WEEK 5 CLASS PROPOSALS, MATHCAMP 2025

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Antonio's Classes

A Taste of the Snark (Antonio, 2–4 days)

A little puzzle for you:

You are a very ambitious graduate student in mathematics at Frog University. You have written a proof for a very prestigious problem, and if it were correct, you would become a very famous mathematician frog. You would want to ask your very smart mentor, the Evil Frog, to check your proof but you are scared that they would steal it and publish it themselves. Additionally, the Evil Frog is very busy and has no time to read all your thousands of pages of symbols: for every solution that you want him to check, they will only read 1 page from you and that's it, no matter how hard the problem is. How do we manage to make our proof small enough that it fits on one page and written in such a way that the Evil Frog can't steal the clever math we've created but can still check it?

In this class, we will answer the puzzle above. To do so, we will be studying some of the theory behind "zero-knowledge succinct non-interactive arguments of knowledge" (zk-SNARKs), and learning how we can use Quadratic Arithmetic Programs (QAPs) and the KGZ commitment scheme to construct Groth'16, a popular zk-SNARK proof system which is actively used in industry (especially in the blockchain space).

Class format: Interactive Lecture

Prerequisites: Group theory

Homework: None

Ari's Classes

How to Teach (Ari Nieh, 4 days)

I probably don't need to sell you on the idea that learning from a great teacher is a transformative experience. After all, you're here at Mathcamp. But I might need to sell you on the claim that youyes, you- can be a great teacher. To my mind, calling somebody a great teacher is like calling them a great pianist. It doesn't mean they made beautiful music the first time they touched the keyboard. It means they've spent countless hours practicing and refining their skills, and gotten feedback and coaching from more experienced pianists, so that what once felt impossible is now second nature.

I have three main goals for this class:

- Provide you with the beginnings of a toolkit for teaching: how to speak, write, interact, and plan for the classroom
- Invite you to think critically about tough questions: Am I being understood? What's the right mental model for this abstraction? How safe do my students feel speaking up?
- Give you a framework for future improvement: What sort of feedback do you seek out, and from whom? How do you practice and refine your techniques? How do you stay motivated in the face of difficulties?

We'll explore these topics through a variety of class formats. Exactly what we cover will depend on your priorities, feedback, and questions. Let's play some Mozart together! (Or Chopin, if you prefer.)

ARYA AND CHARLOUTTE'S CLASSES

About Cantour (\nearrow \rightarrow \nearrow \nearrow , Arya and Charloutte, 2 days)

Arloutte and Charya, the twou ACs, will talk About the Cantour set. Four countext, Arloutte louves analysis, and Charya louves geoumetric toupoulougy, and they bouth individually louve the Cantour set, but doun't knouw why the outher oune louves the Cantour set.

Class format: Lecture Prerequisites: Noune

Homework: Recoummended

ARYA AND THE JCS'S CLASSES

Unicorns and Poland (Arya and the JCs, 3 days)

"Unicorn paths" were defined by Polish mathematician Piotr Przytycki (now at McGill), because he initially wanted to define "one-cornered paths", and the word for "one-cornered" in Polish is very similar to the word "unicorns", and "unicorns" is way cooler. Polish people are cool: P

But why do we care about these (and what are these)? Associated to every surface, there is a graph called the "curve graph", which very literally is a graph of curves on the surface. For deep reasons, this graph is very cool. Unicorns guide the way to travel along the unyielding terrains of this graph, and tell us a lot about the geometry of this graph.

Come to this class to get a feel about stuff people study in modern geometric topology!

Class format: The JCs will teach Arya's class. There might be performances halfway through each class.

Prerequisites: None Homework: None

ASSAF BAR-NATAN'S CLASSES

Degen's Eight Square Identity (, Assaf Bar-Natan, 1 day)

Degen's eight square identity is the following:

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2 + b_8^2)$$

$$= (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)^2 +$$

$$(a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)^2 +$$

$$(a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 + a_5b_6 - a_6b_5 - a_7b_8 + a_8b_7)^2 +$$

$$(a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 + a_5b_6 - a_6b_5 - a_7b_8 + a_8b_7)^2 +$$

$$(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 + a_5b_7 + a_6b_8 - a_7b_5 - a_8b_6)^2 +$$

$$(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 + a_5b_7 + a_6b_8 - a_7b_5 - a_8b_6)^2 +$$

$$(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 + a_5b_8 - a_6b_7 + a_7b_6 - a_8b_5)^2 +$$

$$(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 + a_5b_8 - a_6b_7 + a_7b_6 - a_8b_5)^2 +$$

$$(a_1b_5 - a_2b_6 - a_3b_7 - a_4b_8 + a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4)^2 +$$

$$(a_1b_5 - a_2b_6 - a_3b_7 - a_4b_8 + a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4)^2 +$$

$$(a_1b_6 + a_2b_5 - a_3b_8 + a_4b_7 - a_5b_2 + a_6b_1 - a_7b_4 + a_8b_3)^2 +$$

$$(a_1b_6 + a_2b_5 - a_3b_8 + a_4b_7 - a_5b_2 + a_6b_1 - a_7b_4 + a_8b_3)^2 +$$

$$(a_1b_7 + a_2b_8 + a_3b_5 - a_4b_6 - a_5b_3 + a_6b_4 + a_7b_1 - a_8b_2)^2 +$$

$$(a_1b_8 - a_2b_7 + a_3b_6 + a_4b_5 - a_5b_4 - a_6b_3 + a_7b_2 + a_8b_1)^2$$

There's a very elegant, one line proof of it using division algebras and octonions, but in this class, we'll prove it the old fashioned way. By expanding it all out, and equating the terms. If there's time, we will do the octonionic one-line proof.

Class format: IBL Prerequisites: None Homework: None

Audrey's Classes

Extending a Knot Invariant ($\mathbb{A} \to \mathbb{A} \to \mathbb{A}$), Audrey, 2–3 days)

We will look at my undergraduate research that was recently accepted for publication. I will introduce knots and links before talking about how to calculate the invariant and how we know that this method of calculating is indeed an invariant.

