculus (🏴🚗, Tim!, $\boxed{\text{T}}$WΘF)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Eleni is 5 cubits tall and Krishan is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Eleni's head to the top of Krishan's head that touches the ground in the middle. What is the shortest length of string you can use?
- Della rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an very straight section of the Lake Champlain shoreline. The dog's person stands 20 meters away along the shoreline, and throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- When Mathcamp rented out the movie theater to see the *Barbie* movie, you had the chance to choose the optimal seat. Which seat should you have chosen so to make the screen take up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: We won't use calculus (that's the point), but it would be good if you've seen it for context.

How not to integrate (🏴🚲, Steve, TW$\boxed{\Theta}$F)
The function e^{-x^2} has no elementary antiderivative, but the doubly-improper "Gaussian integral"

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

can be easily computed: it's just $\sqrt{\pi}$. To see this, square it, rewrite the result as a double integral, and change to polar coordinates—this shifts from "$dx\,dy$" to "$r\,dr\,d\theta$," letting us wrap up with a simple u-substitution.

This is a really clever trick, and it's natural to hope that it can be used to solve other integrals. Unfortunately, it can't: in a precise sense, any integral which can be solved this way is just a trivial modification of the Gaussian integral itself. This is a simple argument due to Robert Dawson, and we'll see it in detail. If time remains, we'll examine a follow-up argument by Denis Bell, that (1) there is actually a more general version of the Gaussian integral trick and (2) it's also useless.

Homework: Recommended

Class format: Lecture

Prerequisites: Multivariable calculus; be comfortable with the solution of $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ discussed in the blurb.

Imperfection (🚗, Mia and Nathan, T$\boxed{\text{W}}$ΘF)
While things might be all sunshine, daisies, and perfection in the world of graph theory, life gets yuckier in the land of analysis. Exhibit A: $0.9999\ldots = 1$. Who allowed this nonsense?!?! Maybe they should have thought about *not* including so many darn digits. Why can't 0 and 1 be enough? Which brings us to Exhibit B: Have you ever successfully represented 2 in base 10 using just the numbers 0 and 1? We didn't think so. (And if you have, do let us know.) In short, there is an inherent tension between representing real numbers uniquely and representing every real number. AKA bases suck. In this class, we'll prove why.

Homework: Optional

Class format: Interactive lecture

Prerequisites: Epsilons and deltas

Lastly, choose randomly (✈, Travis, TWΘ$\boxed{\text{F}}$)
Have you been taking a random walk on the Champlain campus, pining over the loss of random choices in Mathcamp classes since week 2? Have you despaired over the dire dearth of combinatorial chaos? Then rejoice as we revive that which you thought you had lost.

In this class, we talk about yet one more method in the probabilistic arsenal for solving combinatorial problems, one which that dastardly devil Travis has thus far hidden from your prying eyes. We'll see how to use the Lovász Local Lemma to solve problems in graph theory, or to color the real number line to break *every single pattern in the world.*

Homework: Optional

Class format: Interactive lecture

Prerequisites: **You do not need to have taken *First, choose randomly* to take this class.** Prior experience with graphs (the combinatorial ones) will help make one application easier to understand.

Let $\varepsilon_0 > 0$ be sufficiently small (🚗, Della, TW$\boxed{\Theta\text{F}}$)
Ordinal numbers are what you get if you decide to count and never stop, not even at infinity. A lot of introductions to ordinal numbers go through the ordinals up to ε_0, and then skip straight past all the countable ordinals to ω_1. But in between ε_0 and ω_1, there's a lot of stuff happening—in fact, an uncountable amount of stuff!

In this class, I'll introduce the ordinal numbers in a somewhat unconventional way, and talk about ways to represent countable ordinals. Then we'll try to construct larger and larger countable ordinals, and get a sense of the flavor of the uncountable amount of stuff that's waiting for us. Of course, we'll only be able to understand a tiny (countable) portion of it, but we can get much further than ε_0.

Homework: Recommended

Class format: Lecture

Prerequisites: 🚗 becomes ✈ if you haven't seen ordinal numbers before.

