WEEK 5 CLASS PROPOSALS, MATHCAMP 2024

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Alan's Classes

Fourier series (🌶🌶, Alan, 4 days)
Around 1800, the French mathematician Jean-Baptiste Joseph Fourier accompanied Napoleon through Egypt. Egypt was very hot, and Fourier became interested in heat, so he developed Fourier series to solve the differential equation known as the "heat equation." (This is a story I heard from Elias Stein, the mathematician who taught me Fourier analysis.)

The central idea of Fourier series is to decompose a periodic function into pure oscillations (i.e., sine waves):

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

This is what our ears do when we listen to music; it explains why the C-sharp of a piano sounds different from same C-sharp of a violin. (In class, we'll see this with some demonstrations using the software Audacity.)

Fourier analysis has wide applications to other areas, including signal processing (e.g., wireless communication), number theory (e.g., Dirichlet's theorem on primes in arithmetic progressions), quantum mechanics (e.g., the Heisenberg uncertainty principle), and Boolean functions.

In this class, we will learn how to find the Fourier series of any periodic function, prove some basic properties, and see how this can be used to solve differential equations.

Class format: Lecture

Prerequisites: single variable calculus, know what a partial derivative is, some linear algebra might be useful but is not required

Homework: Recommended

Fractal projections and a number theory question (🌶🌶, Alan, 4 days)
Let $K_0 \subset \mathbb{R}^2$ be the unit square. Divide K_0 into 16 squares of equal size, and let $K_1 \subset K_0$ be the union of the four corner squares. Repeat the same procedure on each of the four squares of K_1 to get K_2 (a union of sixteen squares), and so on. We define the four corner Cantor set to be the limit set $K = \bigcap_{n=0}^{\infty} K_n$.

In this class, we will discuss some interesting properties of the projections of the four corner Cantor set, including connections to the following number theory fact: If m and n are odd integers, then m/n can be written as the ratio of two numbers of the form $\sum_{j=0}^{\ell} \epsilon_j 4^j$, where $\epsilon_j \in \{-1, 0, 1\}$. (Incidentally, this number theory fact is proved in a paper called "An awful problem about integers in base four.")

Class format: Lecture

Prerequisites: None

Homework: Recommended

ALYONA'S CLASSES

Intro to knot theory (and to Alyona's undergrad thesis) (🌶🌶, Alyona, 4 days)
Why knot learn more about knots? Especially when you canknot find a punnier subject! In this class, we will explore the world of knots and try to figure (8) out how we can tell them apart. Knot theory is still a dynamic branch of mathematics, so in our quest to distinguish all knots, some of our attempts may be trefoiled. But nonetheless, we will valiantly try to distinguish knots using things like Reidemeister moves, as well as numerical, color, and polynomial invariants! *this description is shamelessly stolen from Kayla Wright's 2019 class blurb, because I really like it.*

On the last day, we'll also talk about braid group, its representations, R-matrices, and what it all has to do with knot invariants. (Aka secretly an overview of Alyona's undergraduate thesis)

Class format: Interactive Lecture

Prerequisites: Intro to Linear Algebra for some parts.

Homework: Optional

Arya's Classes

Medians, convex sets and partitions (🌶🌶🌶🌶, Arya, 4 days)
A cube complex is a complex built with cubes. An n-dimensional cube is a copy of $[0,1]^n$. So why, Arya, is this a 4 chilli class with no "cube" in the title?

Turns out you can do a lot of things with cubes. You can cut a cube into two even halves—something you cannot do with a simplex. A cube has geometry—for instance, all angles are right angles. A cube has faces—each of which is a cube of one dimension lower. Trust me, this class description and the class title are relevant to the class, I promise.

In this class, we will go on a journey through ultrafilters, hyperbolic geometry, medians and convexity. At some point, the fact that I'm allergic to cats will become relevant.

Class format: Interactive lectures

Prerequisites: Having attended my Geometric Geometry class would help. You should know what a metric space is (and in particular, be familiar with the Taxicab metric on $\mathbb{R}^2$. Some familiarity with hyperbolic geometry may also help. Come chat with me.

Homework: None

Automorphisms and Russia (🌶🌶🌶🌶, Arya, 4 days)
Russian mathematician Nikolai Ivanov made a "meta conjecture"—Every object naturally associated to a surface with sufficiently rich structure has the extended mapping class group of the surface as its group of automorphisms.

Now this conjecture is true by definition—the question is whether a particular object has "sufficiently rich structure". The meta conjecture says more—Ivanov proved that the curve graph associated to a surface satisfies this property, and the proof associated to any other object should boil down to Ivanov's proof for the curve graph.

In this class, we shall study the "curve graph" of a surface (which is a simplicial graph whose vertices correspond to homotopy classes of curves on a surface), the "mapping class group" of a surface (which is a group of homeomorphisms of the surface with itself). We shall try to understand Ivanov's proof for the curve graph, and perhaps look at some general "objects naturally associated to a surface".

Class format: Interactive lecture.

Prerequisites: Having attended Teichmueller Theory of the Torus would be helpful. Alternatively, some familiarity with the classification of surfaces, and basic definitions of graphs and groups would be assumed.

Homework: Optional

Athina's Classes

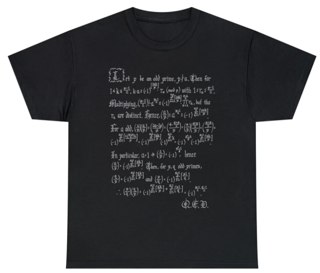
Philosophy and mathematics (🌶, Athina, 2–4 days)
Throughout history, philosophy and mathematics have been intertwined; the phrase "Let no one ignorant of geometry enter" was inscribed above the gate of Plato's Academy, René Descartes gave us "Cogito, ergo sum" as well as the Cartesian plane, and at beginning of the 20th century, mathematicians and philosophers banded together to address the so-called "crisis" in the foundations of mathematics. In this class, we will study some of that history, focusing on the various attempts to answer questions such as "is math invented or discovered?", "what is the nature of mathematical objects?", and "what do we mean when we say a mathematical statement is true?".

Class format: Lecture and some discussion

Prerequisites: None

Homework: None

T-Shirt math (🌶🌶 → 🌶🌶🌶, Athina, 2–4 days)
You might have seen Athina wearing a T-shirt that looks like this:



Come to this class to find out what the math on it means, and why Gauss referred to this result as his "golden theorem"!

Class format: Interactive lecture

Prerequisites: Some elementary number theory, ideally including Fermat's little theorem.

Homework: Optional

Wythoff's game (🌶, Athina, 1 day)
The TAU cookie fairies wanted to shake things up a bit, so they came up with a game. They approach a pair of unsuspecting campers, Alice and Bob, and present them with a plate full of cookies and carrots. Alice and Bob are told that they will be taking turns eating the treats in front of them, according to the following rule: "On your turn, you may eat as many treats as you want (at least one) as long as they are all of the same kind, OR exactly the same number of cookies and the same number of carrots (at least one)". The player who eats the last treat wins. Can you come up with a winning strategy?

Class format: Interactive lecture

Prerequisites: None

Homework: None

Ben's Classes

Cantor's leaky tent (🌶🌶🌶, Ben, 2–3 days)
One of the notorious counterexamples in point-set topology is called "Cantor's Leaky Tent" or the "Knaster–Kuratowski Fan." This space is connected! But there's one particular point that, when removed, makes the space *totally disconnected*. In this class, we'll go over all of these terms, put up our tent, and prove that it does exactly what it's supposed to.

Class format: Interactive Lecture

Prerequisites: Baire's Category Theorem, having seen metric spaces before. Helpful to have seen the Cantor set, but not *strictly* necessary.

Homework: Recommended

Coloring books: A how-to guide (🌶🌶, Ben, 2 days)
Suppose that we have a set of infinitely thin colored markers and a coloring book. Can we fill in

our coloring book? That is, is there a function that lets use fill in a 2-dimensional space with a 1-dimensional curve? If you've gone to certain colloquia this year, you'll know the answer to this—but you might be curious about how it works!

I've run this a few times. In 2021, I also went a little bit nuts with embroidering iterates of Hilbert's Space-Filling Curve onto, well, everything. It's a fun hobby; very meditative.

Class format: Lecture

Prerequisites: Have encountered uniform convergence before; know that the uniform limit of continuous functions is continuous

Homework: Recommended

The most important result for understanding my PhD thesis (We will not discuss my PhD thesis) (🌶️🌶️🌶️, Ben, 3 days)
In some of the topology classes this summer, people talked about something called "compactness." I was one of those people, and other people[1] asked me "why is this important?"

One reason is that compactness results let me get a PhD. I'm not gonna talk much about that. Instead I'll talk about the most important compactness result for my thesis—Arzela–Ascoli, a result that guarantees that some sequences of functions have convergent subsequences.

Class format: Interactive Lecture

Prerequisites: Some understanding of topology (e.g. having seen what metric spaces are) is almost necessary to motivate this, but not to take it, technically.

Homework: Recommended

Chloe's Classes

Elliptic curves with complex multiplication (🌶️🌶️🌶️, Chloe, 2 days)
The goal of this course is to think about endomorphisms of elliptic curves. We'll start by introducing elliptic curves as $\mathbb{C}$ mod a lattice, justifying this connection, and defining an endomorphism of an elliptic curve. Next we'll prove that every elliptic curve has the 'multiplication-by-n' map for all integers n and see examples of this kind of endomorphism. After that we will explore some examples that are not multiplication-by-n and define complex multiplication. To finish the course we'll prove (with some details left out) that an endomorphism of an elliptic curve is multiplication by a number with a complex component if it is not an integer (justifying the name complex multiplication).

