CLASS DESCRIPTIONS—WEEK 1, MATHCAMP 2024

Contents

9:10 am classes

Geometric algebra. 🌶🌶🌶 (*Ari Nieh*) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

In linear algebra, you may have seen the symmetric dot product (which works for vectors in any dimension) and the antisymmetric cross product (which only works in $\mathbb{R}^3$.) But both of these are components of a secret, third thing: the *geometric product* of vectors. We define

$$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}$$

Here $\cdot$ is the usual dot product or inner product, yielding a scalar. $\wedge$ is the *outer* product, yielding a *bivector*. And why are we adding a scalar and bivector? Because *nobody can stop us*.

With the algebra of multivectors, we can do geometry on $\mathbb{R}^n$ in shockingly elegant ways. In particular, we'll see exactly why rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$ arise from multiplication by complex numbers and conjugation by quaternions, and the straightforward extension to general n.

Homework: Recommended.

Class format: Interactive lecture

Prerequisites: Linear Algebra. You don't have to know what a quaternion is.[1]

Intro group theory. 🌶🌶 (Susan) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

Let's list some operations on sets! Let's see... there's addition on the integers, multiplication on the rationals, and taking the average of two real numbers. Those are three good ones. Now let's think of some more exotic operations. How about multiplying square matrices over the complex numbers, or composing continuous functions from the reals to the reals?

Broadly, abstract algebra is the study of operations of sets. A binary operation that satisfies a collection of particularly nice properties (associativity, identity, and inverses) is called a group operation, and the set that it acts on is called a group. Groups pop up all over mathematics. They can be used to explore symmetries of geometric objects, prove the existence of an unsolvable quintic polynomial, and solve tricky counting problems in combinatorics. But they are also a beautiful class of mathematical objects in their own right.

In this class, we'll be getting to know our good friend the group. Starting with a few simple axioms, we'll build a fundamental toolkit of theorems that give us an instinct for how these objects behave.

[1] Corrected June 30, 2024 — A previous version incorrectly omitted linear algebra as a prerequisite for Geometric algebra.

We'll learn what it means for two groups to be secretly "the same," and talk about subgroups and what Lagrange's theorem has to say about their sizes. Come join us!

Homework: [**HW**] Required.

Class format: This will be a lecture-based class with substantial problem sets to work on between classes. Homework is listed as required because campers will get much more out of the class if they engage substantially with the problem sets. Come join my problem solving parties during TAU!

Prerequisites: None.

This class may help prepare you for:

- **Toppling sandpiles** (Travis, in Week 1) — Prerequisites: None. You'll enjoy this class most if you know the definition of a group or are currently taking Intro to Group Theory, but it's not necessary. You can come to just play with graphs and sandpiles, too.
- **MCSP (Markov Chains and Stochastic Processes)** (Alyona & Arya, in Week 2) — Prerequisites: Basics of matrices and linear transformations (from intro to linear algebra), and basic definitions of probability (but the probability aspect we can cover during the class). Some familiarity with groups could be useful for Day 5, but not necessary.
- **Introduction to ring theory** (Eric, in Week 2) — Prerequisites: None. Group theory can be helpful context but is not necessary.
- **Regular Languages & Word Problems** (Sonya, in Week 2) — Prerequisites: Group theory (actually, mostly just the definition of a group, so it might be possible for students to take this class at the same time as group theory)
- **Commutative algebra/algebraic geometry** (Mark, in Week 3) — Prerequisites: Basic familiarity with groups, rings (including polynomial rings), and fields
- **Field extensions and Galois theory** (Mark, in Week 3) — Prerequisites: Basic familiarity with groups, rings (especially polynomial rings), and fields
- **What do we do when we do math?** (Maya, in Week 3) — Prerequisites: You must have seen a few contexts of abstract math, such as group theory
- **Representation theory** (*Aaron Landesman*, in Week 4) — Prerequisites: Linear algebra and group theory. From group theory, familiarity with group actions is essential. One should also be familiar with symmetric groups, conjugacy classes, commutators, and quotient groups, among other things. From linear algebra, one should be familiar with linear transformations, eigenvalues, trace, and direct sum, among other things.
- **Intro to Elliptic Curves** (Chloe, in Week 4) — Prerequisites: Soft prereq: definition of group.
- **Topological Tverberg's Theorem** (*Viv Kuperberg*, in Week 4) — Prerequisites: Linear algebra. Enough group theory to know what a group action on a set is.