Not theory (✈, Steve, $\boxed{\text{TW}}$ΘF)
Classical propositional logic is pretty boring; there isn't a lot you can do with just two truth values. At various points more interesting logical systems have been proposed, with varying levels of

success/interestingness. Eventually a coherent "theory of propositional logics" emerged, treating in a very general way the properties that any "reasonable" deductive system can have. For example, in a precise sense it *is* possible to add a "half-negation" operator to classical logic without breaking it, but it is *not* possible to add an "un-negation" operator to intuitionistic logic (= classical logic without double negation cancelling) without breaking it.

(That is, and in contrast to things Raj may have told you, a square not is simpler than an un-not.)

We'll explore this area of logic, with particular focus on negation.

Homework: Recommended

Class format: Lecture

Prerequisites: Comfort with Boolean algebra (basically, you should know what a truth table is, and what the truth table for "If A then B" looks like). You do **not** need to have heard of intuitionistic logic before!

Perfection (, Mia, TW$\boxed{\Theta}$F)

Commutative algebraists have their excellent rings and algebraic geometers have their wonderful compactifications, but no one achieves perfection like graph theorists. In this class, we will prove none other than the perfect graph theorem which, in addition to having an excellent[2] name, has an exceedingly clever proof. So, what is perfection? In Graph colorings, we proved that $\omega(G) \leq \chi(G)$ and then asked, can we push those two invariants arbitrarily far apart. An alternative question one might ask is, what graphs achieve equality? Or even better, which graphs achieve equality and have that their subgraphs achieve equality too? The answer, perfect graphs! And what's more, the perfect graph theorem gives us an elegant characterization of these graphs.

Note: Graph colorings is not a prerequisite.

Homework: Optional

Class format: Interactive lecture

Prerequisites: Graph theory

Philosophy of math (, Neeraja, TW$\Theta\boxed{\text{F}}$)

What is the relationship between a mathematical proof and our knowledge of the theorem that it proves? Do some mathematical proofs go beyond establishing the truth of their theorems and actually explain why the theorems are true? How has our mathematical knowledge grown throughout history? We'll discuss some or all of these questions. Most of the questions and examples motivating the discussion come from the book *Proofs and Refutations* by Imre Lakatos.

Homework: None

Class format: Some lecture, mainly discussion

Prerequisites: None

Seven trees in one (, Della, T$\boxed{\text{W}}\Theta$F)

A (binary) tree is either a single node, or a root with a subtree on the left and a subtree on the right. Algebraically, if T is the set of all trees, this can be expressed as $T = 1 + T^2$.

If I gave you the equation $x = 1 + x^2$, you would use the quadratic formula to get $x = \frac{1\pm\sqrt{-3}}{2}$. But that doesn't make any sense in the context of trees, right? And the fact that the solutions satisfy $x = x^7$ can't possibly tell us anything about 7-tuples of trees, can it?

Homework: None

Class format: Lecture

[2]Excellent under the English definition, not the algebraic one.

Prerequisites: None

Sophie Germain primes (🏴🚲, Mia, TWΘ$\boxed{\text{F}}$)
Born in 1776, Sophie Germain was a French mathematician, physicist, and philosopher. Despite having to publish under a male pseudonym (in order to have her work recognized) she spent much of her life at the forefront of mathematics and did ground-breaking work studying the Fermat equation $x^q + y^q = z^q$ for primes q with the property that $p = 2q + 1$ is also prime. Such primes are now known as *Sophie Germain primes.*

In this class, we'll study Sophie Germain primes, prove a surprising fact about primitive roots modulo a Sophie Germain prime, and celebrate a seriously awesome female mathematician!

Homework: Optional

Class format: Interactive lecture

Prerequisites: Introduction to number theory

The Ra(n)do(m) graph (🏁🚗, Travis, T$\boxed{\text{W}}$ΘF)
In *First, choose randomly*, we limited ourselves to randomly producing finite graphs. But it turns out that strange things happen once we start to choose *infinite* random graphs, and in this class, I will tell you the story of these graphs. It is surprisingly short—short enough to fit in 50 minutes—but we'll hit upon some serious mathematics, including the 0–1 law for graphs, which says that every "first-order" graph property is either true for almost every graph or false for almost every graph, with no middle ground.

So settle in, my friends, while I tell you the tale of the infinite random graph. Lean back while I weave for you the disparate threads of its history, from probability theory and model theory to the enduring legacy of popular folk singer-songwriter Arlo Guthrie's most enduring song. Come experience the full range of human emotion, shouting with excitement, gasping with amazement, and weeping over what might have been, as you revel in the dramatic legend of the infinite random graph.