Class format: Interactive lectures

Prerequisites: Some familiarity with elliptic curves, especially the definition of points of finite order. Definitions of groups, rings, fields, and homomorphisms

Homework: Optional

Quadratic reciprocity (what I didn't get to yap about in intro to number theory) (🌶️🌶️, Chloe, 1 day)
You really should have probably seen a proof of quadratic reciprocity by the time you've completed an intro to number theory course—but better late than never! Quadratic Reciprocity tells us about exactly when squares appear in modular arithmetic. We'll prove this most elegant result of elementary number theory with just a few simple ingredients!

Class format: Interactive lectures

Prerequisites: Modular arithmetic and Chinese remainder theorem

[1] Who know who they are.

Homework: None

Chloe and Della's Classes

Another weird programming language (🌶🌶, Chloe and Della, 1 day)
Have you ever wanted to program without loops? Is your code always doing things, all the time? Maybe you wished it was more lazy and didn't do that? I have some great news for you! But I'll only tell you if you come to this class.

Class format: Interactive lectures; Group work component

Prerequisites: Some programming may be helpful, but not required

Homework: None

Chloe and Kevin's Classes

Nonunique factorization (🌶🌶, Chloe, Kevin, 2 days)
Everybody loves number theory! There's a good reason for that: every integer can be uniquely factored into primes. Surely, if we move from $\mathbb{Z}$ to $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{2}]$, or even $\mathbb{Z}[\sqrt{-5}]$, nothing can go wrong... After all, everybody loves number theory!

Class format: Lecture

Prerequisites: Concepts from intro to number theory (Euclidean algorithm, unique factorization)

Homework: Optional

Della's Classes

A taste of ramsey theory (🌶, Della, 1 day)
Among six people, there are always three who are either all friends or all not friends. If we want to guarantee a group of 4 who are all friends or all not friends, how many people do we need? What about a group of n?

We will partially answer this and similar questions by finding very loose upper bounds. Ramsey theory is about proving that structure has to exist in large enough objects, and is less concerned with exactly how large the objects need to be. There will be big numbers—some of the biggest natural numbers you've heard of come from Ramsey theory!

Class format: Lecture

Prerequisites: None

Homework: Optional

Monoids in the category of endofunctors (🌶🌶🌶🌶, Della, 1 day)
You might sometimes hear category theory nerds say gibberish phrases like that. But what do they mean? In this class I'll, speedrun category theory definitions until we can make sense of "monoid in the category of endofunctors". If there's time, we'll see what that has to do with burritos.

Class format: Lecture

Prerequisites: None

Homework: None

Slaying the hydra (🌶🌶, Della, 1 day)
When Hercules fought the hydra, he had his nephew cauterize the wound after cutting off each head,

to prevent two more heads from growing back. We're going to be fighting graph-theoretic hydras, where that kind of cheating isn't an option. Can we still win?

Class format: Lecture

Prerequisites: None

Homework: Optional

The inner lives of outer automorphisms (🌶🌶🌶, Della, 2 days)
An *outer automorphism* is an isomorphism from a group to itself which isn't just conjugation by some element. It turns out that S_n doesn't have any outer automorphisms, except that S_6 has exactly one (up to inner automorphism)! We'll prove this fact, and see some cool pictures of the outer automorphism of S_6, including a relationship with rotations of the dodecahedron.

Class format: Lecture

Prerequisites: Basic group theory

Homework: Optional

Will it flatten? (🌶🌶, Della, 1 day)
This is an origami class! We'll think about which crease patterns can be folded perfectly flat. Unfortunately, this is a hard problem[2], much too hard for a 1-day class. To make our lives easier, we will exclusively fold one-dimensional paper!

Class format: Lecture

Prerequisites: None

Homework: None

Wizards in hats (🌶🌶🌶, Della, 1 day)
Ten wizards are each wearing a black or white hat, and can see everyone's hat except their own. One at a time, they guess their hat's color. How can they ensure all but one wizard is right?

Now suppose instead there are (countably) infinitely many wizards, and they have to guess at the same time (so their guess can't depend on what other wizards say). But I'll be generous: instead of only allowing one wizard to be wrong, you can have any finite number of wrong guesses. Is this possible?

In this class, I'll pose a variety of puzzles similar to this one, and see connections to topics like the axiom of choice, the continuum hypothesis, and linear algebra.

Class format: Lecture

Prerequisites: Ordinal numbers, axiom of choice

Homework: Optional

~~Large~~ Small cardinal axioms (🌶🌶🌶, Della, 1 day)
You might have heard set theorists talking about weird things like 'inaccessible', 'measurable', 'ineffable', and 'almost huge' cardinals. These are called *large cardinal axioms*, and they assert that some really big numbers exist—in particular, big enough to imply that normal set theory (ZFC) is consistent.

I'm not going to tell you about any of those. Instead, I will argue that 0 is a large cardinal, and the axiom that the empty set exists is a large cardinal axiom. Then we'll move on to the axioms that let

[2] NP-hard in fact!

us build even bigger cardinals like 6! The point of this class is to introduce all of the axioms of ZF, from the perspective of large cardinals.

Class format: Lecture

Prerequisites: Set theory

Homework: Recommended

Eric's Classes

18446744069414584321 (🌶🌶🌶, Eric, 1 day)
This is a follow-up to my week 3 class "How to multiply numbers reallllly fast". 18446744069414584321 is a prime which is particularly useful in some practical multiplication problems. In this class we'll learn about what makes this prime special, how you can do computations with it efficiently on a computer, how it's relevant for Fast Fourier Transforms, and what other similar primes are out there in the cosmos.

Class format: Interactive lecture

Prerequisites: Mild familiarity with the (Cooley–Tukey) Fast Fourier Transform algorithm, for example as in my week 3 class.

Homework: Recommended

Grammatical group generation (🌶, Eric, 2 days)
Do you like silly word games? Normal subgroups and presentations of groups got you down? Come to this extremely light-hearted romp through the world of grammatically generated groups! In this class, based on a real actual published math paper, we will use group theory to understand how many homophones and anagrams the English language has. If you think this sounds silly, it's because it is silly. But we'll do it anyways. Be prepared for terrible jokes and words you will never see used in any other context.

Class format: 50/50 lecture and worksheets

Prerequisites: Group theory, at the level of having seen definitions of normal subgroups and quotients.

Homework: Optional

Newton polygons (🌶🌶, Eric, 2 days)
This will be a version of the class "What are your numbers worth?" that I taught in 2021 and 2023. We'll learn how to compute the "prime factorization" of any algebraic number, once we figure out what "prime factorization" should mean for irrational algebraic numbers. It turns out that we can do this by drawing a simple picture based on the coefficients of a polynomial the algebraic number solves. We'll learn about these pictures (Newton polygons) and use them to prove some cool facts about polynomials.

Class format: Classes will be worksheet-based

Prerequisites: Basic facts about divisibility in integers; intro number theory is more than enough

Homework: Recommended

The sound of proof (🌶, Eric, 1 day)
Can you hear what a proof sounds like? I'll present five proofs from Euclid's Elements, and then play (recordings of) five pieces of music written to capture each proof in sound. You'll get to try and work out which piece of music lines up with which proof, and then we'll dissect how a couple of the compositions "sonify" the proofs. All of the material I'm drawing on is from an art piece entitled

"The Sound of Proof" by mathematician Marcus du Sautoy and composer Jamie Perera at the Royal Northern College of Music in Manchester.

Class format: Interactive lecture feat. listening to music and filling out a short survey

Prerequisites: None

Homework: None

Glenn, Chloe, and Jennifer's Classes

The Eras Tour: Blank Space (🌶, Glenn, Chloe, Jennifer, 1 day)

Just kidding. We will learn how to write a computer program in the language Whitespace, which only allows space, tab, and newline as meaningful characters. It might drive you a little crazy, but surprisingly, it's actually a decent language to learn how computers work. After you write your program, print it out and ask your friends what it does!

Class format: Programming in the computer lab

Prerequisites: None! (no prior programming experience needed)

Homework: None

Glenn and Jennifer's Classes

The Eras Tour: All Too Well (50 Minute Version) (🌶, Glenn, Jennifer, 1 day)
Some numbers with no apparent relationship just seem to align all too well, like $\pi^2 = 9.86$ and $g = 9.81$ (the gravitational constant near the surface of the earth). It can't just be an accident… can it? This is a story of mathematical coincidences, and a few special times in history when the coincidences can actually be explained.

Class format: A 50-minute song (if we can finish writing it in time)

Prerequisites: None

Homework: None

The Eras Tour: Begin Again (🌶🌶, Glenn, Jennifer, 1 day)
There are two main kinds of loops in programming—the for loop and the while loop. For the vast majority of most intro programming courses, no algorithms truly need the while loop, although it might be convenient. And the while loop is dangerous! If you use it incorrectly, you could make your computer get stuck in an infinite loop. However, the while loop is indeed truly necessary, and we can prove it. This is the story of functions that are provably impossible to compute without while loops.