From Hall's theorem to maximum flows. 🌶🌶🌶 (Mark) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

When you think about combinatorics, do you think about counting? Well, so do I, but in this case the class is not really about counting but about existence (showing that certain things exist) and optimization (showing how to get the best things possible), which are also important themes in combinatorics and (especially optimization) in its applications. Here's an example:

If there is a finite set I of employers, and each employer $i \in I$ has a set of job applicants that would be acceptable to her/him/them, when is it possible for all the employers simultaneously to hire applicants (one per employer) which are acceptable to them? A theorem proved in 1935 by Philip Hall states that this is possible if and only if for every subset J of I, the number of applicants acceptable to at least one employer in J is at least the number of elements of J.

Actually, this is often known as Hall's Marriage Theorem, but phrasing it that way makes it hard to avoid either using or discussing outdated stereotypes about gender, marriage, and so forth - which is likely to make some people unhappy. We'll prove this existence theorem in two quite different ways, the second of which will lead to the topic of matchings in graphs, and then we'll start looking at optimization — first of matchings in general, then of flows in networks. (All these terms will be defined, but intuitively you might think of the flow of goods from Los Angeles to Chicago over the interstate highway network, or the flow of information, such as e-mail, from one computer to another via the World Wide Web.) This will bring us to the famous max flow/min cut theorem, that expresses the maximum "value" of a flow in terms of "capacities" of "cuts", which are roughly potential bottlenecks along the way.

If time permits, we may explore other major theorems, such as König's Theorem and Dilworth's Theorem, that are closely related to Hall's Theorem. Whether or not we get to those, this class should convince you (should you need convincing) that while counting can be both fun and important, there is a lot of other worthwhile combinatorics to explore also!

Homework: Recommended.

Class format: Interactive lecture

Prerequisites: None, beyond some comfort with abstraction and subscript notation

Network algorithms & game theory. (Sonya) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

In this class we will discuss ways to find optimal solutions to various network problems. For example, how are medical students matched to residence programs? How are college dorm rooms distributed? Are eBay auctions fair? For our purposes, a network is any system of agents (like people or organizations) with their preferences or available resources, and the problems we'll talk about involve assigning resources or matching the agents in some way. We will discuss what it means for an outcome to be "fair" or "efficient" (there isn't always one obvious definition!) and which of these properties we might actually want in various real-life problems. We'll cover some of the algorithms for finding optimal outcomes and give proofs that these algorithms are guaranteed to work. During the last day of class, we'll talk about game theory, a mathematical modeling tool used to study strategies and decision-making with applications in economics, computer science and social science. Most of the theory will be built from scratch using only logic and basic proof techniques (whenever any additional preliminaries are needed, they will be covered in class). This is a good class to take if you are interested in how many real-life system organization problems can be viewed as logical puzzles and studied with mathematical tools.

Homework: Optional.

Class format: Interactive lecture

Prerequisites: None.

Are there no-where differentiable continuous functions? (Laithy) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

Analysis is filled with pathologies and counterintuitive ideas that challenge our natural way of seeing things. For instance, our intuition leads us to associate continuity of functions with smoothness and differentiability. In fact, the general belief among mathematicians a long time ago is that continuous functions are differentiable "almost everywhere". It turns out that there exist functions that are continuous everywhere but differentiable nowhere! Together, we will explicitly construct such a function, widely known as the Weirstrass monster function. But it gets even more surprising! We will prove that, in a sense, "almost every" continuous function is actually no-where differentiable. That Weirstrass monster function, despite seeming pathological, ends up being a rather generic continuous function. This is due to a very deep theorem called the Baire Category Theorem that we will study and prove rigorously. It vaguely asserts that one cannot cover a space with countably many "thin" slices of that space. Since its discovery, it quickly became one of the fundamental tools in analysis leading to significant changes and advancements in analysis. If time permits, we will explore more of the consequences of the Baire category theorem.

Homework: Recommended.

Class format: Interactive lecture

Prerequisites: You should be familiar with the precise definition of continuity and differentiability (specifically the $\epsilon - \delta$ formulation of the definition). You should also be comfortable with sequences and series. Familiarity of metric spaces is recommended but not required.

