CLASS DESCRIPTIONS—WEEK 5, MATHCAMP 2024

Contents

9:10 am classes

Philosophy and mathematics. (Athina) TWΘFS

Throughout history, philosophy and mathematics have been intertwined; the phrase "Let no one ignorant of geometry enter" was inscribed above the gate of Plato's Academy, René Descartes gave us "Cogito, ergo sum" as well as the Cartesian plane, and at beginning of the 20th century, mathematicians and philosophers banded together to address the so-called "crisis" in the foundations of mathematics. In this class, we will study some of that history, focusing on the various attempts to answer questions such as "is math invented or discovered?", "what is the nature of mathematical objects?", and "what do we mean when we say a mathematical statement is true?".

Homework: None.

Class format: Lecture and some discussion

Prerequisites: None.

Perfect numbers. (Mark) TWΘFS

Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called *perfect*, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes - a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Homework: None.

Class format: Interactive lecture

Prerequisites: None.

Medians, Convex Sets and Partitions. (Arya) TWΘFS

A cube complex is a complex built with cubes. An n-dimensional cube is a copy of $[0,1]^n$. So why, Arya, is this a 4 chilli class with no "cube" in the title?

Turns out you can do a lot of things with cubes. You can cut a cube into two even halves - something you cannot do with a simplex. A cube has geometry - for instance, all angles are right angles. A cube has faces - each of which is a cube of one dimension lower. Trust me, this class description and the class title are relevant to the class, I promise.

In this class, we will go on a journey through ultrafilters, hyperbolic geometry, medians and convexity. At some point, the fact that I'm allergic to cats will become relevant.

Homework: None.

Class format: Interactive lectures

Prerequisites: Having attended my Geometric Geometry class would help. You should know what a metric space is (and in particular, be familiar with the Taxicab metric on $\mathbb{R}^2$. Some familiarity with hyperbolic geometry may also help. Come chat with me.

n^{n-2} proofs of Cayley's tree theorem. (Zach) $\boxed{\text{TW}\Theta}$FS

Let's cut shapes into other shapes and assemble them into even more How many trees exist with vertices $\{1, 2, \ldots, n\}$? The (perhaps surprising) answer is n^{n-2}, and there are soooo many neat ways to show this, using a wide variety of techniques, including inclusion/exclusion, generating functions, random walks, bijections (in multiple ways), analysis, and more! I've collected different proofs of this theorem for many years, and I'll share as many of my favorites as time allows.

Homework: Recommended.

Class format: Interactive Lecture.

Prerequisites: None.

Problem solving: lecture theory. (Misha) TW$\Theta\boxed{\text{F}}$S

This class will teach you about the dark side of problem-solving: how to make educated guesses, how to use problem statements to your advantage, and how to exploit the one piece of extra information contest writers can't help giving you: that the problem has an answer.

(Due to its nature, this class is primarily focused on US contests like the AMC, AIME, and ARML, where you don't have to prove that your answers are correct.)

Homework: Optional.

Class format: Interactive lecture, with slides.

Prerequisites: None.

Fourier series. (*Alan Chang*) $\boxed{\text{TW}\Theta\text{F}}$S

Around 1800, the French mathematician Jean-Baptiste Joseph Fourier accompanied Napoleon through Egypt. Egypt was very hot, and Fourier became interested in heat, so he developed Fourier series to solve the differential equation known as the "heat equation." (This is a story I heard from Elias Stein, the mathematician who taught me Fourier analysis.)

The central idea of Fourier series is to decompose a periodic function into pure oscillations (i.e., sine waves):

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

This is what our ears do when we listen to music; it explains why the C-sharp of a piano sounds different from same C-sharp of a violin. (In class, we'll see this with some demonstrations using the software Audacity.)

Fourier analysis has wide applications to other areas, including signal processing (e.g., wireless communication), number theory (e.g., Dirichlet's theorem on primes in arithmetic progressions), quantum mechanics (e.g., the Heisenberg uncertainty principle), and Boolean functions.

In this class, we will learn how to find the Fourier series of any periodic function, prove some basic properties, and see how this can be used to solve differential equations.

Homework: Recommended.

Class format: Lecture.

Prerequisites: single variable calculus, know what a partial derivative is, some linear algebra might be useful but is not required.

Cantor's leaky tent. (Ben) $\boxed{\text{TW}\Theta}$FS

One of the notorious counterexamples in point-set topology is called "Cantor's Leaky Tent" or the "Knaster–Kuratowski Fan." This space is connected! But there's one particular point that, when removed, makes the space *totally disconnected.* In this class, we'll go over all of these terms, put up our tent, and prove that it does exactly what it's supposed to.

Homework: Recommended.

Class format: Interactive Lecture.

Prerequisites: Baire's Category Theorem, having seen metric spaces before. Helpful to have seen the Cantor set, but not *strictly* necessary.

10:10 am classes

Ultrafilters and voting. (Krishan) T$\boxed{\text{W}}$ΘFS

Imagine you and your friends are trying to decide where to go for dinner. You all have your own personal ranking of the options but somehow you need to combine your individual rankings into a group ranking. If you were hoping that math could help you with this problem then you're out of luck!

It turns out that there is no "fair" way to solve this type problem. This result is known as Arrow's Impossibility Theorem. In this class we will formulate the theorem precisely and will sketch a proof. Surprisingly this theorem about finite objects it proven using ultrafilters (an abstract infinite typically used in set theory).

Homework: None.

Class format: Interactive lecture.

Prerequisites: None.

Monoids in the category of endofunctors. (Della) TW$\boxed{\Theta}$FS

You might sometimes hear category theory nerds say gibberish phrases like that. But what do they mean? In this class I'll, speedrun category theory definitions until we can make sense of "monoid in the category of endofunctors". If there's time, we'll see what that has to do with burritos.