Class format: Lecture

Prerequisites: Some basic programming (in any language) will help contextualize the class, but not strictly necessary

Homework: None

The Eras Tour: Champagne Problems (🌶, Glenn, Jennifer, 1 day)
Did you know that Abraham de Moivre, of De Moivre's formula fame, was born in the Champagne region of France? But his formula is not his only claim to fame—proclaimed by Newton to "know all these things better than I do", the inventor of a precursor to Stirling's approximation, and even a participant in the Calculus Wars, the only thing he never really managed to accomplish was marriage. To really feel immersed in the history, non-alcoholic champagne will be provided.

Class format: Lecture

Prerequisites: None

Homework: None

The Eras Tour: Love Story (🌶🌶, Glenn, Jennifer, 1 day)
Scooter Braun forces Taylor and Travis to play a cruel game. In the game, Taylor and Travis have to be apart and avoid communication, and they can only see each other again if they win. Scooter designs the game so that he's certain they can win no more than 75% of the time. But through the power of love and connection (i.e. quantum entanglement), Taylor and Travis manage to win 85% of the time! In this class, we'll unravel this love story and explain how they did it.

Class format: Lecture

Prerequisites: Very basic linear algebra (multiplying matrices, visualizing vectors)

Homework: None

The Eras Tour: Mastermind (🌶🌶, Glenn, Jennifer, 1 day)
In 1970, Mordecai Meirowitz invented the board game Mastermind, which would eventually be the inspiration for Wordle! In this class, we'll play Mastermind and discuss mathematical and computational strategies for solving it. It also turns out that Mastermind is an NP-complete problem (that is, an efficient way to solve it would earn you $1 million). It might be surprising how games can have computational complexity, too!

Class format: Lecture

Prerequisites: If you've seen the P vs. NP problem before, this will be more cool. But not strictly necessary.

Homework: None

The Eras Tour: Paper Rings (🌶🌶, Glenn, Jennifer, 1 day)
Why like shiny things when you can just have paper rings? We will physically make paper rings in the shape of your favorite mathematical knots! How does a complicated knot fit nicely on your finger, you ask? It turns out that there is a theorem called Alexander's theorem that allows us to turn any knot into a braid of strings whose ends are connected in a loop. At the end of the class, you can even give your paper ring to a friend!

Class format: Small lecture, plus arts and crafts

Prerequisites: None

Homework: None

Kailee's Classes

Determined to Determinant: Leibniz edition (🌶️, Kailee, 1 day)
Are you sick and tired of calculating determinants by expanding along a row or column and then calculating *more* determinants? Would you rather just play around with graphs and permutations? Come learn about my favorite way to think about the determinant and see why Leibniz's formula will forever live rent-free in my head.

Class format: Interactive lecture/group work

Prerequisites: Permutations, calculating determinants

Homework: None

Determined to Determinant: Perfect matching edition (🌶️ → 🌶️🌶️, Kailee, 1 day)
Are you sick and tired of looking at graphs and trying to see if they have a perfect matching? Look no further! With a little determination, we can find all perfect matchings in a graph just by taking a determinant! We'll see how to build a special kind of adjacency(ish) matrix such that taking its determinant lists out all perfect matchings in our graph.

Class format: Interactive lecture/group work

Prerequisites: Permutations, Leibniz formula for determinant, basic graph theory, familiarity with matchings in graphs helpful but not required

Homework: Recommended

Determined to Determinant: Spanning trees edition (🌶️🌶️ → 🌶️🌶️🌶️, Kailee, 2 days)
Are you sick and tired of wondering how many different spanning subtrees your favorite graph has? Look no further than Tutte's Directed Matrix Tree Theorem, where we'll not only learn how to count, but explicitly list out all spanning subtrees in a graph! The story includes a misunderstood polynomial, who looks scary on the outside but is really just a determinant at heart, and a Romeo and Juliet-esque love, which very importantly ends in tragedy.

Class format: Interactive lecture

Prerequisites: Permutations, Leibniz formula for determinant, basic graph theory (subgraphs, trees, directed graphs)

Homework: None

MCSP: Mermaids Contained in Swimming Pools (Gershgorin circle theorem) (🌶️, Kailee, 1 day)
Who are eigenvalues? Where do they live? Come learn about Gershgorin's circle theorem to find out! (Spoiler Alert: We may not have their exact address, but just looking at each row of our matrix, we can say something about where we might find our eigenvalues hanging out.)

Class format: Interactive lecture

Prerequisites: Basic linear algebra, specifically eigenvalues

Homework: None

Sorting algorithms 101 (🌶️, Kailee, 1 day)
What kind of online shopper are you? Price low to high? Newest to oldest? Rated highest to lowest? Unless you enjoy online shopping in chaos, you've probably benefitted from a sorting algorithm at

some point. How do these work? How fast are they? How do they get cute names like bubble sort? In this class we'll cover several basic sorting algorithms, and how to think about them.

Class format: Interactive lecture

Prerequisites: None

Homework: None

Katie Mann's Classes

Foliation theory (🌶 → 🌶🌶🌶, Katie Mann, 3 days)
A foliation is... roughly... a way to divide a space up nicely into a collection of spaces of lower dimension. High dimensional spaces are hard to understand and visualize, so you might hope that slicing them up into lower dimensional ones solves the problem. But, alas, describing all the possible ways to do this is a whole field of math in itself. In this class I'll talk about classifying and distinguishing 1-dimensional foliations of 2-dimensional surfaces (with your help) and then I will bravely attempt to explain what I am doing in my research right now which circles around foliations of the plane...

Class format: Lecture, with some short activities

Prerequisites: None

Homework: Recommended

Tilings of (hyperbolic) planes and isoperimetric problems (🌶 → 🌶🌶🌶, Katie Mann, 3 days)
Here's a puzzle: suppose you have a collection of tiles, all convex 7-gons, but in 100 different sizes and shapes. Can you possibly use these to tile the plane? (the plane meaning that you have an infinite bathroom floor and you need to tile it. "Can you?" means is there a magic collection that works or are you doomed to fail?)

This course will solve (a quite general!) version of this problem and also describe tilings in various strange geometries, study isopermietric inequalities, and optimistically you'll even see what the puzzle secretly has to do with the field called geometric group theory.

Class format: Lecture, with some short activities

Prerequisites: None

Homework: Recommended

Kevin's Classes

Counting solutions to equations mod p (🌶🌶🌶, Kevin, 2 days)
Given a polynomial equation or system of polynomial equations with integer coefficients, how many solutions are there mod p? One of the greatest achievements of twentieth century mathematics was the formulation and proof of the Weil conjectures, which describe these "point counts" in terms of the topology of the underlying equations. While the Weil conjectures for general systems of equations are way beyond the scope of Mathcamp, we'll see what the Weil conjectures say and count some points in some specific examples, like the case of Fermat curves $x^d + y^d \equiv z^d \pmod{p}$.

Class format: Lecture

Prerequisites: Modular arithmetic is required. Knowing about projective space or deeper topics in modular arithmetic (e.g. primitive roots, Gauss sums) is helpful but not required.

Homework: Optional

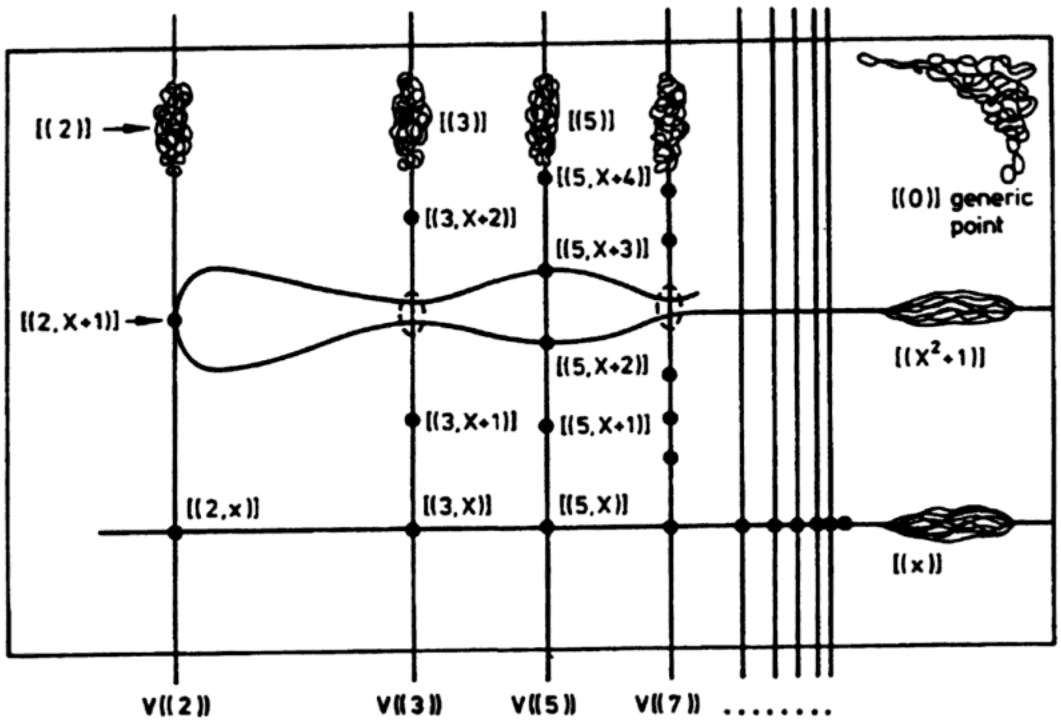
Dimension and transcendence degree (🌶🌶🌶🌶, Kevin, 2 days)
A fundamental notion in algebraic geometry is dimension. A curve is 1-dimensional, a surface is 2-dimensional, and so on. This class is about transcendence degree, an intuitive way to understand dimension on the algebraic side. We'll learn about the relationship between dimension and transcendence degree and prove the Noether normalization lemma, a powerful result describing the structure of coordinate rings of algebraic sets.