10:10 AM CLASSES

Theory of computation. (Athina) TWΘFS 50 minutes

Can you write a computer program that takes as input a polynomial equation with integer coefficients and any (finite) number of unknowns, and outputs whether or not the equation has an integer solution? This is known as Hilbert's 10th problem, and −spoiler alert− the answer is no, such a computer program cannot exist!! Our goal for this class will be to develop a mathematical theory that enables us to reason abstractly about computation and prove such results. We will encounter things like DFAs, NFAs, and Turing Machines! Along the way, we will engage with Big Questions like "What can we compute?", "What can we *not* compute??", "What *is* computation anyway????"

Homework: Recommended.

Class format: Interactive lecture

Prerequisites: None.

Mathcamp crash course. → (Zach) TWΘFS 50 minutes

Math is useless unless it is properly communicated. Most of math communication happens through a toolbox of terminology and proof techniques that provide us with a backbone to understand and talk about mathematics. These proof techniques are often taken for granted in textbooks, math classes (even at Mathcamp!) and lectures. This class is designed to introduce fundamental proof techniques and writing skills in order to make the rest of the wonderful world of mathematics more accessible. This class will cover direct proofs from axioms, proofs using negation, proofs with complicated logical structure, induction proofs, and proofs using cardinality. If you are unfamiliar with these proof techniques, then this class is highly recommended for you. If you have heard of these techniques, but would like to practice using them, this class is also right for you. Here are some problems that can assess your knowledge of proof writing:

- Negate the following sentence without using any negative words ("no", "not", etc.): "If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."
- Given two sets of real numbers A and B, we say that A dominates B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two disjoint, nonempty sets A and B such that A dominates B and B dominates A.
- Prove that there are infinitely many prime numbers.
- Let $f\colon A \to B$ and $g\colon B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
- Define rigorously what it means for a function to be increasing.

- What is wrong with the following argument (aside from the fact that the claim is false)? On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

  Proof. We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. □

If you would not be comfortable writing down proofs or presenting your solutions to these problems, then you can probably benefit from this crash course. If you found this list of questions intimidating or didn't know how to begin thinking about some of them, then you should definitely take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

This class features a mix of lecture and group work. Additionally, because the best way to improve your proof-writing skills is by writing lots of proofs and receiving targeted feedback, we will have required homework and TAU components—please plan to join me for at least 30 minutes during each TAU to work on and discuss homework problems.

Homework: [**HW**] Required.

Class format: Mix of lecture and group work.

Prerequisites: None.

This class may help prepare you for:

- **All other Mathcamp classes!**

King chicken theorems. (Marisa) $\boxed{\text{TW}}\Theta$FS 50 minutes

Chickens can be cruel creatures. Whenever you put a bunch of them together, they will form a pecking order. Perhaps "order" is an exaggeration: the chickens will go around pecking whichever chickens they deem to be weaker than themselves, and whenever chickens encounter one another, it's a peck-or-be-pecked situation. Imagine you're a farmer, and you're observing the behavior of your chickens. You would like to assign blame to the meanest chicken. Is it always possible to identify the meanest chicken? Can there be two equally mean chickens? Are there pecking orders in which all the chickens are equally mean?

Homework: None.

Class format: IBL

Prerequisites: None.

Advanced chickenology. (Misha) TW$\boxed{\Theta\text{FS}}$ 50 minutes

There is much more to learn about chickens and pecking orders! In this class, we will learn—through logic puzzles—about fraud in chicken pecking competitions, about how to find pecking cycles, and about what happens to a pecking order when the number of chickens becomes infinite.

If you've ever seen logic puzzles about truthtellers and liars, especially as popularized in books by Raymond Smullyan, you will find the style of these puzzles familiar. You don't need to take this class to see the logic puzzles; they will be posted on Slack. This class is a time for us to think about and discuss these logic puzzles together, but in a 1-chili, relaxed fashion. (The puzzles themselves range in difficulty.)

Homework: Optional.

Class format: "Interactive puzzle-solving". We will go through logic puzzles about chickens and try to solve them together, discussing the approaches that are good or bad to take, and the mathematical ideas underlying them. If there's interest, we might set aside some time to think about puzzles on your own or in small groups.

Prerequisites: None (in particular, Marisa's class about chickens is not a prerequisite for this class about chickens). However, if you've studied tournaments in graph theory in a more serious context, this class may be too silly for you—check with Misha if you're worried you know too much!

This class can continue as a project: If you decide you enjoy logic puzzles like these ones, you may be interested in doing a project all about writing your own logic puzzles! You could take inspiration from classic puzzles about truthtellers and liars trying to cross a river, or take your favorite topic in math and try to turn it into puzzles.