Homework: None.

Class format: Lecture

Prerequisites: None.

MCSP: Mermaids Contained in Swimming Pools (Gershgorin Circle Theorem). (Kailee) TWΘ$\boxed{\text{F}}$S

Who are eigenvalues? Where do they live? Come learn about Gershgorin's circle theorem to find out! (Spoiler Alert: We may not have their exact address, but just looking at each row of our matrix, we can say something about where we might find our eigenvalues hanging out.)

Homework: None.

Class format: Interactive lecture.

Prerequisites: basic linear algebra, specifically eigenvalues

Fermat is False in Finite Fields (FFFF). (Travis) T$\boxed{\text{W}}$ΘFS

Fermat's Last Theorem says that there are no solutions to the equation

$$a^n + b^n = c^n$$

if $n \geq 3$ and a, b, c are positive integers. It's pretty hard to prove. On the other hand, Fermat's Last Theorem for Finite Fields says that

$$a^n + b^n \equiv c^n \pmod p$$

has no solutions if $n \geq 3$ and a, b, c are not divisible by p. Except this is FALSE.

In this class, we'll use graph theory to prove a result in number theory called Schur's Theorem, and we'll use that (plus a bit of general number theory knowledge) to show that Fermat's Last Theorem is not only false in finite fields, it's *very* false.

NOTE: If you took my graph limits class, you saw a "proof" of a theorem in number theory using a result from graph theory. Unfortunately, we only had time to sketch the proof, since a lot of the intermediate results are hard to prove. In this class, we'll get to go through a *complete* proof in this same style!

Homework: Optional.

Class format: Lecture.

Prerequisites: It will help if you're familiar enough with $\mathbb{Z}/p\mathbb{Z}$ to know that every nonzero element has a multiplicative inverse, but that's not strictly necessary.

4 dimensions is easy. (Travis) TW$\boxed{\Theta}$FS

Among all arrangements of n points in the plane, what is the maximum possible number of pairs of points that are distance 1 apart? This is called the *unit distance problem*, and it's famous (and famously hard). But why stop at the plane? Why not ask the same question in 3 dimensions? (This problem is less famous, and still hard.) Or even 4 dimensions? (This problem is not famous, and is very easy.)

We'll go through this problem and see what makes it so different in 4 dimensions than in 2 or 3 (spoiler: it's because $4 = 2 + 2$) and use that to gain some insight into 4-dimensional space.

Homework: None.

Class format: Interactive lecture.

Prerequisites: Know how to measure distances in $\mathbb{R}^3$. It will help (though it's not strictly necessary) to know what a subspace of $\mathbb{R}^n$ is.

Slaying the hydra. (Della) TWΘ$\boxed{\text{F}}$S

When Hercules fought the hydra, he had his nephew cauterize the wound after cutting off each head, to prevent two more heads from growing back. We're going to be fighting graph-theoretic hydras, where that kind of cheating isn't an option. Can we still win?

Homework: Optional.

Class format: Lecture.

Prerequisites: None.

The most depressing theorem I know. (Mira) T$\boxed{\text{W}}$ΘFS

Say there is a human trait that (e.g. the probability of being in a car accident) on which two groups (e.g. men and women) differ.

Say also that there is a cost or benefit associated with this trait. In our example, if your probability of being in an accident is low, then your car insurance might give you a discount — that's the benefit.

The most depressing theorem I know (Kleinberg et al 2016) says that any algorithm for predicting such a trait will be either biased or unfair toward one of the groups (unless it's 100% accurate, which, realistically, will never happen). I'll explain in the class precisely what I mean by "biased" and "unfair" (they mean very different things in this context), but the upshot is: in a world that starts out unequal, there are essentially no fair algorithms. This is not because the people who write these algorithms are sexist or racist or careless (though that may be true too), but because perfect fairness is just mathematically impossible.

I think this theorem is not just depressing, but also very important: it helps cut through the noise in a lot of political debates where people seem to be talking past each other. Surprisingly, it's not very well known. Let's change that!

Homework: None.

Class format: Interactive lecture.

Prerequisites: None.

Democracy can always be gamed. (Mira) TW$\boxed{\Theta}$FS

> *No one pretends that democracy is perfect or all-wise. Indeed it has been said that democracy is the worst form of Government except for all those other forms that have been tried from time to time...* – Winston Churchill

You may have heard of Arrow's Theorem: it says that if you want your voting system to satisfy certain reasonable-sounding conditions, then your only option is a dictatorship. But this class is *not* about Arrow's Theorem, because Arrow's Theorem is not depressing enough: its definition of a voting system is so restrictive that it barely ever applies in practice.

The Gibbard-Satterthwaite theorem is less famous, but I think it's much more depressing. It says that if a voting system satisfies two very simple criteria,

(a) if candidate A is preferred by all the voters then A wins;
(b) the system is not a dictatorship,

then this system is vulnerable to strategic voting whenever there are more than two candidates. In other words, there is at least one voter who can obtain better results by voting dishonestly than by voting honestly. Democracy can *always* be gamed. In this class, you won't necessarily learn how, but you'll learn why.

Homework: Recommended.

Class format: Interactive lecture.

Prerequisites: None.

The politics of rounding fractions. 🌶 (Mira) TWΘ$\boxed{\text{F}}$S

> *God help the state of Maine when Mathematicks reach for her and undertake to strike her down!*
>
> – Representative Littlefield (R, ME), 1901

The US Constitution mandates that "representatives ... shall be apportioned among the several states ... according to their respective numbers". This is usually taken to mean that the number of representatives in each state should be proportional to its population. But exact proportionality is not possible: for example, California cannot have 54.37 representatives. The same issue arises in countries where seats in parliament are apportioned to parties based on the percentage of votes each party received. Once again, what do you do with fractions of representatives?