Class format: Lecture

Prerequisites: Ring theory is required. Mark's classes on field theory and algebraic geometry are helpful but not required.

Homework: Optional

Spec of a ring (🌶🌶🌶🌶, Kevin, 2–3 days)
Given a ring R, one can associate to it a topological space $\mathrm{Spec}(R)$ whose points are the prime ideals of R. In this class, we'll talk about why this is a reasonable space to do algebraic geometry on by discussing a bunch of examples, and I'll hopefully get to draw a bunch of cool pictures like this one:



Class format: Lecture (maybe a bit of groupwork)

Prerequisites: Ring theory (prime and maximal ideals, homomorphisms) is required. Some exposure to point-set topology (e.g. what is a topological space? basis? continuous function?) is strongly recommended. Also, Mark's class on algebraic geometry would be helpful but is not required.

Homework: Optional

Weyl's equidistribution theorem (🌶🌶🌶, Kevin, 2 days)
A sequence of real numbers (a_n) is equidistributed mod 1 if the fractional parts $\{a_n\}$ appear in each part of $[0, 1)$ equally often. In this class, we'll prove an elegant criterion for equidistribution due to Weyl and use it to prove that (αn) is equidistributed mod 1 for any irrational α.

Class format: Lecture

Prerequisites: Some background in analysis will be required. Knowing Weyl's inequality from my circle method class might be helpful.

Homework: None

Why field theory in characteristic p sucks (mostly) (🌶🌶🌶, Kevin, 2 days)
In a world where $p = 0$, a lot of things go wrong. For example, we can have irreducible polynomials of degree > 1 with only one root when we pass to an algebraic closure. This class is about inseparable polynomials and inseparable extensions, which make field theory in characteristic p a bit more painful than field theory in characteristic 0. I'll also talk about some reasons that we like characteristic p ($(a+b)^p = a^p + b^p$ yay!).

Class format: Lecture

Prerequisites: Know what a field is. Mark's class on field theory is helpful but not required. Also, it would be helpful to know about finite fields, but it's not necessary.

Homework: Optional

Kevin and Laithy's Classes

The inverse function theorem (🌶🌶🌶, Kevin, Laithy, 3 days)
A general theme in geometry and analysis is linearization: Given a complicated nonlinear map, what can we learn about it by looking at a linear approximation? This is done all the time in calculus when we study a function using its derivative. For example, if $f: \mathbb{R} \to \mathbb{R}$ satisfies $f'(x_0) \neq 0$ for some x_0, then $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ for x close to x_0, and so f is locally invertible near x_0. The inverse function theorem is a fundamental result that generalizes this to higher dimensions: If a map $\mathbb{R}^n \to \mathbb{R}^n$ has invertible total derivative at a point (we'll learn what this means), the map itself must be locally invertible near that point.

In this class, we'll prove the inverse function theorem using tools from linear algebra and analysis and, if there's time, see a related result called the implicit function theorem, which describes how equations cut k-dimensional submanifolds out of Euclidean space $\mathbb{R}^n$, forming what we call "foliations."

Class format: Lecture

Prerequisites: Know some fundamental notions from analysis (e.g. completeness, open and closed sets, ϵ-δ definition of limits and continuity) and linear algebra (e.g. what is a linear transformation and when is it invertible?). Also know what a partial derivative is.

Homework: Recommended

Krishan's Classes

How not to prove the continuum hypothesis (🌶🌶🌶, Krishan, 3 days)
Cantor, one of the fathers of set theory, hypothesized that any infinite subset of $\mathbb{R}$ is either countable or is the same size as $\mathbb{R}$. This conjecture is called the continuum hypothesis. In 1900 David Hilbert included this problem in his famous list of millennium problems. Then in 1940, Kurt Godel proved that it is impossible to prove the continuum hypothesis false. But this does not mean that it's true! In 1963, almost 50 years after Cantor died, Paul Cohen showed that it is also impossible to prove that the continuum hypothesis is true! This means that continuum hypothesis is independent of the normal rules of set theory: they simply do not provide enough information to answer the question!

This is a class about, the perfect set program, a failed strategy to prove that the continuum hypothesis is true. The idea was to prove that continuum hypothesis holds for "nice" infinite subsets of $\mathbb{R}$, and then expand the definition of "nice" until all subsets of $\mathbb{R}$ are accounted for. In this class, we'll prove that this strategy works for a while, and we'll discuss how it eventually breaks down for very nasty subsets.

Class format: Mix of lecture and group work

Prerequisites: You should be comfortable the notion of cardinality

Homework: Recommended

How the compactness theorem got its name (🌶🌶🌶🌶, Krishan, 2–3 days)
The extreme value theorem states that any continuous function $f\colon [a,b] \to \mathbb{R}$ attains a maximum and minimum value. A natural question to ask is whether the theorem would still hold if we change the domain of f. For example, does the theorem still hold for continuous functions $f\colon [a,b] \cap \mathbb{Q} \to \mathbb{R}$? It turns out that the answer is **no**: for example consider the function $sin(x)$ on domain $[0,2] \cap \mathbb{Q}$.

The key difference between these domains is that $[a,b]$ is *compact* while $[a,b] \cap \mathbb{Q}$ is not. In this class we'll learn about the concept of *compactness* coming from point-set topology, and we'll see how this is related to the compactness theorem from model theory.

Class format: Interactive lecture

Prerequisites: You should be familiar with the compactness theorem from model theory.

Homework: Recommended

Ultrafilters and voting (🌶🌶, Krishan, 1 day)
Imagine you and your friends are trying to decide where to go for dinner. You all have your own personal ranking of the options but somehow you need to combine your individual rankings into a group ranking. If you were hoping that math could help you with this problem then you're out of luck!

It turns out that there is no "fair" way to solve this type problem. This result is known as Arrow's Impossibility Theorem. In this class we will formulate the theorem precisely and will sketch a proof. Surprisingly this theorem about finite objects it proven using ultrafilters (an abstract infinite typically used in set theory).

Class format: Interactive lecture

Prerequisites: none

Homework: None

Laithy's Classes

Classification of one-manifolds (🌶🌶, Laithy, 2 days)
The only compact, connected, one-dimensional manifolds with boundary are the closed interval and the circle! This might seem obvious, but its proof turns out to be more involved than expected. The idea is simple. Beginning at some particular point, just run along the curve at constant speed. Since the manifold is compact, you cannot run forever over new territory; either you arrive again at your starting point and the curve must be a circle, or else you run into a boundary point and it is an interval. We will sketch the proof of this result and study some of its consequences. In particular, we will prove the Brouwer Fixed Point Theorem, which asserts that any smooth map from the n-dimensional ball to itself must have a fixed point.

Class format: Lectures

Prerequisites: Basic calculus.

Homework: Recommended

Fourier series (🌶🌶, Laithy, 3 days)
We have all seen functions written as an infinite sum of polynomials; but have you seen functions written as an infinite sum of cosines and sines? These are called Fourier series, a powerful tool in analysis used to represent periodic functions. Functions need to be infinitely many times differentiable to be represented as a power series; on the other hand, even non-differentiable functions admit a Fourier series. In fact, sins and cosines form an orthogonal basis of the very large vector space of square-integrable functions on $[a, b]$ denoted by L^2.

We will analyze the convergence properties of Fourier series and distinguish between different types of convergence. We will study some of the applications of Fourier series like the isoperimetric inequality, Weyl's equidistribution theorem, and evaluating sums like $\sum_{n=1}^{\infty} 1/n^{2k}$ and $\sum_{n=1}^{\infty}(-1)^n/(n+a)^2$ for noninteger values of a.

Class format: Interactive lectures

Prerequisites: Basic calculus, sequences and series. Familiarity with uniform convergence is very recommended.

Homework: Recommended

Proof that the universe has a beginning (🌶🌶🌶🌶, Laithy, 4 days)
Astronomical evidence indicates that the universe can be modeled (in smoothed, average form) as a spacetime containing a perfect fluid whose "molecules" are the galaxies. It becomes possible to build quite simple cosmological models whose properties have a reasonable chance of being physically realistic. At the very least, such models provide a testing ground for discovering properties of our universe.

We will study the most famous cosmological model, the Robertson–Walker spacetime. Given a function f on $I \times \mathbb{R}^3$, where I is an interval, the Robertson–Walker metric is $-dt^2 + f(t)^2 g$, where g is a metric on $\mathbb{R}^3$ of "constant curvature" $k = -1, 0$ or 1 (these are the hyperbolic space, flat space, and the 3-d sphere, respectively).

Astronomical data from the Hubble telescope suggests some restrictions on f that lead to the following deep theorems about our universe:

(1) There exists an initial endpoint t_* such that $f \to 0$ and $f' \to \infty$, called the big bang.

(2) If $k = 0, -1$, then $I = (t_*, \infty)$ and as $t \to \infty$, $f \to \infty$. (The universe is expanding indefinitely)

(3) If $k = 1$, then f reaches a maximum followed by a point t^* in which $f \to 0$ and $f' \to -\infty$, called a big crunch. Hence $I = (t_*, t^*)$ and so the universe eventually collapses.

These theorems predict that our universe begins in a colossal explosion and will either expand indefinitely or collapse in finite time. We will then prove a sufficient condition on the matter density that can potentially be confirmed with more astronomical data, which will determine whether or not the universe has an end.