This class is not required to work on this project, but it could provide inspiration.

The circle method and Waring's problem. 🌶🌶🌶🌶 (Kevin) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

Famously, every positive integer is the sum of 4 squares, and slightly less famously, every positive integer is the sum of 9 cubes. In this class, we'll study the following problem: given $n \geq 9$ and a positive integer N, estimate $R_n(N)$, the number of ways to write N as a sum of n cubes. We'll use a cool technique called the Hardy-Ramanujan-Littlewood circle method to analyze some exponential sums and prove an asymptotic for $R_n(N)$.

Homework: Optional.

Class format: Lecture

Prerequisites: Comfort with calculus and number theory (e.g. modular arithmetic, arithmetic functions).

This class can continue as a project: The project will be further reading about the circle method, either covering applications to polynomials of higher degree or the generalization of the method to the function field case, which has applications to algebraic geometry.

A recipe for resolving real riddles. 🌶🌶 (Glenn) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

From being the universe in which we do algebra throughout high school, to describing the coordinate plane for geometry, real numbers show up everywhere in math. They are fundamental, and theorems about them range from easy to downright grueling. Famous mathematicians like Issac Newton struggled to solve some problems about real numbers for centuries.

But surprisingly, we can now completely automate basically everything about real numbers! It turns out that your computer can actually automatically prove practically every theorem about real numbers, and by extension, polynomials, geometry, etc. The goal of this class will be to explore and understand this amazing algorithm of Alfred Tarski.

Homework: Recommended.

Class format: Worksheets (to be finished as homework)

Prerequisites: None! The derivative will make a guest appearance for one day, but it'll be accessible even if you haven't seen it before.

11:10 AM CLASSES

Probabilistic method in graph theory and k-SAT problems. 🌶🌶 (Kailee) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

Theorem: *math theorem here*

Proof: maybe x, maybe y, and maybe z. QED

If we flip a coin, maybe it will land heads up, or maybe tails. Probability is all about saying "maybe". It's pretty surprising that, if we can come up with just the right set of "maybes", we can prove things with certainty. This approach is called the probabilistic method. In this class we'll learn the basics of discrete probability and the probabilistic method. Then, you will use the methods discussed in class to prove several results in graph theory (relating to colorings, Hamiltonian paths, tournaments, independent sets, and more!) and theoretical computer science (specifically k-SAT problems).

Homework: Recommended.

Class format: Interactive lecture and groupwork

Prerequisites: Combinatorial counting rules, factorials, binomial coefficients

🔧 *This class can continue as a project:* The project would be a reading course type of project for students who are interested to learn/see more proofs using probabilistic methods.

Linear algebra (intro) (week 1 of 2). 🌶🌶 (Mark) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

You may have heard that linear algebra involves computations with matrices and vectors - and there is some truth to that. But this point of view makes it seem much less interesting than the subject really is; what's exciting about linear algebra is not those computations themselves, but

1) the conceptual ideas behind them, which are elegant and which crop up throughout mathematics, and
2) the many applications, both inside and outside mathematics.

In this class we'll deal with questions such as: What is the real reason for the addition formulas for sin and cos? What happens to geometric concepts (such as lengths and angles) if you're not in the plane or 3-space, but in higher dimensions? What does "dimension" even mean, and if you're inside a space, how can you tell what its dimension is? What does rotating a vector, say around the origin, have in common with taking the derivative of a function? What happens to areas (in the plane), volumes (in 3-space), etc., when we carry out a linear change of coordinates? If after a sunny day somewhere the next day has an 80% probability of being sunny and a 20% probability of being rainy, while after a rainy day the next day has a 60% probability of being sunny and a 40% probability of being rainy, and if today is sunny, how can you (without taking 365 increasingly painful steps of computation) find the probability that it will be sunny exactly one year from now? If you are given the equation $8x^2+6xy+y^2=19$, how can you quickly tell whether this represents an ellipse, a hyperbola, or a parabola, and how can you then (without technology) get an accurate sketch of the curve? How do astronomers know the chemical composition of distant stars? (We may not get to all these things in the two weeks, but we should cover most of them at least to some extent.)

Homework: Recommended.

Class format: Interactive lecture

Prerequisites: Although the blurb above refers to taking a derivative, you'll be able to get by if you don't know what that means. In fact, there are no prerequisites, *but* if you have no previous exposure to abstract concepts, you should seriously consider taking the Mathcamp Crash Course at the same time.