Note: I taught the first half of this class under the same name last year; this version will cover more material.

Homework: Optional

Class format: Storytime

Prerequisites: Be familiar with basic laws of probability. **You do not need to have taken *First, choose randomly* to take this class.**

The transcendence of a single number (including Liouville's constant) (🏴🚗, Travis, TW$\boxed{\Theta}$F)

Proving the transcendence of many numbers is good, but proving the transcendence of one number is good enough. Plus, for this, we can prove transcendence simply and quickly: in one day only!

In this class, we'll take the quick route to finding an explicit transcendental number (Liouville's constant) and then see why the title of this class is actually not telling the truth and consequently find an uncountable set of real numbers and prove that all of them are transcendental.

Homework: None

Class format: Interactive lecture

Prerequisites: You don't need nuthin'!

1:10 Classes

Axiom of choice (🏳🚲, Narmada, $\boxed{\text{TW}}$ΘF)
Why do mathematicians get so fussy about the axiom of choice? We'll talk a little bit about why

the axiom of choice isn't just obviously true. We'll look at some obviously fake statements that are equivalent to the axiom of choice. We'll see why math without the axiom of choice might be sad sometimes. And by the end of this class, *you* get to be a mathematician who's fussy about the axiom of choice!

(If you are a returning camper, this is the same class I ran last year.)

Homework: None

Class format: Lecture

Prerequisites: None

Galois theory crash course (⚐✈, Mark, $\boxed{\text{TW}\Theta\text{F}}$)
In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully. Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later).

If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we might conceivably be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class.) Even if we don't get that far, the so-called Galois correspondence (which we should be able to get to, and probably prove) is well worth seeing!

Homework: Optional

Class format: Interactive lecture

Prerequisites: Group theory; linear algebra; some familiarity with fields and with polynomial rings

Honey, I shrunk the vectors (⚑🚗, Tanya, TW$\boxed{\Theta}$F)
Imagine you have a high dimensional dataset (e.g. how much each camper likes every possible LN2 ice cream flavor) that you're trying to analyze, however, every algorithm that you attempt to use ends up being much too slow. There are several possible ways to address this issue—use a cleverer, faster algorithm, get a more powerful computer to run your code on, or find a way to shrink your data without losing too much of the original structure. The focus of this class will be the latter technique. We will see a simple randomized procedure for dimensionality reduction while (almost) preserving pairwise distances between the points in your dataset with high probability. If you enjoyed the second half of my Week 2 class comparing randomized and deterministic computation, you will most likely have fun here as well!

Homework: None

Class format: Interactive lecture

Prerequisites: Knowing how matrices act on vectors

How the compactness theorem got its name (⚐✈, Krishan, TWΘ$\boxed{\text{F}}$)
The compactness theorem is one of the most important theorems in the field of model theory. Surprisingly, it gets its name from the topological notion of compactness. In this class we'll see why this

is the case. We'll define a certain topological space associated with a theory, and we'll prove that the compactness theorem is equivalent to the fact that this space is compact.

Homework: None

Class format: Interactive lecture

Prerequisites: Model theory

Percolating through percolation theory (🏴🚗, Tanya, T$\boxed{\text{W}}$ΘF)
Percolation theory is an area of probability theory that studies the structure of infinitely large graphs after edges get randomly removed. We will see that certain properties for such graphs will either hold with probability 0 or 1, however, perhaps surprisingly, we may not be able to easily distinguish between the two situations. We will also learn about some open questions in this field, as they are often rather straightforward to state, yet incredibly challenging to resolve.

Homework: None

Class format: Interactive lecture

Prerequisites: Having seen a bit of probability before might be helpful.

The Chevalley–Warning theorem (🏴🚲, Kevin, TWΘ$\boxed{\text{F}}$)
The Chevalley–Warning theorem is a nice little result that says that the number of solutions to a system of equations mod p must be divisible by p, given that the number of variables is sufficiently high relative to the degrees of the polynomials. We'll see a slick proof of the theorem using Fermat's little theorem, which we'll also introduce in the class.

Homework: None

Class format: Interactive lecture

Prerequisites: Modular arithmetic, polynomials

Why 0 is the biggest prime (🏳🚗, Kevin and Krishan, T$\boxed{\text{WΘ}}$F)
This class will be about a neat application of model theory to algebra called the Lefschetz principle.