Class format: Lectures

Prerequisites: Week 4 Black Holes class.

Homework: Recommended

Winding numbers and the Jordan curve theorem (🌶🌶🌶, Laithy, 4 days)
The Jordan Curve Theorem says that every simple closed curve in $\mathbb{R}^2$ divides the plane into two pieces, the "inside" (a bounded open connected set) and the "outside" (an unbounded open connected set). While this seems obvious, its proof ends up being very technical and involved, using winding numbers and tools in complex analysis. The generalization of this theorem, called the Jordan–Brouwer Separation theorem, asserts that the complement of a compact, connected hypersurface X in $\mathbb{R}^n$ consists of two connected open sets, the "outside" and the "inside". The proof of the more general

theorem is even more involved and relies on concepts in differential topology called "transversality" and "intersection theory".

In this class, we will define the winding number of a closed curve in $\mathbb{C}$ around a point $z \notin \gamma$ to be, vaguely speaking, the number of times γ winds around z, which will be made precise using complex line integrals. We will use this notion to prove rigorously the Jordan curve theorem and will study some of its applications. If time permits, we will give a sketch of the proof of the Jordan–Brouwer Separation theorem.

Class format: Interactive Lectures

Prerequisites: Basic calculus, integrals, complex numbers, topology of $\mathbb{R}^2$ (open, closed, compact, connected).

Homework: Recommended

Mark's Classes

A tour of Hensel's world (🌶🌶 → 🌶🌶🌶, Mark, 1 day)
In one of Euler's less celebrated papers, he started with the formula for the sum of a geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and substituted 2 for x to arrive at the apparently nonsensical formula

$$1 + 2 + 4 + 8 + \cdots = -1.$$

More than a hundred years later, Hensel described a number system in which this formula is perfectly correct. That system and its relatives (for each of which 2 is replaced by a different prime number p), the p-adic numbers, are important in modern mathematics; we'll take a quick look around this strange "world".

Class format: Interactive lecture

Prerequisites: Some experience with the idea of convergent series.

Homework: None

Exploring the Catalan numbers (🌶🌶, Mark, 1 day)
What's the next number in the sequence $1, 2, 5, 14, \ldots$? If this were an "intelligence test" for middle or high schoolers, the answer might be 41; that's the number that continues the pattern in which every number is one less than three times the previous number. If the sequence gives the answer to some combinatorial question, though, the answer is more likely to be 42. We'll look at a few questions that do give rise to this sequence (with 42), and we'll see that the sequence is given by an elegant formula, for which we'll see a lovely combinatorial proof.

Class format: Interactive lecture

Prerequisites: None

Homework: None

Multiplicative functions (🌶🌶 → 🌶🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.

Class format: Interactive lecture

Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is not needed.)

Homework: Optional

Perfect numbers (🌶, Mark, 1 day)
Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called *perfect*, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes—a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Class format: Interactive lecture

Prerequisites: None

Homework: None

Quadratic Reciprocity (🌶🌶🌶, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is q a square modulo p?"
(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Class format: Interactive lecture

Prerequisites: Some basic number theory (if you know Fermat's Little Theorem, you should be OK)

Homework: Optional

Systems of differential equations (🌶🌶🌶, Mark, 3–4 days)
Many models have been devised to try to capture the essential features of phenomena in economics, ecology, and other fields using systems of differential equations. One classic example is given by the Volterra-Lotka equations from the 1920s:

$$\frac{dx}{dt} = -k_1x + k_2xy;$$
$$\frac{dy}{dt} = k_3y - k_4xy,$$

in which x, y are the sizes of a predator and a prey population, respectively, at time t, and k_1 through k_4 are constants. There are two obvious problems with such models. Often the equations are too hard to solve (except, perhaps, numerically); more importantly, they are not actually correct (they can only hope to approximate what really goes on). On the other hand, if we're approximating anyway and we have a system

$$\frac{dx}{dt} = f(x, y);$$
$$\frac{dy}{dt} = g(x, y),$$

why not approximate it by a linear system such as

$$\frac{dx}{dt} = px + qy;$$
$$\frac{dy}{dt} = rx + sy?$$

Systems of that form can be solved using eigenvalues and eigenvectors, and usually (but not always) the general behavior of the solutions is a good indication of what actually happens for the original (nonlinear) system if you look near the right point(s). If this sounds interesting, come find out about concepts like trajectories, stationary points, nodes, saddle points, spiral points, and maybe Lyapunov functions. Expect plenty of pictures.

Class format: Interactive lecture

Prerequisites: Linear algebra (eigenvectors and eigenvalues), calculus, a little bit of multivariable calculus (equation of tangent plane).

Homework: Optional

The Cayley–Hamilton theorem (🌶🌶🌶, Mark, 1 day)
Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Class format: Interactive lecture

Prerequisites: Linear algebra, including a solid grasp of determinants (the "Magic of determinants" class would definitely take care of that).

Homework: None

The Putnam (🌶, Mark, 1 day)
You may well have heard of it, but maybe not by its formal name: The William Lowell Putnam Mathematical Competition. As you may know, this is a challenging annual 6-hour exam for undergraduates in the U.S. and Canada; more often than not, the median score is under 10 (out of a possible 120 points). At various times before I retired, I was involved with the Putnam, as a problem setter and later in a more administrative capacity. In this class (if it runs) I intend to talk some about the history and "culture" of the Putnam, tell some stories about it, answer any questions you may have, and try to dissuade you from taking it while you are still in high school (which is possible), with one exception.

Class format: Interactive lecture; Q and A

Prerequisites: None

Homework: None

The magic of determinants (🌶🌶 → 🌶🌶🌶, Mark, 3–4 days)
This year's linear algebra class barely touched on determinants in general. If that left you feeling

dissatisfied, either about not really having a good definition (by the way, using the Laplace expansion, while it is computationally often handy, leads to a miserable definition—with no intuitive basis at all) or about not having seen many of the properties that determinants have, this may be a good class for you. If all goes well, we'll give a definition of determinant that's both motivated and rigorous, and there will be proofs of all its main properties (such as Laplace expansion), as well as a few applications such as general formulas for the inverse of a matrix and for the solution of n linear equations in n unknowns ("Cramer's Rule").

Class format: Interactive lecture

Prerequisites: Some linear algebra, including linear transformations, matrix multiplication, and determinants of 2×2 matrices.

Homework: Optional

Wedderburn's theorem (🌶️🌶️ → 🌶️🌶️🌶️, Mark, 1–2 days)
Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j.$$

Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field).

In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Class format: Interactive lecture

Prerequisites: Some group theory and some ring theory; familiarity with complex roots of unity would help.

Homework: None

Mira's Classes

Combinatorial game theory (🌶️🌶️, Mira, 3–4 days)
Let's play a berry-eating game (my favorite kind!). We have a plate with 14 blueberries, 20 blackberries, and 17 raspberries, and we're going to take turns eating them. On your turn, you may eat as many berries as you want (as long as you eat at least one), but they all have to be berries of the same type. Then it's my turn. The winner of the game is the person who eats the last berry. Who's going to win?

This is the game of Nim. It turns out that this innocent-looking game has enormous significance: all other games where the two players have the same options (unlike, say, chess, where can you only move the black pieces and I can only move the white ones) are equivalent to Nim! In this class, we'll discover how to win at Nim, prove that it is the universal impartial game, and use this fact to develop strategies for lots of other games. I will try to guide you through this process rather than giving away the punchline: discovering this beautiful theory by yourself is half the fun.

Class format: Combination interactive lecture and in-class problem solving

Prerequisites: None

Homework: Optional

Democracy can always be gamed (🌶️🌶️, Mira, 1 day)

> *No one pretends that democracy is perfect or all-wise. Indeed it has been said that democracy is the worst form of Government except for all those other forms that have been tried from time to time. . .* – Winston Churchill

You may have heard of Arrow's Theorem: it says that if you want your voting system to satisfy certain reasonable-sounding conditions, then your only option is a dictatorship. But this class is *not* about Arrow's Theorem, because Arrow's Theorem is not depressing enough: its definition of a voting system is so restrictive that it barely ever applies in practice.

The Gibbard–Satterthwaite theorem is less famous, but I think it's much more depressing. It says that if a voting system satisfies two very simple criteria,

(a) if candidate A is preferred by all the voters then A wins;
(b) the system is not a dictatorship,

then this system is vulnerable to strategic voting whenever there are more than two candidates. In other words, there is at least one voter who can obtain better results by voting dishonestly than by voting honestly. Democracy can *always* be gamed. In this class, you won't necessarily learn how, but you'll learn why.

Class format: Interactive lecture

Prerequisites: None

Homework: None

Dominoes and a bizarre formula (🌶🌶🌶, Mira, 3 days)

Question: How many ways are there to tile an $M \times N$ chessboard with 2×1 dominoes?

Answer:

$$\prod_{m=1}^{M}\prod_{n=1}^{N}\left(4\cos^2\frac{m\pi}{M+1}+4\cos^2\frac{n\pi}{N+1}\right)^{1/4}$$

Yes, this is always an integer! (Check it by hand for $M = 1, N = 2$ or other small values if you'd like.) For $M = N = 8$, the formula gives 12,988,816.

Come and find out where this bizarre formula comes from. There'll be some beautiful graph theory and linear algebra along the way, including a deep dive into determinants. If there's time, we'll look at the extension of this problem to chessboards on a torus.