This is the *problem of apportionment*, and it's a lot trickier and more interesting than might appear at first glance. Over the course of US history, Congress went through five different apportionment methods, always accompanied by fierce political debates. The method that we currently use was proposed in 1921 by a Harvard mathematician (!), and its was adopted by Congress on the recommendation of the US Academy of Sciences (!!). As far as I know, the US is the only country in the world that uses a method of apportionment that was derived by a mathematician from first principles!

Homework: Optional.

Class format: Interactive lecture.

Prerequisites: None.

A tour of Hensel's world. 🌶🌶 (Mark) T$\boxed{\text{W}}$ΘFS

In one of Euler's less celebrated papers, he started with the formula for the sum of a geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and substituted 2 for x to arrive at the apparently nonsensical formula

$$1 + 2 + 4 + 8 + \cdots = -1.$$

More than a hundred years later, Hensel described a number system in which this formula is perfectly correct. That system and its relatives (for each of which 2 is replaced by a different prime number p), the p-adic numbers, are important in modern mathematics; we'll take a quick look around this strange "world".

Homework: None.

Class format: Interactive lecture.

Prerequisites: Some experience with the idea of convergent series.

The fastest algorithm nobody uses. 🌶🌶🌶 (*Shimon Schlessinger*) TW$\boxed{\Theta}$FS

Computer scientists spend a great deal of time and effort searching for faster algorithms to solve computational problems. Some of the most pressing open questions in the field —like the quickest way to factor numbers, plan routes, play games like chess, and the infamous P vs. NP problem— all boil down to one fundamental question: "What is the quickest way for a computer to solve this problem?" However, this is not actually the right way to think about these questions. In fact, what if I told you that we already know the fastest algorithm for each of these problems? This class will introduce my favorite computer algorithm, which also happens to be the fastest solution to *every* computational question! If you don't believe me (and you probably shouldn't!), come to this class to find out how it works!

Homework: None.

Class format: Lecture.

Prerequisites: None — some sense of how computers work/programming experience is nice but definitely not required.

Statistical mechanics. (*Max Misterka*) TWΘFS

If you have taken high school physics, then you've probably studied the behavior of physical systems with a small number of rigid objects. But what if we want to study much larger systems like the air in a room, which contains around 10^{27} molecules? This system has so many particles that even the most powerful computers in existence today cannot fully simulate it! So physicists had to get clever and invent a new set of tools called statistical mechanics to deal with huge systems like this.

Statistical mechanics is a field of physics that applies the powerful tools of probability and statistics to systems that contain many particles so that we can understand their behavior without keeping track of the locations of every particle at once. The first half of this one-day class will introduce you to several of the most important ideas of statistical mechanics and will build up enough mathematical machinery to rigorously prove results such as the fact that heat flows from higher temperature to lower temperature. The second half of the class will focus on a specific physical system called the two-state system and will derive some *very* counterintuitive facts about temperature.

There will likely be an unofficial second day of this class posted on the scheduleboard in Week 5, where we will go deeper into statistical mechanics at a 4 chili pace. We will derive even more cool results, possibly including the exact distribution of speeds of the molecules in a gas or (if we have time) the existence of a fifth state of matter called a Bose–Einstein condensate!

Homework: Recommended.

Class format: Interactive lecture.

Prerequisites: Familiarity with probability and partial derivatives. Basic physics is helpful but not required.

11:10 AM CLASSES

The magic of determinants. (Mark) TWΘFS

This year's linear algebra class barely touched on determinants in general. If that left you feeling dissatisfied, either about not really having a good definition (by the way, using the Laplace expansion, while it is computationally often handy, leads to a miserable definition — with no intuitive basis at all) or about not having seen many of the properties that determinants have, this may be a good class for you. If all goes well, we'll give a definition of determinant that's both motivated and rigorous, and there will be proofs of all its main properties (such as Laplace expansion), as well as a few applications such as general formulas for the inverse of a matrix and for the solution of n linear equations in n unknowns ("Cramer's Rule"). Prerequisites: Some linear algebra, including linear transformations, matrix multiplication, and determinants of 2×2 matrices.

Homework: Optional.

Class format: Interactive lecture.

Prerequisites: Some linear algebra, including linear transformations, matrix multiplication, and determinants of 2×2 matrices.

The Cayley-Hamilton theorem. (Mark) TWΘFS

Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X{*}4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Homework: None.

Class format: Interactive lecture.

Prerequisites: Linear algebra, including a solid grasp of determinants (the "Magic of determinants" class would definitely take care of that).

Intro to knot theory (and to Alyona's undergrad thesis). (Alyona) TWΘFS

Why knot learn more about knots? Especially when you canknot find a punnier subject! In this class, we will explore the world of knots and try to figure (8) out how we can tell them apart. Knot theory is still a dynamic branch of mathematics, so in our quest to distinguish all knots, some of our attempts may be trefoiled. But nonetheless, we will valiantly try to distinguish knots using things like Reidemeister moves, as well as numerical, color, and polynomial invariants! *this description is shamelessly stolen from Kayla Wright's 2019 class blurb, because I really like it.*

On the last day, we'll also talk about braid group, its representations, R-matrices, and what it all has to do with knot invariants. (Aka secretly an overview of Alyona's undergraduate thesis)

Homework: Optional.

Class format: Interactive lecture.

Prerequisites: Intro to linear algebra for some parts.

Unlikely maths. (Misha) TWΘFS

A popular way to "construct" a combinatorial object we want to see is to instead give up and choose it at random. This works great if the properties we want from it actually hold for almost any objects we choose. But sometimes we are greedy and want so much from our construction that a randomly chosen object has, at best, an exponentially small chance of making us happy.