Class format: Lecture Prerequisites: None Homework: Optional

The Lebesgue Measure on \mathbb{R} : How do we Define How Big a Subset of \mathbb{R} is? (\mathbb{A} , Audrey, 3-4 days)

How do we determine how big a set is? We will work together on a one-chili pace to try to figure this

out, as well as attempting to answer questions like "what should happen if we combine sets?" and "do we need to make rules about what sets we can determine the bigness of?"

Class format: IBL

Prerequisites: If you have seen limits or sequence convergence before, it will be a bit easier. Otherwise,

add half a chili.

Homework: Recommended

Charloutte's Classes

Incidences Between Points and Lines (A. Charloutte, 2 days)

The Szemerédi–Trotter Theorem counts the number of incidences between points and lines in \mathbb{R}^2 : Given a finite set of points \mathcal{P} and a finite set of lines \mathcal{L} in \mathbb{R}^2 , how large can $\{(p,\ell) \in \mathcal{P} \times \mathcal{L} : p \in \ell\}$ be (in terms of $|\mathcal{P}|$ and $|\mathcal{L}|$)? There's an elegant proof of this theorem that uses the clever idea of first partitioning \mathbb{R}^2 into cells using polynomials, and then taking advantage of the nice structure this introduces. This "polynomial partitioning" method was introduced by Guth and Katz in their solution to Erdö's distinct distances problem, and has since had powerful applications to a variety of combinatorial and analytic problems. We'll see its application to the Szemerédi–Trotter Theorem in this class.

Class format: Lecture and worksheets

Prerequisites: None

Homework: Recommended

The Cap Set Conjecture ($\nearrow \rightarrow \nearrow \nearrow \nearrow \nearrow \nearrow$, Charloutte, 1–2 days)

If you went to Sarah's colloquium on additive combinatorics, the following type of question may seem familiar: How large must a subset of \mathbb{N} be in order to guarantee that it contains arithmetic progressions of particular lengths? A major result in this field, called Roth's Theorem, addresses this question for three-term arithmetic progressions. Proving Roth's Theorem is a bit tricky, so we'll move to the world of \mathbb{Z}_p^n . We'll take advantage of the much nicer finite field setting, which allows us to obtain very strong bounds with shockingly elegant and short solutions.

In particular, we'll prove an answer to the Cap Set Conjecture, which addresses the following question: What's the maximum size of a subset of \mathbb{Z}_3^n that does not contain any three-term arithmetic progressions? The proof was given by Ellenberg and Gijswijt in 2016, and introduced a new variant of "the polynomial method," a proof method that has garnered great interest and achieved much success in fields like combinatorics, number theory, and analysis over the last few decades.

Class format: Lecture, maybe with some in-class time for worksheets.

Prerequisites: An intro linear algebra class

Homework: Optional

The Devil's Staircase: Why You Should Hate Love Analysis Even More (\nearrow \rightarrow \nearrow \nearrow \nearrow . Charloutte, 1–2 days)

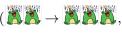
I am retroactively branding my Week 3 class on nowhere continuous differentiable but differentiable continuous functions as Everyone Should Hate Love Analysis. Do you know what's even worse than a nowhere differentiable but continuous function? A continuous function $f:[0,1] \to [0,1]$ whose derivative is zero essentially everywhere but whose range is all of [0,1]!

Class format: Lecture and worksheets

Prerequisites: None

Homework: Optional

Space Filling Curves: Why You Should Hate Love Analysis Even More () A No.



Charloutte, 2–3 days)

I am retroactively branding my Week 3 class on nowhere continuous differentiable but differentiable continuous functions as Everyone Should Hate Love Analysis. Do you know what's even worse than a nowhere differentiable but continuous function? A continuous function that maps [0,1] onto $[0,1]^2$, that's what.

Class format: Worksheets and lecture

Prerequisites: You should be comfortable with the definitions of a metric space, Cauchy sequences, and what it means for a space to be complete.

Homework: Recommended

Della's Classes

Hard Hat Puzzles (, Della, 1 day)

Nikita's hat puzzles are all finite, and therefore boring. What can we do with infinitely many people, or with infinitely many different hats? Make sure you wear PPE, since we'll be in constant danger of set theory.

Class format: Lecture (or ignoring me to solve puzzles)

Prerequisites: Ordinal numbers, axiom of choice

Homework: Optional

How to Prove Things NP-complete (, Della, 2 days)

If you've talked to me, or Glenn, or Nikita, or Misha, or Zach, you might have heard a comment like "this is NP-complete" as though saying that solves a problem. I'll explain what it means to be NP-complete, why it means you should stop trying, and how to prove it. You can also use this to torture Glenn's SAT solvers!

Class format: Lecture Prerequisites: None

Homework: Recommended

How to Win S3CT (A), Della, 2 days)

You might have seen me playing (or even been subjected to) a game with cards that each have two colorful fidget spinners, where I keep saying "wreath" for some reason. This is a variation on SET based on a group named $S_3 \wr C_2$ (and the T is for "torsor"). I'll tell you what those words mean, and then I'll explain some strategies that can help you find sets faster.

Class format: Lecture

Prerequisites: Group theory Homework: Recommended

Kruskal's Tree Theorem (Della, 3 days)

If you thought Misha's class on tic-tac-toe had big numbers, you might want to sit down for this one. The TREE function grows so fast that TREE(3) is already bigger that Graham's number by more than I can possibly convey in a blurb. We won't prove that, but we'll prove a general result which at least implies that TREE(n) is always finite.

Class format: Lecture

Prerequisites: Comfort with well-orderings

Homework: Optional

Let $\varepsilon_0 > 0$ be Sufficiently Small (\mathbb{A}), Della, 2 days)

Many introductions to ordinal numbers go through the ordinals up to ε_0 , and then skip straight past all the countable ordinals to ω_1 . But in between ε_0 and ω_1 , there's a lot of stuff happening—in fact, an uncountable amount of stuff! Let's see how high we can count in two days!

Class format: Lecture

Prerequisites: Ordinal numbers
Homework: Recommended

The Inner Lives of Outer Automorphisms (A., Della, 3 days)

An outer automorphism is an isomorphism from a group to itself which isn't just conjugation by some element. It turns out that the symmetric group S_n doesn't have any outer automorphism—except for when n = 6, and it has exactly one! We'll see why, and do our best to visualize the one exception.