Class format: Interactive lecture

Prerequisites: Linear algebra (eigenvectors and eigenvalues). The Week 1 linear algebra course is sufficient.

Homework: None

The most depressing theorem I know (🌶, Mira, 1 day)

Note: I am also proposing a Week 5 class about another very depressing theorem (Gibbard–Satterthwaite)—but this one is much worse.

Say there is a human trait that (e.g. the probability of being in a car accident) on which two groups (e.g. men and women) differ.

Say also that there is a cost or benefit associated with this trait. In our example, if your probability of being in an accident is low, then your car insurance might give you a discount—that's the benefit.

The most depressing theorem I know (Kleinberg et al. 2016) says that any algorithm for predicting such a trait will be either biased or unfair toward one of the groups (unless it's 100% accurate, which, realistically, will never happen). I'll explain in the class precisely what I mean by "biased" and "unfair" (they mean very different things in this context), but the upshot is: in a world that starts out unequal, there are essentially no fair algorithms. This is not because the people who write these algorithms are sexist or racist or careless (though that may be true too), but because perfect fairness is just mathematically impossible.

I think this theorem is not just depressing, but also very important: it helps cut through the noise in a lot of political debates where people seem to be talking past each other. Surprisingly, it's not very well known. Let's change that!

Class format: Interactive lecture

Prerequisites: None

Homework: None

The politics of rounding fractions (🌶, Mira, 1–2 days)

> *God help the state of Maine when Mathematicks reach for her and undertake to strike her down!*
>
> – Representative Littlefield (R, ME), 1901

The US Constitution mandates that "representatives ... shall be apportioned among the several states ... according to their respective numbers". This is usually taken to mean that the number of representatives in each state should be proportional to its population. But exact proportionality is not possible: for example, California cannot have 54.37 representatives. The same issue arises in countries where seats in parliament are apportioned to parties based on the percentage of votes each party received. Once again, what do you do with fractions of representatives?

This is the *problem of apportionment*, and it's a lot trickier and more interesting than might appear at first glance. Over the course of US history, Congress went through five different apportionment methods, always accompanied by fierce political debates. The method that we currently use was proposed in 1921 by a Harvard mathematician (!), and its was adopted by Congress on the recommendation of the US Academy of Sciences (!!). As far as I know, the US is the only country in the world that uses a method of apportionment that was derived by a mathematician from first principles!

Class format: Interactive lecture if 1 day; more in-class exploration if two days

Prerequisites: None

Homework: None

Misha's Classes

Problem solving: lecture theory (🌶, Misha, 1 day)
This class will teach you about the dark side of problem-solving: how to make educated guesses, how to use problem statements to your advantage, and how to exploit the one piece of extra information contest writers can't help giving you: that the problem has an answer.

(Due to its nature, this class is primarily focused on US contests like the AMC, AIME, and ARML, where you don't have to prove that your answers are correct.)

Class format: Interactive lecture, with slides

Prerequisites: None

Homework: Optional

Problem solving: rigid transformations (🌶🌶, Misha, 2 days)
In this problem-solving class, we'll learn how to tackle olympiad-style geometry problems by applying a rigid transformation of the plane: a translation, a rotation, or a reflection.

These are situational but powerful techniques that are very satisfying to apply—but it's often very hard to spot how they can be used. In class, we will learn about how to use these transformations, and how to spot when they can be used, by solving problems together. There will be problems left to solve on your own. You won't need to solve these to keep up with the class, but you should, because solving problems on your own is critical to learning problem-solving.

Class format: Interactive problem-solving

Prerequisites: Comfort with the basic notions of Euclidean geometry; if you've taken any kind of geometry class in school, you should be fine.

Homework: Recommended

Problem solving: floppy transformations (🌶🌶🌶, Misha, 2 days)
A "floppy transformation" is not an official mathematical term; I'm using to refer to the useful transformations of the plane that *do not* preserve distance. The ones we'll look at in this class specifically are similarities, spiral similarities, and inversions.

I will briefly introduce how these techniques work, but our main goal in this class will be to spot when they can be useful. How will we do this? By solving problems together, of course! There will also be problems left to solve on your own. You won't need to solve these to keep up with the class, but you should, because solving problems on your own is critical to learning problem-solving.

Class format: Interactive problem-solving

Prerequisites: Comfort with the basic notions of Euclidean geometry; if you've taken any kind of geometry class in school, you should be fine.

Homework: Recommended

Problem solving: tetrahedra (🌶🌶🌶🌶, Misha, 1 day)
In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

Class format: Interactive problem-solving

Prerequisites: Comfort with the basic notions of Euclidean geometry; if you've taken any kind of geometry class in school, you should be fine. It is *very* important to know what a tetrahedron is.

Homework: None

Ramsey theory (🌶🌶 → 🌶🌶🌶, Misha, 1–4 days)
To a first approximation, Ramsey theory is about proving theorems that say, "If we color all the whatsits of a sufficiently large thingy with yea many colors, then we will be able to find a monochromatic doodad."

We'll follow a meandering path between results of this kind, pausing to see the sights, and going a bit out of our way to ask slightly different questions than you might normally end up asking.

Topics of interest include upper and lower bounds, clever constructions that everyone should see at least once, and connections to number theory and geometry.

Class format: Interactive lecture

Prerequisites: None

Homework: Optional

The axioms of geometry (🌶 → 🌶🌶, Misha, 1–4 days)
Many people call Euclid's Elements a cornerstone of mathematical rigor, but it was still written two thousand years ago, so Euclid's standards are not the same as ours. Euclid begins by defining a

point as "that which has no part", and even though he begins with 23 definitions, 5 postulates, and 5 common notions, the very first thing he proves with them relies on an unstated assumption.

In this class, we will instead take a closer look at Hilbert's axioms of plane geometry, which were built closer to the modern day and can withstand greater scrutiny. By "take a closer look", though, I don't mean "accept". Instead, I want to start with nothing and add Hilbert's axioms in one at a time—and admire the strange new universes we encounter along the way.

Class format: Interactive lecture

Prerequisites: You must be willing to pretend that you've never heard of Euclidean geometry.

Homework: Optional

Unlikely maths (🌶🌶🌶 → 🌶🌶🌶🌶, Misha, 1–4 days)
A popular way to "construct" a combinatorial object we want to see is to instead give up and choose it at random. This works great if the properties we want from it actually hold for almost any objects we choose. But sometimes we are greedy and want so much from our construction that a randomly chosen object has, at best, an exponentially small chance of making us happy.

This class is about the Lovász Local Lemma: one of the ways to prove that this exponentially small chance is still positive. (If you've taken Kailee's class in week 1, you may have already seen the LLL in action.) In this class, we will see a few unusual uses of the LLL, develop some intuition for why its statement is plausible, and if time permits, look into how to take the last step: actually finding the unlikely object we wanted.

Class format: Interactive lecture

Prerequisites: You should be acquainted with a variety of concepts from graph theory: vertex degrees, paths, trees, cliques, and subgraphs. You should be comfortable with expected value and conditional probability.

Homework: Optional

Narmada's Classes

Approximating lattice points in polytopes (🌶🌶🌶, Narmada, 2 days)
Have you ever wondered what exactly a graduate student in mathematics does all day? Let me tell you about a research problem I'm currently working on: if I fix some parameters of a lattice polytope in $\mathbb{R}^n$, can I say something about its integer points? Almost all known bounds involve monstrous constants in terms of the dimension n. In this class, we will reject Big Oh and try to find explicit bounds for these numbers. The goal of the class is to prove the Approximate Caratheodory theorem and then to talk about whether a stronger result is possible.

Class format: Lecture

Prerequisites: Know what a convex polytope is, know some elementary discrete probability (expectation, variance, Markov's inequality)

Homework: Optional

How to solve the Riemann hypothesis (for finite fields) (🌶🌶🌶🌶, Narmada, 3 days)
I can't resist a good clickbait title. This class is actually about Dwork's proof of the rationality of the zeta function over finite fields. The full proof requires a lot of heavy mathematical machinery—as you can see from the prerequisites—so we will have to skip some details. However, by the end of the class you should have a good understanding of why this result is important and what the key ingredients of Dwork's proof are.

Class format: lecture

Prerequisites: know what groups and group homomorphisms are; know what a ring is, particularly the ring of formal power series; know what the radius of convergence of a power series is; know the characterization of finite fields; know what the trace of a linear map is; have some intuition for vector spaces over finite fields

Homework: Optional

Not all abelian groups can be free (🌶🌶, Narmada, 1 day)
We all know and love the groups $\mathbb{Z}^n$, but mathematicians are never satisfied with finite numbers. How can we define $\mathbb{Z}^\infty$? It turns out that there are two ways to do that: the direct sum and the direct product. In this class, we'll prove that only one of these groups can be free using a neat counting argument.

Class format: Lecture

Prerequisites: Know what a group is, know what a group homomorphism is, know what finite direct sums of groups are

Homework: None

The finite field Kakeya conjecture (🌶, Narmada, 2 days)
The Kakeya conjecture over $\mathbb{R}$ is a longstanding open problem about the fractal dimension of Kakeya sets. For a long time, people assumed that the Kakeya conjecture over finite fields would be equally hard to prove. In 2008, Zeev Dvir shocked these people by proving the finite field Kakeya conjecture in just one simple paragraph. In doing so, he opened the floodgates to a whole new technique in mathematics: polynomial methods in combinatorics.

In this class, we'll see Dvir's original proof of the Kakeya conjecture and look at some other applications of the polynomial method.