This class may help prepare you for:

- **MCSP (Markov Chains and Stochastic Processes)** (Alyona & Arya, in Week 2) — Prerequisites: Basics of matrices and linear transformations (from intro to linear algebra), and basic definitions of probability (but the probability aspect we can cover during the class). Some familiarity with groups could be useful for Day 5, but not necessary.
- $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (Kevin, in Week 2) — Prerequisites: Comfort with linear algebra (eigenvalues, inner products)
- **Alice and Bob go quantum** (Narmada, in Week 2) — Prerequisites: linear algebra
- **Application of Linear Algebra and Projective Geometry** (Tim!, in Week 2) — Prerequisites: Linear Algebra
- **How to multiply numbers realllllly fast** (Eric, in Week 3) — Prerequisites: Intro linear algebra, at the level of knowing that linear transformations are invertible iff the determinants of their corresponding matrices are non-zero. Basic computer science (big O notation) is helpful context but is not necessary, consider the class as 3 chilis if you don't have CS background.

  Alternatively if we have an intro CS course this could have that as a formal prereq and I'd spent less time talking about big O notation and complexity analysis.
- **Numerical Analysis: How Computers Do Calculus & Differential Equations** (Sonya, in Week 3) — Prerequisites: Linear algebra (actually, just matrix multiplication). Also calculus.
- **Representation theory** (*Aaron Landesman*, in Week 4) — Prerequisites: Linear algebra and group theory. From group theory, familiarity with group actions is essential. One should also be familiar with symmetric groups, conjugacy classes, commutators, and quotient groups, among other things. From linear algebra, one should be familiar with linear transformations, eigenvalues, trace, and direct sum, among other things.
- **Error-Correcting Codes and Sphere Packing** (Kailee, in Week 4) — Prerequisites: Linear algebra (matrix multiplication, linear independence), Combinatorics (counting rules, binomial coefficients)
- **The first black hole: Schwarzschild spacetime** (Laithy, in Week 4) — Prerequisites: Linear algebra and calculus. Special relativity is recommended but not required.
- **Totally positive, dude** (*Mia Smith*, in Week 4) — Prerequisites: Linear algebra (familiarity with bases, matrices, and determinants)
- **Ghostly graphs** (Travis, in Week 4) — Prerequisites: You should know: the dot product for real vectors, what an orthonormal basis is, and the spectral theorem for real matrices. (The Introduction to Linear Algebra class at Mathcamp will be sufficient, if you've never seen linear algebra before.) If you've seen some of these things but not others, talk to me—I might be able to catch you up.
- **Topological Tverberg's Theorem** (*Viv Kuperberg*, in Week 4) — Prerequisites: Linear algebra. Enough group theory to know what a group action on a set is.
- **Arithmetic complexity** (*Yuval Wigderson*, in Week 4) — Prerequisites: It would be helpful, but not strictly necessary, to know what determinants are and to have a rough sense of what the P vs. NP question asks. It would also be helpful to have seen big-O notation.

Ordinals and cardinals: a mellow introduction to infinity. (Krishan) TWΘFS 50 minutes

In this class we'll learn how to count past infinity! In particular, we'll learn about two types of infinite numbers: the ordinals and the cardinals. Both of these can be used to compare infinite sets in different ways. Cardinal numbers tell us about the size of a set ("I'm taking three classes"). On the other hand, ordinal numbers let us keep track of the position of an element in a list ("This is my third class today").

When talking about finite sets, these concepts are not too different. There is only one way to take three classes in a day: you take the first, then the second and finally the third. However, this would not be the case if you were taking infinitely many classes instead of three. We'll see that there are meaningfully different ways to take infinitely classes in one day, so these notions diverge in the infinite case. Don't worry if this doesn't make sense to you now, it will after you take this class!

This is a fairly abstract topic for a one-chili class (infinity is hard to wrap your head around)! But my goal is to make it as accessible as possible: we'll take our time and will focus on big ideas and intuition over details.

Homework: None.

Class format: Mix of lecture and group work

Prerequisites: None.

Surreal numbers. 🌶 (Krishan) TWΘ$\boxed{\text{FS}}$ 50 minutes

The goal of this class is for you, the reader, to uncover the secrets of a brand new number system: the surreal numbers. The surreals were invented by John Conway and then popularized by David Knuth in a rare piece of mathematical fiction. Knuth's book tells the story of a couple, Bill and Alice, who uncover an ancient stone with mysterious writing. The stone gave them the barest hints of how this "ancient" number system worked, but with perseverance, the pair was able to uncover its secrets.