This class is about the Lovász Local Lemma: one of the ways to prove that this exponentially small chance is still positive. (If you've taken Kailee's class in week 1, you may have already seen the LLL in action.) In this class, we will see a few unusual uses of the LLL, develop some intuition for why its statement is plausible, and if time permits, look into how to take the last step: actually finding the unlikely object we wanted.

Homework: Optional.

Class format: Interactive lecture.

Prerequisites: You should be acquainted with a variety of concepts from graph theory: vertex degrees, paths, trees, cliques, and subgraphs. You should be comfortable with expected value and conditional probability.

Foliation theory. (*Katie Mann*) TWΘFS

A foliation is. . . roughly. . . a way to divide a space up nicely into a collection of spaces of lower dimension. High dimensional spaces are hard to understand and visualize, so you might hope that slicing them up into lower dimensional ones solves the problem. But, alas, describing all the possible ways to do this is a whole field of math in itself. In this class I'll talk about classifying and distinguishing 1-dimensional foliations of 2-dimensional surfaces (with your help) and then I will bravely attempt to explain what I am doing in my research right now which circles around foliations of the plane...

Homework: Recommended.

Class format: Lecture, with some short activities.

Prerequisites: None.

The seven circles theorem. 🌶🌶 (Zach) TWΘ$\boxed{\text{F}}$S

We'll show two proofs of a beautiful fact in Euclidean geometry about six tangent circles enclosed within a 7th circle. The first proof relies primarily on TPS from this year, while the second involves a whirlwind tour through Hyperbolic Geometry, where we will discuss equivalences between two different representations of hyperbolic geometry in the unit disk, as well as try to make sense of the perimeter of infinite hyperbolic polygons.

Homework: None.

Class format: Lecture.

Prerequisites: Knowledge of inversion will be helpful, but not strictly necessary.

Proof that the universe has a beginning. 🌶🌶🌶🌶 (Laithy) $\boxed{\text{TWΘF}}$S

Astronomical evidence indicates that the universe can be modeled (in smoothed, average form) as a spacetime containing a perfect fluid whose "molecules" are the galaxies. It becomes possible to build quite simple cosmological models whose properties have a reasonable chance of being physically realistic. At the very least, such models provide a testing ground for discovering properties of our universe.

We will study the most famous cosmological model, the Robertson-Walker spacetime. Given a function f on $I \times \mathbb{R}^3$, where I is an interval, the Robertson-Walker metric is $-dt^2 + f(t)^2 g$, where g is a metric on $\mathbb{R}^3$ of "constant curvature" $k = -1, 0$ or 1 (these are the hyperbolic space, flat space, and the 3-d sphere, respectively).

Astronomical data from the Hubble telescope suggests some restrictions on f that lead to the following deep theorems about our universe:

(1) There exists an initial endpoint t_* such that $f \to 0$ and $f' \to \infty$, called the big bang.

(2) If $k = 0, -1$, then $I = (t_*, \infty)$ and as $t \to \infty$, $f \to \infty$. (The universe is expanding indefinitely)

(3) If $k = 1$, then f reaches a maximum followed by a point t^* in which $f \to 0$ and $f' \to -\infty$, called a big crunch. Hence $I = (t_*, t^*)$ and so the universe eventually collapses.

These theorems predict that our universe begins in a colossal explosion and will either expand indefinitely or collapse in finite time. We will then prove a sufficient condition on the matter density that can potentially be confirmed with more astronomical data, which will determine whether or not the universe has an end.

Homework: Recommended.

Class format: Lecture.

Prerequisites: Week 4 Black Holes class.

1:10 PM CLASSES

Grammatical group generation. 🌶️ (Eric) TWΘFS
Do you like silly word games? Normal subgroups and presentations of groups got you down? Come to this extremely light-hearted romp through the world of grammatically generated groups! In this class, based on a real actual published math paper, we will use group theory to understand how many homophones and anagrams the English language has. If you think this sounds silly, it's because it is silly. But we'll do it anyways. Be prepared for terrible jokes and words you will never see used in any other context.

Homework: Optional.

Class format: 50/50 lecture and worksheets.

Prerequisites: Group theory, at the level of having seen definitions of normal subgroups and quotients.

Elliptic curves with complex multiplication. 🌶️🌶️🌶️ (Chloe) TWΘFS
The goal of this course is to think about endomorphisms of elliptic curves. We'll start by introducing elliptic curves as C mod a lattice, justifying this connection, and defining an endomorphism of an elliptic curve. Next we'll prove that every elliptic curve has the 'multiplication-by-n' map for all integers n and see examples of this kind of endomorphism. After that we will explore some examples that are not multiplication-by-n and define complex multiplication. To finish the course we'll prove (with some details left out) that an endomorphism of an elliptic curve is multiplication by a number with a complex component if it is not an integer (justifying the name complex multiplication).

Homework: Optional.

Class format: Interactive lecture.

Prerequisites: Some familiarity with elliptic curves, especially the definition of points of finite order. Definitions of groups, rings, fields, and homomorphisms.

Probabilistic programming: human intelligence as computation (hour 1 of 2). 🌶️🌶️ (*Vikash Mansinghka & Josh Tenenbaum*) TWΘFS
This class meets for two hours: 1:10–3 pm
Can a machine be programmed to think the way the human mind does? Is it possible to explain human intelligence as a kind of computation, and to model brains as a kind of computer? If so, how? This class will present a way to answer these questions, and give you hands-on experience with a new kind of programming approach for modeling human intelligence that we use in our classes and research groups at MIT.