Class format: Lecture + group work

Prerequisites: Know that $\mathbb{Z}_p$ is a field

Homework: Recommended

The geometry of Banach spaces (🌶🌶🌶, Narmada, 2 days)
Functional analysis is, loosely speaking, the study of "nice" functions on vector spaces. Introductory linear algebra defines "nice" as "linear", but what if it meant "convex"? We'll look at the relationship between convex sets, convex functions, and norms on vector spaces. Then, we'll use the Hahn–Banach theorem to talk about convexity in infinite-dimensional vector spaces.

Class format: Lecture

Prerequisites: Linear algebra: know some examples of vector spaces that aren't just $\mathbb{R}^n$ or $\mathbb{C}^n$, know what a linear map is, have some intuition for the terms "convex" and "compact"

Homework: None

Sonya's Classes

Colorful puzzles & Dehn functions (🌶🌶, Sonya, 2 days)
Consider the following puzzle: You have a bunch of puzzle pieces that are flexible polygons with colorful edges and a big polygonal frame also with colorful edges. You want to fill in the frame with puzzle pieces so that only edges that touch have the same color. For a given configuration of puzzle pieces and frame, what is the minimal number of pieces necessary to solve the puzzle? How can we figure out if a solution exists at all? Take this class to find out how this is a visual representation

of Dehn's function. We will compute Dehn's function for a few groups, prove some facts about its behavior, and draw a connection to manifolds.

Class format: Interactive lecture

Prerequisites: Knowing what a group is

Homework: Optional

Dynamical degrees and rational points (🌶🌶🌶, Sonya, 1–2 days)
Let f be is a polynomial; what is the degree of f^n? Easy, it's $\deg(f)^n$. The situation gets a bit more complicated if we want to keep track of the powers of a pair of polynomials in two variables: $(f, g), (f(f, g), g(f, g)), \ldots$. And it gets even more interesting if we use rational functions instead of polynomials. A useful measure of degree growth is called dynamical degree. But it's not the only way to qualify how the function acts on the space. Dynamical degree is related to entropy, which is a measure of how much the map "mixes up" points. It is also related to arithmetic degree, which tracks the effect of the function on the complexity of rational points. For instance, if f is monic of degree n, $f(1/2)$ will definitely have a 2^n in the denominator, so it increases the complexity by a power of n, same as its regular degree and dynamical degree. For maps in higher dimensions and rational points restricted to curves or surfaces, the equality of dynamical and arithmetic degrees is a long-standing conjecture. In this class, we will cover the basics of dynamical degrees using just algebra—no knowledge of algebraic geometry necessary.

Class format: Interactive lecture

Prerequisites: None, but you'll get more out of the class if you know what projective space is or and what p-adic numbers are

Homework: Optional

The cyclic polytope (🌶🌶, Sonya, 2–3 days)
What is the maximum number of faces for a polyhedron with 13 vertices? And is there a better way of constructing an example than just moving vertices around until you get the desired result? According to a key result in polyhedral combinatorics, both questions have really satisfying answers (and not just in our regular 3-dimensional space). Join this class for an introduction to the study of polytopes leading up to the upper bound theorem. Or just to draw nice pictures of polyhedra.

Class format: Interactive lecture

Prerequisites: None

Homework: Optional

The Staff's Classes

Kursed Counterexamples (🌶🌶 → 🌶🌶🌶, The Staff, 1–4 days)
Most classes here at Mathcamp focus on the beautiful side of math. This is not one of those classes. In this class members of the staff will expose you to some of the strangest, ugliest, and most badly behaved mathematical objects you'll ever see. We'll cover kursed counterexamples from across mathematics with varying levels of spiciness. Come watch if you dare!

Class format: Interactive Lecture

Prerequisites: None.

Homework: None

Tim!'s Classes

Calculus without calculus (, Tim!, 1–4 days)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Phoebe is 5 cubits tall and Laithy is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Phoebe's head to the top of Laithy's head that touches the ground in the middle. What is the shortest length of string you can use?
- Athina rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along a very straight section of the Puget Sound shoreline. The dog's person stands 20 meters away along the shoreline, and throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- If you went to Laboba to get boba or to Metropolitan Market to get The Cookie, you may have also stopped by the movie theater. Which seat should you have chosen so to make the screen take up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Class format: Interactive Lecture

Prerequisites: We won't use calculus (that's the point), but it would be good if you've seen it for context.

Homework: Recommended

Discrete derivatives (, Tim!, 1–4 days)
Usually, we define the derivative of f to be the limit of $\frac{f(x+h)-f(x)}{h}$ as h goes to 0. But suppose we're feeling lazy, and instead of taking a limit we just plug in $h = 1$ and call it a day. The thing we get is kind of a janky derivative: it's definitely not a derivative, but it acts sort of like one. It has its own version of the power rule, the product rule, and integration by parts, and it even prefers a different value of e. We'll take an expedition into this bizarre parallel universe. If we have three or four days, we'll apply what we find to problems in our own universe: we'll talk about Stirling numbers, and we'll solve difference equations and other problems involving sequences.

Class format: Interactive lecture.

Prerequisites: Calculus (derivatives).

Homework: Optional

Dynamic programming (, Tim!, 1–3 days)
Dynamic programming is a reliable way to get efficient algorithms for all sorts of problems. Here are a few such problems:

- If you have k eggs and an n story building, find a strategy to determine the highest floor of the building you can drop an egg from without it breaking.

- If you have a bunch of text that you would like to typeset (for instance, in LaTeX), how should you break the text into lines so that it is the most aesthetically pleasing?
- Given a collection of items each with a given integer weight and value, find a set of items that weighs at most 100 kilograms and has the maximum possible value.
- Given a level of Super Mario Bros., determine whether it is possible to beat the level (noting that any part of the level that scrolls off-screen gets reset).

Come find some quick algorithms!

Class format: Interactive lecture and problem solving.

Prerequisites: None.

Homework: Recommended

Evasiveness (🌶🌶🌶 → 🌶🌶🌶🌶, Tim!, 4 days)
We'll explore a conjecture in computer science that has been open for over 40 years, concerning the complexity of graph properties. One way to measure the complexity of a problem (like "Does this graph have a Hamiltonian cycle?") is by its time complexity — roughly, how long it takes a computer to solve it. Another important way to measure complexity is query complexity — roughly, how many questions you need to ask about the graph to sole the problem. The graph properties with maximum query complexity are called *evasive*, and the conjecture is that a huge class of graph properties — specifically, all those that are nontrivial and monotone — are evasive.

We'll trace the story of this conjecture through time from its conception in 1973 to more recent results. Along the way, we'll see scorpion graphs, clever counting, collapsible simplicial complexes, transitive permutation groups, and hypergraph properties.

This class is directly related to my research, and in class we might get to see a result of mine, along with its proof (or if we don't get to it, you'll know enough about the area for us to be able to discuss it at TAU).

Class format: Interactive lecture.

Prerequisites: If you haven't seen group theory or graph theory, talk to me first.

Homework: Required

Intersecting polynomials (🌶🌶, Tim!, 2 days)
You might think that everything there is to know about one-variable real polynomials has been known for hundreds of years. Except, in 2009, while bored at a faculty meeting, Kontsevich scribbled down a brand new fact about polynomials. You'll discover it.

Class format: IBL—you'll discover this result from start to finish in groups.

Prerequisites: None.

Homework: None

Sperner's lemma (🌶🌶, Tim!, 2–3 days)
Suppose that after Mathcamp you and your new friends decide to hold a reunion in Alaska. You rent an igloo to share, and you will all chip in to pay for it. But how do you decide who gets to sleep where, and how much each person should pay? You'd be willing to pay more to sleep in a bed rather than a couch, and that's still worth more than the frozen floor. Different people have different preferences—some may rather get a nice bed, while others might not care where they sleep as long as they save money. Can you arrange it so that nobody is jealous of another's sleeping spot (with its corresponding price tag)?

Especially if there's a lot of people, it's not clear that you can come up with envy-free room assignments. But it's possible! The justification relies on an aesthetically-pleasing result about coloring points in a triangle. And this unassuming lemma (whose proof is particularly cute) can prove a whole host of other facts too! It gives a way to fairly divide a cake among friends. It can prove Brouwer's fixed-point theorem (if I crumple up a map of Champlain College and throw it on the ground, some point on the map will be on top of the real-life point it represents). It can prove that a square cannot be divided into an odd number of equal-area triangles, but you have to see it to believe it.

Come experience what this lemma can do!

Class format: Interactive lecture.

Prerequisites: None.

Homework: None

Travis's Classes

4 dimensions is easy (🌶️🌶️, Travis, 1 day)
Among all arrangements of n points in the plane, what is the maximum possible number of pairs of points that are distance 1 apart? This is called the *unit distance problem*, and it's famous (and famously hard). But why stop at the plane? Why not ask the same question in 3 dimensions? (This problem is less famous, and still hard.) Or even 4 dimensions? (This problem is not famous, and is very easy.)

We'll go through this problem and see what makes it so different in 4 dimensions than in 2 or 3 (spoiler: it's because $4 = 2 + 2$) and use that to gain some insight into 4-dimensional space.

Class format: Interactive lecture

Prerequisites: Know how to measure distances in $\mathbb{R}^3$. It will help (though it's not strictly necessary) to know what a subspace of $\mathbb{R}^n$ is.