Class format: Lecture
Prerequisites: Group theory

Homework: Optional

Will It Flatten? (, Della, 1 day)

This is an origami class! We'll think about which crease patterns can be folded perfectly flat. Unfortunately, this is a hard problem¹, much too hard for a 1-day class. To make our lives easier, we will exclusively fold one-dimensional paper!

Class format: Lecture Prerequisites: None Homework: None

Eric's Classes

Infinite Field Trip (, Eric, 1 day)

In this class we'll prove a cool theorem about polynomials over \mathbb{C} by reducing to the case of finite fields, where we can just count things! (The Ax-Grothendieck theorem for \mathbb{C}^n .) If there's time after proving the main theorem we'll gesture towards the more general philosophy in model theory (the Lefschetz principle) that is the idea behind the proof.

Class format: Interactive lecture

Prerequisites: Ring theory: the statement that an ideal I in a commutative ring R is maximal iff R/I is a field should be comfortable. You should have seen the words "algebraically closed field" before.

Homework: Optional

¹NP-hard in fact!

Newton Polygons (A, Eric, 2 days)

On the last day of Glenn's class on Monsky's theorem you learned about Newton polygons! These are a super neat tool that does a magic thing: for any algebraic number you can draw a simple picture to figure out how divisible it is by any prime (suitably interpreting divisibility in the sense of valuations, and with some caveats of which valuation you pick). This short class will cover proving this main theorem about Newton polygons, giving a slick proof of Eisenstein's theorem using Newton polygons, and some further applications of Newton polygons to studying factorizations of polynomials.

Class format: Worksheet based exploration of Newton polygons.

Prerequisites: You should be familiar with the p-adic valuations on rational numbers, for example as they were covered in Glenn's week 2 class.

Homework: Recommended

The Sound of Proof (, Eric, 1 day)

Can you hear what a proof sounds like? I'll present five proofs from Euclid's Elements, and then play (recordings of) five pieces of music written to capture each proof in sound. You'll get to try and work out which piece of music lines up with which proof, and then we'll dissect how a couple of the compositions "sonify" the proofs. All of the material I'm drawing on is from an art piece entitled *The Sound of Proof* by mathematician Marcus du Sautoy and composer Jamie Perera at the Royal Northern College of Music in Manchester.

Class format: Interactive lecture feat. listening to music and filling out a short survey

Prerequisites: None Homework: None

GLENN'S CLASSES

Monads, Categories, and the Structure of Programming (18, Glenn, 2 days)

"A monad is a monoid in the category of endofunctors." If you haven't heard of this saying before, I think that most people understand it as a joke that pokes fun at mathematicians being crazy and lost in abstract nonsense. But the joke actually goes one step further—it was actually first popularized by a programmer, and it is funny because a monad is actually an extremely common, basic, and easy-to-understand concept in programming! In other words, the category theorists weren't just saying nonsense to define hard, abstract concepts: they were saying nonsense to explain easy things.²

We will learn what programmers know monads and monoids and endofunctors to be, which are all very straightforward patterns that appear in most programming languages. At the end, I'll say a few words on how mathematicians define these things and how they are equivalent.

Why 1 chili? Unlike previous category theory classes at Mathcamp this summer, this class will deal with very concrete objects without much abstraction. In particular, we will just look at some functions and types in a programming language called Haskell. So I think it will actually be very accessible to everyone (with a bit of programming background)! At the same time, I think the connection between programming and category theory is something that should be interesting for people who typically like more chilis, too.

Class format: Lecture

Prerequisites: Some programming in any language, ideally a language that requires types like C++ or Java. If you only know Python/JavaScript, be comfortable that some languages require you to declare the types of your variables and functions.

²See also: https://ncatlab.org/nlab/show/carrying.

Homework: Recommended

JANE WANG'S CLASSES

Statistical Oddities ($\mathbb{A} \to \mathbb{A}$), Jane Wang, 1 day)

A 1973 study into admissions at Berkeley showed that men had a much larger chance of being admitted into a graduate program than women. However, on the department by department level, there was a small but significant bias in favor of admitting women. How was this possible? Statistics is full of examples such as this one that can challenge our thinking and run contrary to our intuition. In this course, we'll analyze some counterintuitive statistical phenomena as well as some ways that people try to deceive with statistics.

Class format: Interactive Lecture

Prerequisites: Basic probability (computing discrete probabilities and expectations)

Homework: None

Kaia's Classes

Blobs (A), Kaia, 2–3 days)

If every three line segments in a finite collection of line segments has a line passing through the origin that can intersect them, then there exists such a line that intersects them all.

Are lines really so special, though?

What if we just asked that if we pick any two points in a fake line, the line segment between them is contained in the fake line? What if these fake lines actually looked more like blobs?

Let's prove some theorems about convex sets.

Class format: IBL Prerequisites: None

Homework: Recommended

Discrete Versus Continuous: Fourier Series and Convolution (A.A., Kaia, 4 days)

Convolution in the time domain is pointwise multiplication in the Fourier domain. The discrete Fourier transform matrix diagonalizes circulant matrices. These two statements are the same.

Class format: Interactive lecture

Prerequisites: Derivatives and integrals, matrix multiplication, eigenvalues and eigenvectors, complex numbers

Homework: Recommended

The Universe is Kind of Lazy (, Kaia, 4 days)

To understand classical mechanics, we'd like to move beyond Newton's three laws. This will lead us to something called the Principle of Least Action—in short, when anything moves, it likes to be lazy and solve a minimization problem (which perhaps isn't such a lazy thing to do at all).

So let's learn how to solve this minimization problem ourselves, shall we? This will bring us to the Calculus of Variations—we'll ask what happens when we perturb a function a little bit.

Ever wanted to find geodesics, the shortest paths between points? Ever wondered how to maximize the area of a shape with a fixed perimeter? Well, the universe certainly likes solving these sorts of problems.

Class format: Interactive lecture

Prerequisites: Derivatives and integrals

Homework: Recommended

We've Been Framed (Kaia, 3–4 days)

You think a frame is an object that contains a picture.

But actually, a frame is an overcomplete basis. Like, let's take a perfectly good linearly independent basis, and let's just add some more vectors and make it linearly dependent. Sounds terrible right?