We'll introduce the concepts, mathematics and techniques of *probabilistic programming*, a field at the intersection of computer science, probability theory and data science, which also draws on and informs computational cognitive science and neuroscience. The key idea is to think of intelligence as a rational framework for making good guesses and good bets about the world — that's where the "probabilistic" part comes in — using symbolic programs that can adaptively simulate how the world works, as well as the processes you use to sense the world and act in it. Probabilistic programming is a set of tools for building, learning and using these *generative world models* (or what a cognitive scientist calls a "mental model") to efficiently and accurately extrapolate the parts of the world you haven't observed based on the parts that you have. It rests on some elegant mathematics as well as some pretty cool programming language ideas and software engineering, which we'll touch on. It is being used by robotics and data science researchers to make their systems smarter in more human-like ways, and it is being used to build the first quantitative models of human intelligence that accurately predict behavior and also the structure and dynamics of how neural circuits work in the brain.

In this class we will introduce you to the probabilistic programming language Gen, and examples drawn from time-series analysis (how can a program predict the future of any function that a person can?), 3D computer vision (how can a machine see the shape of objects and scenes in three dimensions, given only a sequence of two-dimensional frames in a movie?), robotic navigation (how can an agent figure out where it is in a complex spatial environment, based only on sparse perceptual data like a robot's directional depth sensor or a mouse's whiskers?), and conversational AI (how can a machine answer questions that are based on rational inferences about the world, rather than just the patterns of what people say?).

The class meets over two days, with two sessions on each day. The second session on each day is **optional** and will focus on programming examples that we guide you through. If you want to participate in the programming part of the class, it would be helpful to bring a laptop. But it's okay if you don't have one; come anyway and we'll make sure you can use ours or others. You can come to just the first hour of each day, or both hours each day. You can also just come to the first hour of the first day, if you want to get a taste for what this field is about don't want to commit to more yet.

Homework: Optional.

Class format: Interactive lecture (hour 1), hands-on exercises (hour 2).

Prerequisites: Familiarity with coding in Python or a similar language, and a little calculus (the concept of a derivative and integral). Experience with probability and Bayesian inference is valuable but you'll learn what you need in this class.

Counting solutions to equations mod p. 🌶🌶🌶 (Kevin) TW$\boxed{\Theta\text{F}}$S

Given a polynomial equation or system of polynomial equations with integer coefficients, how many solutions are there mod p? One of the greatest achievements of twentieth century mathematics was the formulation and proof of the Weil conjectures, which describe these "point counts" in terms of the topology of the underlying equations. While the Weil conjectures for general systems of equations are way beyond the scope of Mathcamp, we'll see what the Weil conjectures say and count some points in some specific examples, like the case of Fermat curves $x^d + y^d \equiv z^d \pmod{p}$.

Homework: Optional.

Class format: Lecture.

Prerequisites: Modular arithmetic is required. Knowing about projective space or deeper topics in modular arithmetic (e.g. primitive roots, Gauss sums) is helpful but not required.

Exploring the Catalan numbers. 🌶🌶 (Mark) $\boxed{\text{T}}$WΘFS

What's the next number in the sequence $1, 2, 5, 14, \ldots$? If this were an "intelligence test" for middle or high schoolers, the answer might be 41; that's the number that continues the pattern in which every number is one less than three times the previous number. If the sequence gives the answer to some combinatorial question, though, the answer is more likely to be 42. We'll look at a few questions that do give rise to this sequence (with 42), and we'll see that the sequence is given by an elegant formula, for which we'll see a lovely combinatorial proof.

Homework: None.

Class format: Interactive lecture.

Prerequisites: None.

Wizards in hats. 🌶🌶🌶 → 🌶🌶🌶 (Della) T$\boxed{\text{W}}$ΘFS

Ten wizards are each wearing a black or white hat, and can see everyone's hat except their own. One at a time, they guess their hat's color. How can they ensure all but one wizard is right?

Now suppose instead there are (countably) infinitely many wizards, and they have to guess at the same time (so their guess can't depend on what other wizards say). But I'll be generous: instead of only allowing one wizard to be wrong, you can have any finite number of wrong guesses. Is this possible?

In this class, I'll pose a variety of puzzles similar to this one, and see connections to topics like the axiom of choice, the continuum hypothesis, and linear algebra.

Homework: Recommended.

Class format: Lecture.

Prerequisites: Ordinal numbers, axiom of choice.

Colorful puzzles & Dehn functions. 🌶🌶 (Sonya) TW$\boxed{\text{ΘF}}$S

Consider the following puzzle: You have a bunch of puzzle pieces that are flexible polygons with colorful edges and a big polygonal frame also with colorful edges. You want to fill in the frame with puzzle pieces so that only edges that touch have the same color. For a given configuration of puzzle pieces and frame, what is the minimal number of pieces necessary to solve the puzzle? How can we figure out if a solution exists at all? Take this class to find out how this is a visual representation of Dehn's function. We will compute Dehn's function for a few groups, prove some facts about its behavior, and draw a connection to manifolds.

Homework: Optional.

Class format: Interactive lecture.

Prerequisites: Knowing what a group is.

Calculus without calculus. 🌶🌶 (Tim!) $\boxed{\text{TW}}$ΘFS

If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Phoebe is 5 cubits tall and Laithy is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Phoebe's head to the top of Laithy's head that touches the ground in the middle. What is the shortest length of string you can use?
- Athina rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along a very straight section of the Puget Sound shoreline. The dog's person stands 20 meters away along the shoreline, and throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- If you went to Laboba to get boba or to Metropolitan Market to get The Cookie, you may have also stopped by the movie theater. Which seat should you have chosen so to make the screen take up the largest angle of your vision?

- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Recommended.

Class format: Interactive lecture.

Prerequisites: We won't use calculus (that's the point), but it would be good if you've seen it for context.