Working in groups, you'll follow in Alice and Bill's footsteps to see how the surreals are built. You'll start with nothing and end up with a number system containing the reals as well as infinite numbers, infinitesimals (numbers which are larger than zero, but smaller than every positive real) and much much more!

Homework: None.

Class format: Mostly group work with a bit of lecture

Prerequisites: Induction (you'll be okay if you take the Crash Course alongside this one)

How fast can we Banach this Tarski? 🌶🌶🌶🌶 (Narmada) $\boxed{\text{TW}\Theta\text{FS}}$ 50 minutes

The short answer is: very slowly. And the truthful answer is: I have no idea, but unpacking the question is far more interesting!

We will use the Banach–Tarski paradox as our entry into the world of measurable combinatorics, which solves equideconposition problems (measurable) using infinite graphs (combinatorics). When life gives you a graph, run an algorithm on it. It turns out that asking, "Can I find a continuous coloring of this infinite graph?" is basically the same as asking, "Can I find a fast LOCAL algorithm to color finite graphs?" The goal of this class is to convince you that these two questions are, indeed, the same.

All of the big results we'll see are very recent, which unfortunately means that we will not see their proofs. But if you're excited to learn about a very active and super cool research area, consider taking this class!

Homework: Optional.

Class format: lecture

Prerequisites: basic graph theory, know what a continuous function is

Geometric geometry. 🌶️🌶️🌶️ (Arya) TWΘFS 50 minutes

There are many ways to study geometry, and many different interpretations of what "studying geometry" means. In this class, we shall study geometry geometrically, the way it was intended. We shall study and construct various geometric spaces, in various levels of abstraction.

For example, we have studied a lot of properties of Euclidean geometry growing up in our youthful days. What would it mean for a generic metric space to "be Euclidean"? What other geometries can you define on the plane, and how would they behave? More broadly, what would it mean for two different geometric structures to be "different"?

The first two days of the class would involve abstracting familiar geometric concepts such as metrics and curvature to general metric spaces. The last three days would involve working with hyperbolic geometry, aka the best geometry, and the idea of studying geometric spaces "coarsely" to make them behave like hyperbolic geometry.

Homework: Recommended.

Class format: Interactive lecture plus tons of group work / homework exercises.

Prerequisites: None.

1:10 PM CLASSES

Graph inequalities by magic. 🌶️🌶️ (Travis) TWΘFS 50 minutes

A lot of extremal graph theory is about counting, which is something that babies can do. Not so EXTREME after all, eh? But does a baby know that any graph with k edges has at most $k^{3/2}$ triangles? I don't think so. That result is called the Kruskal–Katona theorem, and in this class we'll do math by drawing pictures and prove not only this result, but many others that babies only wish they could dream of.

Homework: Recommended.

Class format: Mixed interactive lecture and groupwork

Prerequisites: None.

Calculus wars. 🌶️ (Travis) TWΘFS 50 minutes

The opening years of calculus are filled with secrecy, bitterness, hostility, political maneuvering, xenophobia, religious persecution, and plague. What a time to be alive.

Among all of that, Isaac Newton invented calculus. And Gottfried Leibniz *also* invented calculus. Everyone grooved along happily for a bit, until egos clashed, splitting the continent of Europe apart in a political and mathematical war over the *true* inventor of calculus.

Settle into a seat, get comfy, and listen to the tale of Newton, Leibniz, and the origins of calculus. It's the story of the beginning of science and the end of alchemy, a story full of interesting asides: Parliamentary shenanigans, rampant counterfeiting, Newton's sins, mathematical challenges, courtly intrigue, schemes for aristocratic enrichment, that time Newton stuck a needle in his eye, the Great Fire of London, and personal deception. And, somewhere along the way, the invention of calculus.

Homework: None.

Class format: Storytime

Prerequisites: None.

Toppling sandpiles. 🌶🌶 (Travis) TWΘ[FS] 50 minutes

It's beach day! Grab your swimsuit, sunglasses, sunscreen—and don't forget your sand pail and architect's hat, because we're gonna build cities of sandcastles and topple them to observe what happens when society collapses. But it'll be math!