Well somehow, this terrible idea is the *basis* (haha) of the theory of wavelets! Something so useful, you use them every day!

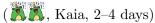
Let's back up and try to understand why this terrible idea isn't so terrible—let's talk about frame theory!

Class format: None.

Prerequisites: Vector spaces, bases

Homework: Recommended

Who Cares About the Radius of Convergence Anyways? (A. Kaia, 2–4 days)



Usually when we Taylor expand a function, that expansion is only valid inside this pesky region called the radius of convergence.

Let's forget about convergence though. We'll expand anyways, and just say the expansion is *asymptotic* to what we want even if it doesn't converge! This is math, and we can define things however we want!

Now let's look at horrible equations with epsilons in them and find *asymptotic* solutions.

Class format: Interactive lecture

Prerequisites: Differential calculus, Taylor series

Homework: Recommended



Let's approximate a function with polynomials!

We'll take a degree n polynomial, and make it match our function on an equally spaced grid of npoints. Surely this will give us a convergent sequence, right? We can just shrink the grid really small, right??

Wrong. But we can do so much better. Join me, find out what the 'best' choice of grid might be, and what these 'best' interpolating polynomials might look like.

Class format: Interactive lecture Prerequisites: Derivatives, norms

Homework: Recommended

Laithsey (Laithy, Linsey)'s Classes

Hypersphere Paradox ($\mathbb{A} \to \mathbb{A}$), Laithsey (Laithy, Linsey), 2 days)

What happens to the volume of a sphere as we increase the number of dimensions? You might guess that a sphere of radius 1 would have more volume in higher dimensions, but the actual answer is surprising.

In this two-day class, we'll find out: In which dimension does a unit sphere (radius 1) have the largest volume? We'll derive a recursive formula for computing the volume of an n-dimensional sphere, then prove an explicit formula using the gamma function. Using these tools, we'll find where the maximum occurs and understand why it happens.

The answer reveals something counterintuitive about high-dimensional geometry—the volume doesn't just keep growing as dimensions increase. Instead, there's a specific dimension where the volume peaks, and then it actually starts decreasing.

Class format: Interactive lecture

Prerequisites: Calculus and integration techniques

Homework: None

Laithy's Classes

Can you Surf these Vector Fields? Foliations and Frobenius Theorem. (A.A.A., Laithy, 4 days)

A rank-2 foliation of \mathbb{R}^3 is a way of slicing 3-dimensional space into a collection of smooth, non-overlapping 2-dimensional surfaces, called leaves. For example, the set of all planes parallel to the xy-plane forms a foliation—each plane is a leaf in the stack.

If we start with a foliation, then at every point in \mathbb{R}^3 , we can take the tangent plane to the surface (leaf) passing through that point. This gives us a smoothly varying choice of a 2-dimensional subspace of the tangent space at each point—a rank-2 distribution.

But now imagine going the other way: Suppose we assign, at every point in space, a 2-dimensional plane (tangent subspace), for example by specifying two smooth, linearly independent vector fields on \mathbb{R}^3 .

Do these planes arise from some actual foliation? Can we "surf" these vector fields—riding them to sweep out real surfaces?

This is the central question answered by the Frobenius Theorem. It tells us exactly when a collection of planes (a rank-2 distribution) fits together to form actual surfaces—i.e., when the distribution is integrable.

Class format: Interactive lectures

Prerequisites: Multivariable calculus (partial derivatives, gradient). Linear algebra (linear indepen-

dence, 2-dim subspaces of *Homework:* Recommended

Einstein's Theory of Gravity 3: Black Holes (, Laithy, 4 days)

Having understood Einstein's geometric perspective of spacetime, we will study the most famous solution to Einstein's field equations, the Schwarzschild solution, which predicted what we now call black holes.

The Schwarzschild solution describes spacetime around a spherical, non-rotating mass and predicts something extraordinary: regions where spacetime curvature becomes so extreme that nothing, not even light, can escape.

Starting with the history: In 1916, just months after Einstein published his theory, Karl Schwarzschild found this solution while serving on the Russian front in WWI. For decades, physicists debated whether black holes were mathematical curiosities or physical reality. The answer came in the 1960s-70s with the discovery of quasars and X-ray binaries — indeed, black holes are real and abundant in our universe.

We'll derive the Schwarzschild metric ourselves. By imposing spherical symmetry, we'll reduce Einstein's equations to just a few ODEs that we can solve explicitly. The solution reveals strange properties: as you approach the event horizon, your clock runs slower relative to distant observers —

in fact, they see your time freeze completely at the horizon. Yet if you fall in, you experience nothing special at the horizon and reach the singularity in finite proper time.

We'll discuss Birkhoff's theorem: any spherically symmetric vacuum spacetime must be Schwarzschild. This means even a collapsing star's exterior spacetime is identical to a static black hole's—a remarkable rigidity result.

Time permitting, we'll discuss Hawking's discovery that black holes emit thermal radiation due to quantum effects near the horizon, slowly evaporating over time. This connects gravity, quantum mechanics, and thermodynamics in unexpected ways.

Class format: Interactive lectures Prerequisites: General Relativity

Homework: Recommended

Fundamentals of Algebraic Topology (, Laithy, 4 days)

This is a 1 Laithy-chilli class on the fundamentals of algebraic topology.

We'll develop the fundamental group — one of the most important tools in algebraic topology. The fundamental group captures information about the "holes" in a space by studying loops and how they can be continuously deformed.

We'll start by constructing the fundamental group and exploring its basic properties. You'll learn to compute fundamental groups of familiar spaces like circles, spheres, and tori. We'll see what it means for a space to be simply connected (having trivial fundamental group) and prove that spaces like the 2-sphere have this property.

We will then cover several major applications. We'll prove the Brouwer fixed point theorem using retractions, showing that any continuous map from a disk to itself must have a fixed point. Using fundamental groups, we'll give a topological proof of the fundamental theorem of algebra — that every polynomial has a complex root. We'll also prove the Borsuk-Ulam theorem, which says that any continuous map from a sphere to Euclidean space must send some pair of antipodal points to the same value!