In the first day we'll use model theory to prove the Lefschetz principle, which states that a first order property is true in $\mathbb{C}$ if and only if its true in algebraically closed fields of with arbitrarily large characteristic. We'll use a bunch of concepts from the model theory class in the second week, but a camper who didn't take the class will be able to understand the big ideas.

In the second day, Kevin will talk about how the Lefschetz principle can be used to be prove some cool results about algebra and algebraic geometry, such as (1) the fact that any injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is surjective (2) Hilbert's Nullstellensatz, which describes all maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$.

Homework: None

Class format: Interactive lecture

Prerequisites: Ring theory (required). 🚗 becomes ✈ if you haven't taken model theory.

Zeroes of recurrence sequences through p-adics (🏳🚗, Eric, TW$\boxed{\text{ΘF}}$)
In this class we'll aim to prove (a simple case of) the Skolem–Mahler–Lech theorem: for an integer recurrence sequence a_n, the set of indices n where $a_n = 0$ can be at worst the union of a finite set and some arithmetic progressions. This theorem is really cool because the only known proofs rely on p-adic analysis in a crucial way. I'll introduce some minimal amount of necessary facts about the p-adic numbers and we'll prove the Skolem–Mahler–Lech theorem using p-adic infinite series.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra to the point of being comfortable with diagonalizing 2×2 matrices. Comfort with the idea of defining functions by infinite series, having seen series expansions of e^x and $\log(x)$ before is useful. Modular arithmetic, at the level of a is invertible mod n if and only if $\gcd(a, n) = 1$.

2:10 Classes

Ben teaches Susan's class (🏁🚲, Ben, T$\boxed{\text{W}}$ΘF)

Five minutes before class time, Susan will send Ben a slide deck. Good luck, Ben!

Homework: None

Class format: Slideshow Telephone

Prerequisites: None. Prerequisites would be too big a hint.

Everything Ben knows about nonmeasurable sets (🏳🚗, Ben, TW$\boxed{\Theta\text{F}}$)

Here are a few questions. Some are easier and some are harder! Here, $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable sets of $\mathbb{R}$. Also, $\mathcal{N}$ denotes the set of Lebesgue non-measurable sets of $\mathbb{R}$. For convenience.

(1) How large is $\mathcal{L}$?
(2) Suppose we declare two sets A, B equivalent if their symmetric difference $(A \setminus B) \cup (B \setminus A)$ has measure zero. How many equivalence classes does $\mathcal{L}$ have?
(3) Same as the previous one. How many equivalence classes does $\mathcal{N}$ have?

This class will answer all of those in pretty abrupt fashion, by breaking out some comically overpowered tools, mostly from logic. Some other topics might include the fact that we can use nonprincipal ultrafilters to build nonmeasurable sets and Bernstein sets, the sets which don't contain or miss any perfect sets.

Homework: Recommended

Class format: Lecture

Prerequisites: Having seen measure (e.g. in Week 3) is probably necessary for this to make sense. It will help to have seen transfinite induction before.

Fair division using topology (🏁🚗, Jane Wang, $\boxed{\text{TW}}$ΘF)

How can we fairly divide a cake among multiple people when each person values frosting, edges, etc. differently? We can answer this question using tools from topology, the study of continuous functions and properties that are preserved under continuous deformation. It turns out that topology has many surprising applications to fields ranging from economics to combinatorics to data science. In this short course, we will survey some applications to problems of fair division (of cakes, necklaces, rent, and more). No prior knowledge of topology will be assumed.

Homework: Optional

Class format: Interactive lecture

Prerequisites: None

From the Sato–Tate conjecture to murmurations (🏳✈, David Roe, T$\boxed{\text{W}\Theta\text{F}}$)

The Sato–Tate conjecture was made in the 1960s based on numerical experimentation with elliptic curves, and proven in 2011 (though generalizations to higher genus curves are still open). It focuses on counting solutions to equations like $y^2 + y = x^3 - x$ modulo different primes p. We prove Hasse's theorem, which tells us that the number of solutions is about $p+1$, with an error of at most $2\sqrt{p}$. The Sato–Tate conjecture focuses on the distribution of the error term as p varies, providing two possible specific limiting distributions which you can see numerically with cool histograms.