The finite field Kakeya conjecture. (Narmada) TW$\boxed{\Theta\text{F}}$S

The Kakeya conjecture over $\mathbb{R}$ is a longstanding open problem about the fractal dimension of Kakeya sets. For a long time, people assumed that the Kakeya conjecture over finite fields would be equally hard to prove. In 2008, Zeev Dvir shocked these people by proving the finite field Kakeya conjecture in just one simple paragraph. In doing so, he opened the floodgates to a whole new technique in mathematics: polynomial methods in combinatorics.

In this class, we'll see Dvir's original proof of the Kakeya conjecture and look at some other applications of the polynomial method.

Homework: Recommended.

Class format: Lecture + group work.

Prerequisites: Know that $\mathbb{Z}_p$ is a field

The Eras Tour: Blank Space. (Glenn, Chloe, & Jennifer) $\boxed{\text{T}}$WΘFS

It's Day 1 of the Eras Tour, so get ready for an action-packed programming puzzle adventure about Taylor Swift! In *Blank Space*, you will use the esoteric programming language Whitespace to craft tools in order to save Taylor Swift after being kidnapped by Scooter Braun (if you're unfamiliar with the IRL drama, he's the person who bought the rights to Taylor's music and caused her to start releasing Taylor's Versions). Whitespace is a language where the only meaningful characters are tab, space, and new line—everything else is completely uninterpreted. A programming environment custom-built for this class will hopefully ease some of the pains, but can you figure out all the techniques and tools needed to save Taylor? She's waiting for you!

Here's a sample of the flavor text from stage 1:

Taylor Swift is on her Eras Tour, and she's making a surprise stop in Tacoma just in time for Mathcamp! The crowd goes wild as she sings, "Got a long list of ex-lovers, they'll tell—"

All of a sudden, the lights go out. Scooter Braun appears on stage, and proclaims, "There is nothing I do better than revenge... mwa ha ha ha!" He kidnapped her!

The venue spontaneously combusts into nothing but a sea of blank space, fog as far as the eye can see. And it's your job to save Taylor from her predicament!

Homework: None.

Class format: Programming in the computer lab.

Prerequisites: None! (no prior programming experience needed)

The Eras Tour: Begin Again. (Glenn & Jennifer) T$\boxed{\text{W}}$ΘFS

In Day 2 of the Eras Tour, Taylor is back on stage, after you saved her from the kidnapping in Day 1. The crowd goes wild as she sings, "But on a Wednesday, in a cafe, I watched it—"

The lights go out again, and when they come back, the stage is set for *Miss Americana and the Heartbreak Prince*. We're back to the start of the show again, oh no! It seems like Scooter Braun surreptitiously messed with the code controlling the stage, and now we're stuck in an infinite loop!

There are two main kinds of loops in programming - the for loop and the while loop. The for loop is safe because you can't create infinite loops with it, and although the while loop may be convenient, the vast majority of problems in most programming courses can be completed without any while loops. It seems like Scooter used the infinite power of the while loop to mess with Taylor's show again! Can we take this power away from him?

Sadly, it turns out the answer is no. There are problems that provably require a while loop to compute, and we'll see some of them in this course. We'll need to find a different way to defeat Scooter this time!

Homework: None.

Class format: Lecture.

Prerequisites: Some basic programming (in any language) will help contextualize the class, but not strictly necessary.

The Eras Tour: Love Story. (Glenn & Jennifer) TW$\boxed{\Theta}$FS

Day 3 of the Eras Tour, and look who decided to show up — Travis Kelce! What a supportive boyfriend. Taylor looks Travis in the eye and sings, "It's a love story, baby just say—." She point the microphone at Travis, beckoning him to finish the lyric.

But Scooter Braun is at it again, whisking Taylor and Travis away before Travis has a chance to say yes! In his cruel game, Taylor and Travis have to stay very far apart and avoid communication, and they can only see each other again if they win. Scooter designs the game so that he's certain they can win no more than 75% of the time. But through the power of love and connection (i.e. quantum entanglement), Taylor and Travis realize they can actually win 85% of the time! In this class, we'll unravel this love story and explain how they did it.

Homework: None.

Class format: Lecture.

Prerequisites: Very basic linear algebra (multiplying matrices, visualizing vectors).

The Eras Tour: All Too Well (50 Minute Version). (Glenn & Jennifer) TW$\Theta\boxed{\text{F}}$S

It's the final act of the Eras Tour, and Scooter Braun is finally defeated. All that's left is to sing *All Too Well (10 Minute Version)*. But this is a special Eras Tour stop, in Tacoma, just for Mathcamp! So it's time for a very special rendition of *All Too Well*, with some more mathematical content.

The instructors of this course will produce the lecture entirely in song, in several parodies of *All Too Well*. The topics will be about mathematical coincidences, in other words, times when two numbers line up all too well to be mere chance. Put on your tinfoil hats, and then take them off again, because it turns out that several of these coincidences have mathematical reasons for existing! Find out why $\pi^2 = 9.86$ is so close to $g = 9.81$ (the gravitational constant near the surface of the earth) and more at this special concert.

Disclaimer: We will try to write as many lyrics as possible, but no guarantees that we'll be able to pull of of lyrics by Friday. We promise at least 10 minutes, and hopefully significantly more than that.

Homework: None.

Class format: Lecture, but completely in song.

Prerequisites: None.

2:10 pm classes

Determined to Determinant: Leibniz Edition. 🌶 (Kailee) $\boxed{\text{T}}$WΘFS

Are you sick and tired of calculating determinants by expanding along a row or column and then calculating *more* determinants? Would you rather just play around with graphs and permutations? Come learn about my favorite way to think about the determinant and see why Leibniz's formula will forever live rent-free in my head.

Homework: None.

Class format: Interactive lecture/group work.

Prerequisites: Permutations, calculating determinants.