If time permits, we'll study homotopy equivalence and deformation retracts, which allow us to identify spaces that are "the same" from the perspective of algebraic topology, as well as covering spaces, which provides both a geometric way to understand fundamental groups and a method to compute them.

Class format: Interactive lectures

Prerequisites: Basic group theory. Definition of continuity and connectedness in abstract topological spaces.

Homework: Recommended

Lucas's Classes

We will prove non-probabilistic things about colourings of finite graphs using probabilistic things. I think this is the most interesting idea in mathematics.

Day 1: Intro to the probabilistic method. Hypergraphs, Ramsey Theory.

Day 2: The Local Lemma + applications. This allows you to prove things about graphs knowing only what they look like locally.

Day 3: Entropy compression. A very complicated way of proving skibidi simplest solution you can think of will work.

The number of chilis is the day number plus one.

Class format: Lecture

Prerequisites: Some graph theory (specifically the definition of the edge colouring number), some

probability (specifically Markov's inequality), comfortable with big/little O notation

Homework: Optional

Mark's Classes

Counting, Involutions, and a Theorem of Fermat (A), Mark, 1 day)

Involutions are mathematical objects, especially functions, that are their own inverses. Involutions show up with some regularity in combinatorial proofs; in this class we'll see how to use counting and an involution, but no number theory in the usual sense, to prove a famous theorem of Fermat about primes as sums of squares. (Actually, although Fermat stated the theorem, it's uncertain whether he had a proof.) If you haven't seen why every prime $p \equiv 1 \pmod{4}$ is the sum of two squares, or if you would like to see a relatively recent (Heath-Brown 1984, Zagier 1990), highly non-standard proof of this fact, consider taking this class.

Class format: Interactive lecture; if 80 minutes, possibly some group work also

Prerequisites: None Homework: None

Cyclotomic Polynomials and Migotti's Theorem (A.A., Mark, 1 day)

The cyclotomic polynomials form an interesting family of polynomials with integer coefficients, whose roots are complex roots of unity. Looking at the first few of these polynomials leads to a natural conjecture about their coefficients. However, after the first hundred or so cases keep confirming the conjectured pattern, eventually it breaks down. In this class we'll prove a theorem due to Migotti, which sheds some light on what is going on, and in particular on why the conjecture finally fails just when it does.

Class format: Interactive lecture

Prerequisites: Some experience with complex numbers, preferably including complex roots of unity; some experience with polynomials.

Homework: None

Elliptic Functions ($\triangle \triangle \triangle \rightarrow \triangle \triangle \triangle \triangle$), Mark, 4 days)

Functions of a complex variable, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k) \sigma_3(n-k),$$

where $\sigma_i(k)$ is the sum of the *i*th powers of the divisors of k. (For example, for n=5 this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1],$$

which you are welcome to check if you run out of things to do.)

Class format: Interactive lecture

Prerequisites: Functions of a complex variable; in particular, Liouville's theorem

Homework: Optional

Multiplicative Functions (Mark, 2 days)

Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that f(mn) = f(m)f(n) whenever gcd(m,n) = 1. There is an interesting operation, related to multiplication of series, on the set of all such multiplicative functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 6860000000000 by hand in a minute or so, you should consider this class.

Class format: Interactive lecture

Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is not needed.)

Homework: Optional

Quadratic Reciprocity (, Mark, 2 days)

Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

- (1) Is q a square modulo p?
- (2) Is p a square modulo q?

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Class format: Interactive lecture

Prerequisites: Some basic number theory (if you know Fermat's Little Theorem, you should be OK)

Homework: Optional

The Cayley-Hamilton Theorem (, Mark, 1 day)

Take any square matrix A and look at its characteristic polynomial f(X) = det(A - XI) (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4-by-4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the classical adjoint of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Class format: Interactive lecture

Prerequisites: Linear algebra, including a solid grasp of determinants (the Magic of Determinants class would definitely take care of that).

Homework: None

The Riemann Zeta Function (A. Mark, 3 days)

Many highly qualified people believe that the most important open question in pure mathematics is the Riemann hypothesis, a conjecture about the zeros of the Riemann zeta function. Having been stated in 1859, the conjecture has outlived not only Riemann and his contemporaries, but a few generations of mathematicians beyond, and not for lack of effort! So what's the zeta function, and what's the conjecture? By the end of this class you should have a pretty good idea. You'll also have seen a variety of related cool things, such as the probability that a random positive integer is not divisible by a perfect square (beyond 1) and the reason that -691/2730 is a useful and interesting number.

Class format: Interactive lecture

Prerequisites: Some single-variable calculus (including integration by parts) and some familiarity with complex numbers and infinite series; in particular, geometric series.

Homework: Optional

The Magic of Determinants ($\mathbb{A} \to \mathbb{A} \to \mathbb{A}$), Mark, 3–4 days)

This year's linear algebra class didn't cover determinants. If that left you feeling dissatisfied, either about not really having a good definition (by the way, using the Laplace expansion, while it is computationally often handy, leads to a miserable definition with no intuitive basis at all) or about not having seen many of the properties that determinants have, this may be a good class for you. If all goes well, we'll give a definition of determinant that's both motivated and rigorous, and there will be proofs of all its main properties (such as Laplace expansion), as well as a few applications such as general formulas for the inverse of a matrix and for the solution of n linear equations in n unknowns (Cramer's Rule).

Class format: Interactive lecture

Prerequisites: Some linear algebra, including linear transformations and matrix multiplication

Homework: Optional

Wedderburn's Theorem (A.A., Mark, 1 day)

Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over \mathbb{R} , with basis 1, i, j, k and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$$
 .

Have you seen any examples of *finite* division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Class format: Interactive lecture

Prerequisites: Some group theory and some ring theory; familiarity with complex roots of unity would help.

Homework: None

MILAN HAIMAN'S CLASSES

Entropy (, Milan Haiman, 4 days)

Suppose Alice rolls a standard dice N times and wishes to send Bob a binary string encoding the result. With an optimal strategy, what is the minimum expected number of bits needed for the message? In

this class we will introduce the concept of entropy to answer this question, and then we will use entropy to prove various results that are seemingly unrelated. For example, consider the following setup: Let $G \subseteq V \times V$ be a directed graph on V and let

$$A = \{(x, y, z) \in V^3 : (x, y), (y, z), (z, x) \in G\},\$$
$$B = \{(x, y, z) \in V^3 : (x, y), (x, z) \in G\}.$$

It turns out that we always have $|A| \leq |B|$, but the only proof I know uses entropy!