Last year, a related phenomenon was observed by several authors, including an undergraduate at the University of Connecticut. Instead of fixing an elliptic curve and allowing p to vary, they ordered elliptic curves with rational coefficients in a specific way and looked at how a certain average value of the error term varied. The resulting plots showed an unexpected oscillation, and behave differently based on whether the elliptic curve has finitely many or infinitely many rational points (rather than points modulo p).

Homework: Optional

Class format: Interactive lecture

Prerequisites: Elliptic curves and Introduction to group theory. Finite fields and Functions of a complex variable will be helpful for a few parts of the course, but not required.

Spherical geometry (⚑🚲, Kira Lewis, TW$\boxed{\Theta}$F)
What if I told you that:

- Parallel lines do not exist?
- The sum of angles in a triangle is strictly $> 180°$?
- The area of any polygon can be calculated in terms of its angles?

This may all sound ridiculous in normal geometry, but in the spooky world of spherical geometry, it's all true. Spherical geometry is where points, lines, and circles live on the sphere instead of in the plane. Come join this class for a tour through some of the craziest theorems in spherical geometry and to see proofs of the statements above.

Homework: None

Class format: Lecture

Prerequisites: None

Taming the grouchy Grassmannian (🏁🚗, Kayla, TW$\boxed{\Theta\text{F}}$)
The Grassmannian is a mathematical object that is natural place to work in. As a set, it is comprised of k-dimensional vector space in some n-dimensional space. However, even with its innocent definition, this space is notoriously grouchy for its complicated associated structures.

Namely, the Grassmannian has topological and geometric structure we can endow it with. For instance, we can view it as a *projective variety* using a set of coordinates called Plücker relations or as a compact smooth manifold by looking at something called its Schubert decomposition.

We will be exploring the dark sides of the Grassmannian and taming its grouchiness with *combinatorics.* Come see some tableaux, plabic graphs and perfect matchings tame this grouchy Grassmannian!

Homework: None

Class format: Interactive lecture

Prerequisites: None (some linear algebra exposure is good e.g. reduced row echelon form, determinants)

(THICCC) Triangles, Hyperbolas, Isogonal Conjugates, and Certain Circles (🏁✈, Anthony Wang and Nathan Cho, $\boxed{\text{TW}}$ΘF)
Come explore the rich, diverse, and endlessly surprising world of triangle conic geometry! In this class, we'll develop some theory relating circumconics of triangles with whatever the heck an isogonal conjugate is, and we'll use that theory to prove Feuerbach's theorem, a theorem purely about circles. To do this, we will push plane geometry to its limit, and we'll encounter many cool and relatively unknown tidbits of conic and classical geometry along the way.

Homework: Recommended

Class format: Lecture

Prerequisites: Experience with Euclidean geometry, some familiarity with conics

Van Roomen's problem (, Philip Yao, TW$\Theta$$\boxed{\text{F}}$)
Find the smallest positive x that satisfies:

$$x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} - 14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{25} + 483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} - 34512075x^{11} + 7811375x^{9} - 1138500x^{7} + 95634x^{5} - 3795x^{3} + 45x = \sqrt{\frac{7}{4} - \sqrt{\frac{5}{16}} - \sqrt{\frac{15}{8} - \sqrt{\frac{45}{64}}}}$$

Homework: None

Class format: Lecture

Prerequisites: Trig identities (triple angle formula and angle sum/difference should be fine)

Colloquia

Think different (*Po-Shen Loh*, Tuesday)
One of the most fascinating things in math is when a problem is solved via an apparently-unrelated idea. The talk will start off with two examples within math itself, which were a source of inspiration to the speaker early in his career, when he used the technique on math research, and throughout his teaching.

The second part of the talk will go into the speaker's current work (https://live.poshenloh.com), which focuses on a large-scale real-world problem: teaching people how to generate creative math ideas. This part of the talk is designed specifically for math people who might have interest in doing real-world projects someday. It will highlight places where the math background (particularly out-of-the-box math thinking) ended up being instrumental in inventing new real world solutions. Along the way, the philosophy of game theory and graph theory will make cameo appearances.

A magic show (*Tadashi Tokieda*, Wednesday)
"There cannot be any abstract," Tadashi writes. We will find out what happens on Wednesday!

Project fair (Campers, Friday)
Project fair is an occasion for campers to present the results of their project. Sometimes campers make a poster about what they've done, sometimes the output of the project is more complicated than that.

If you are interested in presenting at project fair, talk to the staff member(s) supervising your project by Tuesday!