Homework: None

Chickens at the polls (🌶️, Travis, 1 day)
What's a democracy of chickens to do when it needs to make a decision from the flock's disparate preferences? In this class, we'll talk about the *voting theory* of tournaments. We'll prove that every tournament has a Hamiltonian path and see the minimum number of chickens we need to throw democracy into utter chaos.

Class format: Groupwork

Prerequisites: none. (You do not need to have taken either of the previous chicken classes.)

Homework: Optional

Equilateral sets in the soup norm (🌶️🌶️, Travis, 1 day)
The s(o)up norm in $\mathbb{R}^n$ is defined by $\|(x_1, x_2, \ldots, x_n)\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$. In this class, we'll answer the question: What is the maximum size of a set of points in $\mathbb{R}^n$ such that the difference of every pair of points has soup norm exactly one? (In other words, we'll find the largest possible size of an *equilateral set* in the soup norm.) And with the remaining time, we'll answer similar questions for other norms.

Class format: Lecture

Prerequisites: You should know the dot product and linear independence of vectors.

Homework: Optional

Fermat is False in Finite Fields (FFFF) (🌶🌶🌶, Travis, 1 day)
Fermat's Last Theorem says that there are no solutions to the equation

$$a^n + b^n = c^n$$

if $n \geq 3$ and a, b, c are positive integers. It's pretty hard to prove. On the other hand, Fermat's Last Theorem for Finite Fields says that

$$a^n + b^n \equiv c^n \pmod{p}$$

has no solutions if $n \geq 3$ and a, b, c are not divisible by p. Except this is FALSE.

In this class, we'll use graph theory to prove a result in number theory called Schur's Theorem, and we'll use that (plus a bit of general number theory knowledge) to show that Fermat's Last Theorem is not only false in finite fields, it's *very* false.

NOTE: If you took my graph limits class, you saw a "proof" of a theorem in number theory using a result from graph theory. Unfortunately, we only had time to sketch the proof, since a lot of the intermediate results are hard to prove. In this class, we'll get to go through a *complete* proof in this same style!

Class format: Lecture

Prerequisites: It will help if you're familiar enough with $\mathbb{Z}/p\mathbb{Z}$ to know that every nonzero element has a multiplicative inverse, but that's not strictly necessary.

Homework: Optional

How to geometry reallllllly fast (🌶🌶, Travis, 1 day)
Eric's week 3 class used "evaluation at roots of unity" to quickly multiply numbers. What's lying behind this trick is something called the Discrete Fourier Transform, which is very powerful tool in number theory. In this class, we'll use it to prove this nice theorem in geometry:

Take a polygon P in the plane with vertices $v_1, v_2, \ldots, v_n$. (The polygon does not need to be convex!) Let P_1 be the polygon whose vertices are the midpoints of the edges of P; and let P_2 be the polygon whose vertices are the midpoints of the edges of P_1; and so on. As $n \to \infty$, the polygons P_n converge to a single point, which is the centroid of P.

Class format: Lecture

Prerequisites: You do not need to have taken Eric's class (though if you have, the definition of the Discrete Fourier Transform will seem slightly more familiar).

Homework: Optional

One Helly of a theorem (🌶, Travis, 1 day)
If you take one million convex[3] sets and put them in the plane so that every three of these sets has a point in common, is it possible that no four of the sets has a point in common? In fact, not only is that impossible, but it's *very* impossible: *All one million sets* have a point in common.

This result is called Helly's theorem, and we'll prove it, as well as the extension to 3 dimensions (and maybe even 4 dimensions and more!). If we have extra time, we'll talk about some fun applications of Helly's theorem in geometry.

Class format: Interactive lecture

Prerequisites: None

Homework: Optional

[3] meaning they are connected and don't have any indents or holes

Vikash Mansinghka and Josh Tenenbaum's Classes

Probabilistic programming: human intelligence as computation (🌶🌶, Vikash Mansinghka, Josh Tenenbaum, 2 days)
Can a machine be programmed to think the way the human mind does? Is it possible to explain human intelligence as a kind of computation, and to model brains as a kind of computer? If so, how? This class will present a way to answer these questions, and give you hands-on experience with a new kind of programming approach for modeling human intelligence that we use in our classes and research groups at MIT.

We'll introduce the concepts, mathematics and techniques of *probabilistic programming*, a field at the intersection of computer science, probability theory and data science, which also draws on and informs computational cognitive science and neuroscience. The key idea is to think of intelligence as a rational framework for making good guesses and good bets about the world—that's where the "probabilistic" part comes in—using symbolic programs that can adaptively simulate how the world works, as well as the processes you use to sense the world and act in it. Probabilistic programming is a set of tools for building, learning and using these *generative world models* (or what a cognitive scientist calls a "mental model") to efficiently and accurately extrapolate the parts of the world you haven't observed based on the parts that you have. It rests on some elegant mathematics as well as some pretty cool programming language ideas and software engineering, which we'll touch on. It is being used by robotics and data science researchers to make their systems smarter in more human-like ways, and it is being used to build the first quantitative models of human intelligence that accurately predict behavior and also the structure and dynamics of how neural circuits work in the brain.

In this class we will introduce you to the probabilistic programming language Gen, and examples drawn from time-series analysis (how can a program predict the future of any function that a person can?), 3D computer vision (how can a machine see the shape of objects and scenes in three dimensions, given only a sequence of two-dimensional frames in a movie?), robotic navigation (how can an agent figure out where it is in a complex spatial environment, based only on sparse perceptual data like a robot's directional depth sensor or a mouse's whiskers?), and conversational AI (how can a machine answer questions that are based on rational inferences about the world, rather than just the patterns of what people say?).

The class meets over two days, with two sessions on each day. The second session on each day is **optional** and will focus on programming examples that we guide you through. If you want to participate in the programming part of the class, it would be helpful to bring a laptop. But it's okay if you don't have one; come anyway and we'll make sure you can use ours or others. You can come to just the first hour of each day, or both hours each day. You can also just come to the first hour of the first day, if you want to get a taste for what this field is about don't want to commit to more yet.

Class format: Interactive lecture (hour 1), hands-on exercises (hour 2)

Prerequisites: Familiarity with coding in Python or a similar language, and a little calculus (the concept of a derivative and integral). Experience with probability and Bayesian inference is valuable but you'll learn what you need in this class

Homework: Optional

Zach's Classes

n^{n-2} **Proofs of Cayley's Tree Theorem** (🌶🌶, Zach, 1–3 days)
How many trees exist with vertices $\{1, 2, \ldots, n\}$? The (perhaps surprising) answer is n^{n-2}, and there are soooo many neat ways to show this, using a wide variety of techniques, including inclusion/exclusion, generating functions, random walks, bijections (in multiple ways), analysis, and more!

I've collected different proofs of this theorem for many years, and I'll share as many of my favorites as time allows.

Class format: Lecture

Prerequisites: Knowledge of bijections and inclusion/exclusion.

Homework: Optional

Cut That Out! (🌶🌶, Zach, 1–4 days)
Let's cut shapes into other shapes and assemble them into even more shapes! We'll have a plethora of pretty pictures and a panoply of perplexing puzzles, possibly including:

- Can you divide a square into (any number of) polygons that are similar to each other but all have different sizes? Can you do it with just 2 pieces? 3? Try it! (Hint: the answers are different for 2 and 3.)
- Can you divide a disk into n congruent pieces that *don't* all touch the center? (Hint: yes!)
- Can you dissect a square into finitely many pieces, *change the size of the pieces*, and then rearrange them into a disk? (Hint: just 4 pieces is enough!)

If you enjoy thinking about puzzles like this, or enjoyed Narmada's class on Hilbert's 3rd Problem, this might be the class for you! (Narmada's class is not a prereq.)

Class format: Interactive Lecture

Prerequisites: None

Homework: Recommended

Permuting Conditionally Convergent Series (🌶🌶🌶, Zach, 1–2 days)
The startng point for this class is Riemann's Rearrangement Theorem, which Laithy taught us about earlier in camp: given any conditionally convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers—that is, the sum converges but $\sum_{n=1}^{\infty} |a_n|$ does not—you can permute the terms so that the new sum converges to *any* desired value. But what if the a_n are instead allowed to be *complex*? Can you permute to obtain any *complex* sum? Not always: for the sequence $a_n = (-1)^n \cdot i/n$, the achievable sums lie on the line $i \cdot \mathbb{R} \subset \mathbb{C}$. What other subsets of $\mathbb{C}$ arise in this way? And what about series in $\mathbb{R}^k$? We'll discuss and prove this beautiful classification.

We'll also consider the problem from the permutations' perspective: what are the permutations on $\mathbb{N}$ that always turn convergent series into different convergent series (not necessarily with the same sum)? Amazingly, there are some permutations that do better than $\mathbb{N}$ itself: they turn convergent series into convergent series, but they also transform some *divergent* series into convergent ones!

Class format: Lecture

Prerequisites: Epsilon-delta definition of limits

Homework: Optional

The Seven Circles Theorem (🌶🌶, Zach, 1 day)
We'll show two proofs of a beautiful fact in Euclidean geometry about six tangent circles enclosed within a 7th circle. The first proof relies primarily on TPS from this year, while the second involves a whirlwind tour through Hyperbolic Geometry, where we will discuss equivalences between two different representations of hyperbolic geometry in the unit disk, as well as try to make sense of the perimeter of infinite hyperbolic polygons.

Class format: Lecture

Prerequisites: Knowledge of inversion will be helpful, but not strictly necessary.

Homework: None