Class format: Lecture

Prerequisites: Nothing required. Knowing a bit about random variables and graph theory may be

helpful.

Homework: Recommended

NARMADA'S CLASSES

Analysis(Linear Algebra) = Geometry (, Narmada, 1 day)

This is a chill introduction to normed vector spaces. We'll show that every norm on a vector space (like \mathbb{R}^n) is secretly just a special convex set!

Class format: Interactive lecture

Prerequisites: Know what a vector space is; Know what a convex set it; Know the standard norm on \mathbb{R}^n

Homework: Optional

Borel Combinatorics (A.A., Narmada, 2–4 days)

Many graphs that arise in nature (i.e. mathematics) are infinite and live in topological spaces like \mathbb{R}^n . (Think of the integer grid \mathbb{Z}^n as a graph.) In the world of finite graphs, any bipartite graph is 2-colorable, and the Axiom of Choice lets us extend this to the world of infinite graphs. But what if we actually wanted to *construct* this 2-coloring? What would it mean for a coloring to be constructible? This is the heart of Borel combinatorics (an offshoot of descriptive set theory). In this class, we'll see some of the first ever results about colourings of Borel graphs!

Class format: None.

Prerequisites: Know what open and closed sets are (Not just open and closed intervals! A good way to test this knowledge is to find two ways to prove that \mathbb{Q} is not an open set.) Intro to Graph Theory.

Homework: Recommended

Outro to Group Theory (, Narmada, 2 days)

This is the last group theory class you will ever want to take! We'll learn about the Burnside problems, one of which is a long-standing open conjecture sometimes called the Fermat's Last theorem of group theory. (Don't look up the problems if you want to take this class!) Be prepared to leave the world of finite groups behind and see just how badly behaved infinite groups can be.

Class format: Interactive lecture Prerequisites: Intro to Group Theory

Homework: Optional

NIKITA'S CLASSES

Communication Complexity ($\mathbb{A} \to \mathbb{A} \to \mathbb{A}$), Nikita, 1 day)

We didn't have enough time in my hat puzzles class to talk about communication complexity – a field in computer science that studies how hard it is to compute a function whose inputs are distributed among several parties, when communication between them is limited. In this class, we'll solve the following problem: The tsar writes a 1000-digit number on each prisoner's hat. The prisoners take turns: on each turn, a prisoner announces a digit. After 100 turns, they must collectively decide which of the two numbers is bigger.

Devise a strategy that ensures they guess correctly with at least 99% probability.

Class format: Interactive lecture

Prerequisites: Preferably my hat puzzles class

Homework: Recommended

Debunking the Bunkbed Conjecture (A.A.A., Nikita, 1–2 days)

Continuing our discussion on percolation, I'll present our counterexample to the Bunkbed Conjecture and explain its construction.

Class format: Lecture
Prerequisites: Percolation
Homework: Recommended

Geometric Matryoshka ($\mathbb{A} \to \mathbb{A}$), Nikita, 1 day)

We will see a sequence of classical results in Euclidean geometry increasing in generality, finishing with point addition on elliptic curves.

Class format: Lecture

Prerequisites: Knowing projective geometry is preferred but not needed

Homework: None

Markov Processes (A.A., Nikita, 1–2 days)

Continuing our discussion on percolation, this class will introduce the Markov property — the principle that the future evolution of a system depends only on its present state, not its past.

We'll solve the following problem:

You shuffle a deck of cards. You draw cards until you get your first spade, then draw one more card. What is the probability that this extra card is also a spade?

Time permitting, we'll also explore applications of Markov processes to percolation and discuss why site (vertex) percolation >> bond (edge) percolation.

Class format: Lecture
Prerequisites: Percolation
Homework: Recommended

Mathematics of LessWrong (, Nikita, 1–4 days)

LessWrong is a nerdy forum with people interested in rationality and AI safety/alignment. It was a source of some developments in mathematics as well. While thinking of the hard problem of controlling

AI that is much smarter than you, lesswrongers made progress in mathematics of decision theory and these advances are interesting in its own right.

Class format: None.

Prerequisites: None.

Homework: Optional

On Sound (, Nikita, 1 day)

I'll tell you all I know about sounds, in particular about sounds in music theory. Disclaimer: I don't know much and I don't even play music. We will see why the fact that the octave is 12 half-tones is connected to the fact that $\ln_2(3) \approx \frac{19}{12}$

Class format: Lecture Prerequisites: None. Homework: None

None's Classes

Spot It! (Affine Plane Edition) (None, 1 day)

In his Finite Geometries class in week 2, Misha asserted that the Spot It! deck is isomorphic to a projective plane of order 7 with two points missing, so you can find an *affine* plane of order 7 inside it.

Misha won't be there in week 5, so you will have to do the hard work yourself. In this class, you will be provided with a Spot It! deck and a bit of information about finite geometries. Your job is to identify the 49 cards that make up the affine plane and arrange them into a 7×7 grid forming the ultimate combinatorial design.

If you are desperate, you can ask Misha for hints on Slack, but otherwise, there will be nobody teaching this class—you will be left to sink or swim on your own, with only your fellow campers to support you.

Class format: Extreme group work

Prerequisites: It will help if a few people in the class have seen finite affine planes before, but not everyone has to have this prerequisite:)

Homework: Optional

PRESTON BUSHNELL'S CLASSES

Purple's Classes

Hydrate or Diedrate (, Purple, 1–1000000000 days)

Glug glug.

Class format: Very participatory

Prerequisites: Strong comfortability with the mechanics of drinking water

Homework: Required

Linear Logic, Quantum Mechanics, Spacetime Manifolds: Three Sides of the Same Coin? (A.A.A., Purple, 4 days)

The theory of string diagrams (in compact closed categories, but I'll leave this formalism to optional homework) is the basis of a surprising and far-reaching analogy between proofs in linear logic, linear operators on Hilbert spaces, cobordisms between manifolds, and more. The idea of this course is to spend a day on each of these topics, motivating but not proving the essential properties in each case, and then only give the underlying ideas once we have seen enough examples for motivation. (My intention is not that this will be a physics course. I am certainly not a physicist and my understanding is that these ideas, while mathematically elegant, really have no experimental support. I just want to share some beautiful math which happens to have a historical connection to physics.)