Determined to Determinant: Perfect Matching Edition. 🌶 (Kailee) T$\boxed{\text{W}}$ΘFS

Are you sick and tired of looking at graphs and trying to see if they have a perfect matching? Look no further! With a little determination, we can find all perfect matchings in a graph just by taking a determinant! We'll see how to build a special kind of adjacency(ish) matrix such that taking its determinant lists out all perfect matchings in our graph.

Homework: Recommended.

Class format: Permutations, Leibniz formula for determinant, basic graph theory, familiarity with matchings in graphs helpful but not required.

Prerequisites: Interactive lecture/group work.

18446744069414584321. 🌶🌶🌶 (Eric) TW$\boxed{\Theta}$FS

This is a follow-up to my week 3 class "How to multiply numbers realllllly fast". 18446744069414584321 is a prime which is particularly useful in some practical multiplication problems. In this class we'll learn about what makes this prime special, how you can do computations with it efficiently on a computer, how it's relevant for Fast Fourier Transforms, and what other similar primes are out there in the cosmos.

Homework: Recommended.

Class format: Interactive lecture.

Prerequisites: Mild familiarity with the (Cooley-Tukey) Fast Fourier Transform algorithm, for example as in my week 3 class.

Burnside's Lemma. 🌶🌶🌶 (*William Burnside*) TWΘ$\boxed{\text{F}}$S

How many different necklaces can you make with six beads, if you have a large number of black and white beads? The answer is not 2^6, because if two necklaces are the same up to rotation or reflection, we count them as the same necklace.

You can probably answer this question just by carefully enumerating all the possibilities. But what if we are making necklaces with 20 beads and we have beads of 8 different colors? Fortunately there is a nifty little result in group theory that allows us to solve this type of problem very quickly and without a calculator. Come improve your counting skills!

Homework: None.

Class format: Interactive lecture.

Prerequisites: Basic group theory.

Kursed Counterexamples. 🌶🌶 (Kevin, Krishan, & Zach) TW$\boxed{\Theta}$FS

Most classes here at Mathcamp focus on the beautiful side of math. This is not one of those classes. In this class members of the staff will expose you to some of the strangest, ugliest, and most badly behaved mathematical objects you'll ever see. We'll cover kursed counterexamples from across mathematics with varying levels of spiciness. Come watch if you dare!

Homework: None.

Class format: Interactive lecture.

Prerequisites: None.

Fractal Dimensions. ($\log_2$ 🌶🌶🌶) (*Hermann Minkowski*) TW$\Theta\boxed{\text{F}}$S

A line has dimension 1, a plane has dimension 2, and the space we live in has dimension 3. Can you think of something of dimension 1.5? What does it mean to have dimension 1.5? Actually, what does it even mean to have dimension 2?

In this class, we'll discuss one possible definition of dimension and we will compute the dimension of a few objects, including some with non-integer dimensions.

Homework: None.

Class format: Interactive lecture.

Prerequisites: You need to be comfortable with logarithms and limits.

The cyclic polytope. 🌶🌶 (Sonya) $\boxed{\text{TW}}\Theta$FS

What is the maximum number of faces for a polyhedron with 13 vertices? And is there a better way of constructing an example than just moving vertices around until you get the desired result? According to a key result in polyhedral combinatorics, both questions have really satisfying answers (and not just in our regular 3-dimensional space). Join this class for an introduction to the study of polytopes leading up to the upper bound theorem. Or just to draw nice pictures of polyhedra.

Homework: Optional.

Class format: Interactive lecture.

Prerequisites: None.

Problem solving: tetrahedra. 🌶🌶🌶🌶 (Misha) TW$\boxed{\Theta}$FS

In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

Homework: None.

Class format: Interactive problem-solving.

Prerequisites: Comfort with the basic notions of Euclidean geometry; if you've taken any kind of geometry class in school, you should be fine. It is *very* important to know what a tetrahedron is.

The mathematics of polygamy (and bankruptcy). 🌶 (*Rabbi Judah ha-Nasi*) TW$\Theta\boxed{\text{F}}$S

Here is a passage from the *Mishnah*, the 2nd century codex of Jewish law (with modern formatting):

A man with three wives dies. The first wife, according to her prenuptial agreement, is owed 100 [dinar], the second is owed 200, and the third is owed 300.

- *If the entire estate is only 100, they split it equally.*
- *If the estate is 200, the first wife gets 50 and the other two get 75 each.*
- *If the estate is 300, the first wife gets 50, the second wife gets 100, and the third wife gets 150.*

Similarly, three people who made a joint investment which then made a loss or a profit — this is how they should split the money. (Ketubot 93a)

For 1800 years, this passage had baffled scholars: what could possibly be the logic behind the Mishnah's totally different ways of distributing the estate in the three cases? Then, in 1985, a pair of mathematical economists (and future Nobel laureates) produced a beautifully simple explanation based on ideas from game theory. They showed that for any number of creditors and for any estate size, there is a unique distribution that satisfies certain criteria, and it turns out to be exactly the distribution proposed in the Mishnah. The proof is very cool, based on an analogy with a simple physical system! See if you can figure out this ancient puzzle for yourself, or come to class and find out.