There's something called the *abelian sandpile model* that physicist call a "simple model of self-organized criticality", but we're not physicists, so we'll call it "beach math". The idea is that we'll take a graph and build a sandpile on top of each vertex. But if a pile gets too big, it topples, spreading its sand to nearby vertices. Simple enough, but there's enough complexity that this model connects to probability and statistical physics, combinatorics, linear algebra, and even algebraic geometry. But what we'll focus on is that there's a *group* hiding somewhere here, and it's very good at disguising itself: Even once we define it, we won't know what the identity element is! (But we'll figure it out once we put on our private investigator monocles.)

So, in summary, this class is 1 part beach, 2 parts graph theory, 2 parts algebra, and 1 part undercover detective work.

NOTE: This class will not actually take place at the beach.

Homework: Optional.

Class format: Mostly groupwork

Prerequisites: None. You'll enjoy this class most if you know the definition of a group or are currently taking Intro to Group Theory, but it's not necessary. You can come to just play with graphs and sandpiles, too.

Special relativity. 🌶🌶 (*Nic Ford*) [TW]Θ[FS] 50 minutes

Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but if you took the equations literally they implied some bizarre things about the structure of space and time: depending on their relative velocities, different observers could disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

For a long time, a lot of creative excuses were invented for why we *shouldn't* take the equations literally (including one with the incredibly Victorian name "luminiferous aether") but, in what was probably the second most unsettling event in early twentieth-century physics, all of them failed. The physics community was left with only one viable conclusion: space and time really do behave that way!

In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics had to be rebuilt to accommodate them. We will see how, as the physicist Hermann Minkowski said, "Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." At the end, we'll also briefly look at how to revise the classical definitions of momentum and energy and see why we should believe that $E = mc^2$.

Homework: Optional.

Class format: Interactive lecture

Prerequisites: enough physics to know what momentum and kinetic energy are, but no more than that

Problem solving: inequalities. 🌶🌶🌶 (Misha) [TWΘFS] 80 minutes

Olympiad problem authors try to come up with problems that require a lot of cleverness but very little background. However, for inequalities, that often doesn't work out: they can be nearly impossible to solve if you don't know the necessary tool, and very easy if you do know the right inequality and have some practice using it.

On the first four days, we will cover how to use the following tools to solve olympiad inequality problems:

(1) The Cauchy–Schwarz inequality
(2) The AM-GM inequality
(3) Convexity and Jensen's inequality
(4) Rearrangement and majorization

On the last day of class, we will look at some inequalities outside olympiad math:

(5) Convexity in combinatorics

Almost all of these problems are about proof-writing, and the best way to practice proof-writing is to practice on your own. Homework for this class is not required, but I am happy to give feedback on your solutions if you write them up.

Homework: Recommended.

Class format: "Interactive problem-solving". Some time will be set aside for lecture, but more class time will be spent solving problems together, and I will also give you some time to work on problems on your own or in groups before going over a few of the more significant solutions at the end of class.

Prerequisites: None.

Intro to number theory. (Chloe) TW⊖FS 80 minutes

Explore the magic of the integers and make discoveries about them by gathering data, making conjectures (precisely worded guesses) about your observations, and proving your own conjectures! We will discuss topics relating to divisibility, greatest common divisors, modular arithmetic, and simultaneous congruences. If these topics are unfamiliar to you, then this is a great class for you! On the other hand, if you have seen these topics before it may not be the best fit for you. We will move fairly slowly through our topics and have a strong component of collaborative work.

A secondary focus of the course is to be an introduction to proof writing. We will illuminate the subtle complexity of facts we use everyday by considering axioms or basic facts we must take as true. From these surprisingly few and simple facts, we will see how you can build the elegant structure of elementary number theory.

I leave you with a compelling, although uninformative definition of number theory from Gauss: "Mathematics is the queen of sciences and number theory is the queen of mathematics."

Homework: Recommended.

Class format: IBL

Prerequisites: None.

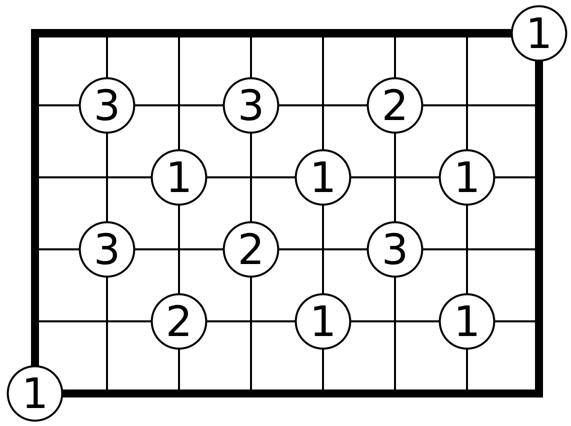
This class can continue as a project: I would be happy to run a reading project that extended a topic from this course. Taking this course is not required for the project.