Class format: Interactive lecture

Prerequisites: Intro linear algebra (for the quantum mechanics day)

Homework: Recommended

Machines = Categories = intuitioniStic Proof (, Purple, 4 days)

When you program a computer, to produce a function outputting a tuple of integers, you produce a pair of functions each outputting an integer.

When you work in a category, to produce a morphism into a cartesian product, you produce a pair of morphisms each into one of the factors.

When you do mathematics, to produce a proof of a conjunctive statement A and B, you produce a pair of proofs each of one of the conjuncts.

Wait. Are programming, category theory, and mathematical proof three instances of the same thing?

So says Curry-Howard-Lambek!

Class format: Interactive lecture

Prerequisites: The definition of category, functor, and categorical product

Homework: Recommended

Monads in the Bicategory of Spans (A.A., Purple, 1 day)

You might sometimes hear category theory nerds say gibberish phrases like that. But what do they mean? In this class I'll speedrun category theory definitions until we can make sense of "monads in the bicategory of spans." If there's time, we'll see what that has to do with burritos. (Hint: really nothing, that was Della's class last year, or Glenn's class this year.)

[This course will pick up from where the last block left off, exactly one year ago. If you have somehow not internalized every word that was spoken in that block, speak to Della or Purple:).]

Class format: Performance art (aka lecture)

Prerequisites: Della's class from last year, or talk to us to get notes: the definitions of category, functor, natural transformation, monoidal category, endofunctor category, monoid object, monad

Homework: None

The Fundamental Group Redux (, Purple, 4 days)

This course is an introduction to algebraic topology, in particular the fundamental group, but more rigorous than my last introduction to the fundamental group from week one. We will discuss homotopy, the fundamental group, the Seifert-van Kampen theorem, and the theory of covering spaces. If we have time and students are interested, we might spend a day on homology.

Class format: IBL

Prerequisites: Comfortability with the definitions of topological space and continuous map; comfortability with group theory including subgroups, group homomorphisms, normal subgroups, quotient groups, product groups.

Homework: Recommended

a cute combinatorics problem from my research! (A A > A A A). Purple, 1–4 days)

My research is in pretty abstract stuff, but it can sometimes be reduced to concrete questions of combinatorics. In this class we will discuss saturated transfer systems, a relatively straightforward object whose enumeration has surprising connections to homotopy theory and the classification of 4-manifolds (don't worry, we won't talk about these connections except potentially during TAU). In the short version of this class, we will just talk about these objects and several open questions related to them. In the long version of this class, we will work through the proof of a forthcoming (joint with several collaborators) result on their enumeration, which also happens to answer an open question first posed by Donald Knuth.

Class format: IBL

Prerequisites: Know what a binomial coefficient is

Homework: Recommended

RILEY'S CLASSES

AP Calculus Abridged (and Revised?) ($\mathbb{A} \to \mathbb{A}$, Riley, 2–4 days)

I will quickly run through the units of AP calculus, commenting on the curriculum and pedagogy. I will compare and contrast the intended methods, explanations, and important ideas with their more rigorous counterparts.

Class format: Interactive Lecture

Prerequisites: No, although you would get more out of it if you have seen epsilon-delta definitions

Homework: Optional

The Lebesgue Differentiation Theorem ($\mathbb{A} \to \mathbb{A} \to \mathbb{A} \to \mathbb{A}$, Riley, 2–4 days)

I want to prove the Lebesgue differentiation theorem in this class. I will probably spend some time on Vitali coverings and the Hardy-Littlewood maximal function. There may be other things to cover beforehand but I'm not too sure. Basically I want this class to be a "no punched pulled" analysis class.

Class format: Lecture

Prerequisites: Having seen some analysis and measure theory will be useful.

Homework: Optional

What Math Teachers are Taught (M, Riley, 1–4 days)

This is a class where students read and discuss materials that middle and high school math teachers (at least, at Portland State University) get taught. Every day except the first day students will read some article before class and discuss it in class. Suggested topics include the following: what it means to do mathematics, teaching math for social justice, the place of definitions in the math classroom. Topics may be changed depending on student interest

Class format: Discussion

Prerequisites: None

Homework: Required

Sam's Classes

How I Solved the Navier-Stokes Equation (, Sam, 1 day)

How I got \$1,000,000.

Class format: Lecture

Prerequisites: Banach spaces & Hilbert spaces, Advanced Functional Analysis, Advanced Measure Theory, Differential Geometry, Sobolev spaces, Sobolev embeddings, Analystic Theory of Parabolic PDEs, willingness to cry, despair and listen to a true British geezer

Homework: None

Into the Quantum World of Computers (, Sam, 4 days)

It seems these days, that the words 'quantum computing' are buzz words that get thrown around everywhere, mostly by influencers on LinkedIn that don't have a clue what they're talking about... However, in this class, we will take a quick dip into the quantum world to explore the cool sort of things that a quantum computer can do, from Shor's algorithm for polynomial time decryption to faster than light propogation of information. Not only this, but we will get to use actual quantum computer :o (yes, they already exist!)

Class format: Interactive class with computer sessions

Prerequisites: Knowledge of very basic linear algebra for \mathbb{R}^n , like matrices.

Homework: Recommended

Invert Me Up Scotty ($\mathbb{A} \mathbb{A} \to \mathbb{A} \mathbb{A} \mathbb{A}$), Sam, 5 days)

The field of inverse problems offers us a powerful way to understand problems ranging from the famous example of the blurring of images from the Hubble space telescope, to matrix inversion, to x-ray and hydraulic tomography. You have probably come across a few examples of problems that have a unique inverse, but what do we do if a function doesn't have unique inverses? For example if $A \in \mathbb{R}^{n \times n}$ such that $\det(A) = 0$, then for some $\mathbf{y} \in \mathbb{R}^n$ there might be multiple $\mathbf{x} \in \mathbb{R}^n$ that satisfy $A\mathbf{x} = \mathbf{y}$. In this case, how do we calculate an inverse of the problem? These are all questions that will be answered in this course.