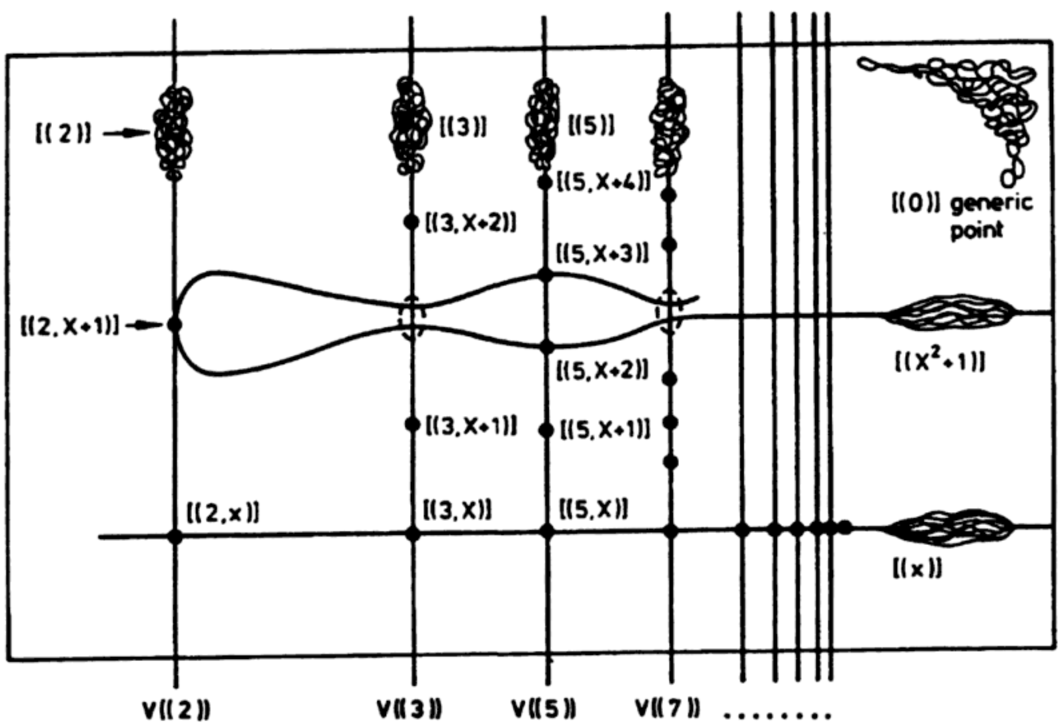
Homework: None.

Class format: Interactive Lecture.

Prerequisites: None.

Spec of a ring. (Kevin) TWΘFS

Given a ring R, one can associate to it a topological space $\mathrm{Spec}(R)$ whose points are the prime ideals of R. In this class, we'll talk about why this is a reasonable space to do algebraic geometry on by discussing a bunch of examples, and I'll hopefully get to draw a bunch of cool pictures like this one:



Homework: Optional.

Class format: Lecture (maybe a bit of groupwork).

Prerequisites: Ring theory (prime and maximal ideals, homomorphisms) is required. Some exposure to point-set topology (e.g. what is a topological space? basis? continuous function?) is strongly recommended. Also, Mark's class on algebraic geometry would be helpful but is not required.

The Putnam. (Mark) TW$\boxed{\Theta}$FS

You may well have heard of it, but maybe not by its formal name: The William Lowell Putnam Mathematical Competition. As you may know, this is a challenging annual 6-hour exam for undergraduates in the U.S. and Canada; more often than not, the median score is under 10 (out of a possible 120 points). At various times before I retired, I was involved with the Putnam, as a problem setter and later in a more administrative capacity. In this class (if it runs) I intend to talk some about the history and "culture" of the Putnam, tell some stories about it, answer any questions you may have, and try to dissuade you from taking it while you are still in high school (which is possible), with one exception.

Homework: None.

Class format: Interactive lecture; Q and A.

Prerequisites: None.

A game you can't play (but would win if you could). (*Ernst Zermelo*) TW$\Theta\boxed{\text{F}}$S

Once upon an infinity, in the Kingdom of Aleph, King Alephonso decides to put his 100 advisors to a test. He has 100 identical rooms constructed in his palace. In each room the king places an infinite sequence of boxes; in each box he puts a real number. The sequence of numbers is exactly the same in each room, but otherwise completely arbitrary.

The king tells his advisors that when they are ready, each of them will be locked in one of the 100 rooms. Each of them will be allowed to open all but one of the boxes in the room. (This will, of course, take an infinite amount of time, but in the Kingdom of Aleph, they're pretty cavalier about infinity.) Finally, each advisor will be required to name the number in the box that they did not open. If more than one advisor names the wrong number, all of them will lose their jobs and their lives.

There is no reason for anyone to hurry in the Kingdom of Aleph, and the king gives his advisors an infinite amount of time to work out a strategy. Do they have any hope of making it through his cruel test alive? What should they do?

Homework: None.

Class format: Interactive lecture.

Prerequisites: None.

Colloquia

From rotating needles to projections of fractals. (*Alan Chang*) $\boxed{\text{T}}$WΘFS

The Kakeya needle problem asks the following question: Suppose you have a unit line segment (a "needle") in the plane and you'd like to rotate it 180 degrees, so that it points in the opposite direction. What is the area of the smallest region you can do this in? In this colloquium talk, we will discuss the surprising connections between the Kakeya needle problem and the projections of fractals.

Orderable groups. (*Katie Mann*) T$\boxed{\text{W}}\Theta$FS

Probably you take for granted the fact that the integers have an "ordering", namely that $-1 < 0 < 1 < 2 < 3\ldots$ etc. But what happens if we try to make similar coherent "orders" on other sets? (for the purposes of this colloquium, if a set has an operation like $+$, then the order should satisfy

$a + b < c + b$ whenever $a < c$). It turns out sometimes this works, sometimes this fails horribly, and sometimes understanding what's going on is a deep problem that you could do a whole PhD on. No prerequisites required at all, but there will be some extra bonus content if you know a tiny bit of group theory.

Project fair. (in Thompson) TWΘ$\boxed{\text{F}}$S

Project fair is an occasion for campers to present the results of their project. Sometimes campers make a poster about what they've done, sometimes the output of the project is more complicated than that.

Come and see what everyone at camp has done!

If you are interested in presenting at project fair, talk to the staff member(s) supervising your project by Tuesday!