This class may help prepare you for:

- **The circle method and Waring's problem** (Kevin, in Week 1) — Prerequisites: Comfort with calculus and number theory (e.g. modular arithmetic, arithmetic functions).
- **Public Key Cryptography** (Athina & Chloe, in Week 2) — Prerequisites: Some elementary number theory. Soft prereq: some coding
- **Continued Fractions and Pell's Equation** (Athina, in Week 3) — Prerequisites: Euclidean algorithm

- **Algorithms for Large Primes** (Zach, in Week 4) — Prerequisites: Modular arithmetic: should understand modular inverses and Fermat's Little Theorem. I plan *not* to assume or use any knowledge of abstract algebra.

Mathematical Concepts for Solving Puzzles. 🌶️🌶️🌶️ → 🌶️🌶️🌶️🌶️ (Della) TWΘFS 80 minutes
Here's a puzzle for you: shade some cells such that the unshaded cells are connected, and every number counts the shaded cells among the (up to) four cells around it.



This is a series of five independent one-day classes. Each day I will give you some puzzles similar to the one above, and to solve them you'll need to apply some lemma or think about them from a particular mathematical perspective. While the ideas involved are related to some deep math, the class will focus on the puzzles and we'll only talk about the math to the extent we need it.

Homework: Optional.

Class format: Mostly campers solving puzzles

Prerequisites: For the last two days, it will be helpful if you've seen basic graph theory.

This class can continue as a project: Write some of your own puzzles! You can do this without the class.

The most beautiful equation in math. 🌶️🌶️ (*Po-Shen Loh*) TWΘFS 50 minutes
What is e? What is π? What is i? What is 1? What is 0? Why radians? What do they all have to do with each other, and why? The last 10 minutes of this talk will use Calculus, but the rest will make sense to anyone who knows about logs that are not made of wood.

Homework: None.

Class format:

Prerequisites: None.

Colloquia

My favorite prime. (Zach) TWΘFS 50 minutes
Meet my favorite prime number! All twelve thousand(ish) digits of it.

Wait, that's a massive number. How do you know it's actually prime? My computer said so! *But how does your computer know?* By asking the Fibonacci numbers, or something like that. *How did you even* find *a prime so large, anyway?* Patience, mostly. *And why is this your favorite prime?* Because I'm its favorite person!

We'll provide more precise answers to the puzzles posed in the previous paragraph, and play with a plethora of (perhaps punny) prime-number possibilities proffered by the procedures that procured the prime in the pfigure.

The quest for atonal spheres. (Travis) T W ΘFS 50 minutes

In the early 20th century, composers went wild and burned the accumulated history of Western music theory in order to compose music in strange and wild new forms (needless to say, they did NOT have staff supervision). For some reason, this style never caught on in popular music, but it raged like wildfire in music theory academia. The music theorists, desperate to deconstruct and analyze these bizarre new creations, had to construct a brand-new type of theory to analyze music like this, and in doing so, they noticed a curious coincidence about collections of 6 notes. A music theorist with a PhD in mathematics and a grad student to pester was the first team to rigorously prove that this coincidence, now called Babbitt's Hexachord Thorem, always held. Mathematicians saw a theorem hiding in the land of music, inspected it with a magnifying glass, and spotted some clues that led them on a wild adventure, from circles to spheres to the liminal and primordial world metric probability spaces.

This is their story.

Guest colloquium. (*Po-Shen Loh*) TW$\boxed{\Theta}$FS 50 minutes

Impostor phenomenon. (Staff) TWΘ$\boxed{\text{F}}$S 50 minutes

Have you ever had the experience of being in a conversation about something — say, about maths — and thinking to yourself "Wow, all these other people are really great at this, and I'm just so-so . . . "? Or received a really nice compliment, and thought to yourself "Well, it's nice they said that, but they don't really know what they're talking about . . . "? In STEM fields, and in maths especially, it's very common to feel like an impostor: as if everybody else belongs, but you don't really. In this event, we'll share some ideas, stories, research, and discuss the experience of so-called "impostor syndrome."