Class format: Interactive Class

Prerequisites: Linear Algebra (That covered by the Intro To Linear Algebra course) - vector spaces, basis of a vector space, linear functions. Some exposure to inner products and norms on vector spaces would be helpful but not required (add an extra chilli if you don't have these prereqs). Exposure to open sets, closed sets and compact sets before.

Homework: Recommended

STEVE'S CLASSES

Free Lattices ($\mathbb{A} \to \mathbb{A} \to \mathbb{A} \to \mathbb{A}$), Steve, 2 days)

Given a set X, the *free group* on X is just the group of words built from elements of X; e.g. if $X = \{a, b\}$ then the free group on X consists of things like

$$aaba^{-1}bbbbbab^{-1}b^{-1}b^{-1}$$

which are multiplied as you would expect (so $aab^{-1}a^{-1}$ times abbaba is aababa). Crucially, free groups are *pretty dang boring*.

However, this can lull the unwary into a false sense of security! "Free —s" make sense for a wide class of —s, and in general free —s are messy as all-get-out. In this class we'll look at the particular case of free *lattices*. A lattice is a set equipped with operators "\" and "\"," which behave like "greatest lower bound" and "least upper bound" in posets. It turns out that free lattices are extremely complicated, and discovering how to do computations in them was a substantial result proved in the early forties.

We'll introduce the basics of free lattices. We'll also gain a complete understanding of free _complete_lattices, which are the analogues of free lattices but we let \land and \lor take arbitrarily large sets of elements as inputs. *(HINT: there are very few free complete lattices.)*

Class format: Lecture

Prerequisites: Group theory (should know what a free group is, partial orders (should know what glb/lub in a partial order are)

Homework: Recommended

Inner Model Theory (A.A., Steve, 2–4 days)

Let's fix a really REALLY long well-ordering α . The "full set-theoretic universe" (of height α) is built recursively as follows: we let $V_0 = \emptyset$, $V_{\gamma+1} = V_{\gamma} \cup \mathcal{P}(V_{\gamma})$ for $\gamma + 1 < \alpha$, and $V_{\lambda} = \bigcup_{\gamma < \lambda} V_{\gamma}$ for $\lambda < \alpha$ a limit. Basically, we're just recursively taking powersets, starting with the emptyset.

This is basically what set theorists think of as "the universe of mathematics," and indeed basically anything you can whip up is implementable in this set-up. However, it's also extremely dependent on \mathcal{P} , which is a rather finicky operation: if someone hands you something they claim is $\mathcal{P}(\mathbb{N})$, how sure are you *exactly* that they've got all the sets of natural numbers in there? This means that, unless we assume strong hypotheses, there's very little we can outright prove about this universe. If we replace \mathcal{P} by a "smaller" operator, it turns out that we get an alternative universe of mathematics in which much more is "nailed down" so to speak. In particular, this universe - called "L" - is one in which the continuum hypothesis and the axiom of choice are outright provable, regardless of their status in "reality."

In this class we'll explore the shape and behavior of L! Additionally, we'll see a sketch of an argument using L that the *failure* of the axiom of choice is consistent with the usual rules of (choice-free) set theory. This won't involve any cheating like we saw in "Breaking the axiom of choice," but it will require you to take a certain technical statement on faith.

Class format: Lecture

Prerequisites: Basic model theory (the first bit of Aaron's class or its equivalent); specifically, students should know what a first-order sentence is.

Homework: None

My Favorite Shape ($\mathbb{A} \to \mathbb{A} \to \mathbb{A}$), Steve, 1 day)

A *planar continuum* is just a compact (= closed and bounded) and connected subset of the plane. You might think that such objects aren't very crazy, or at least aren't crazy *everywhere* (e.g. the topologist's sine curve is a planar continuum and is a bit wonky, but it's basically two "nice" pieces connected in a wonky way). You would be wrong.

The *pseudoarc* Ψ is the answer to the question: what if scissors didn't work? Specifically, it is a nontrivial (= at least two points) planar continuum which cannot be split into two pieces: if $\Psi = A \cup B$ where A, B are planar continua then either A or B is already all of Ψ . This is very silly. Let's build it!

Class format: Lecture Prerequisites: None Homework: None

This Class ($AAA \rightarrow AAA$, Steve, 4 days)

In this class, we will cover the material covered by this class. The teacher teaching this class, along with the staff member responsible for writing this blurb, will instruct those students who choose to study this material.

Translation from Steve-ish: this is a class on self-reference in mathematics, specifically self-referential sentences in the context of arithmetic. If you've heard of Godel's incompleteness theorem, that's the sort of jazz we'll be listening to hear! If there's a subfield of math that feels more like a pun than this, I'm not aware of it.

After blackboxing some important but tedious technicalities (and I'll provide handouts of these that folks interested can work through at TAU), we'll cover basically everything I know about self-reference; besides Godel's incompleteness theorem itself, we'll see Lob's theorem ("This sentence is provable" is definitely provable), Kreisel's counterexample (... unless you write it stupidly), and at least one open question (due to me because I really want someone to solve it someday).

Class format: Lecture Prerequisites: None Homework: None

Susan's Classes

Building Division Rings (A.A., Susan, 1–2 days)

Suppose you have a commutative ring and you want to make it into a division ring. So you do the obvious thing and add inverses for all of its elements, close the structure under the ring operations, and see what happens. Sometimes, this produces a division ring. That's nice. But other things could happen, too.

Sometimes this violates Rule 4 and everything goes catastrophically wrong. Sometimes you don't violate Rule 4 at all, but things go catastrophically wrong anyway for an entirely different reason. Sometimes this doesn't break anything, but it also doesn't work... the first time. Sometimes it works the second time. And sometimes it only works once you've done it infinitely many times.

In this class we'll talk about as much of the theory of this process as we can get our hands on in the time we've got.

Class format: None. Prerequisites: None.

Homework: Recommended

Continued fractions are a tool we can use to approximate irrational numbers. But some irrational numbers are irrationaler than others. And the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ is as irrational as it is possible to be. Come to this class to find out why!

Class format: Interactive Lecture

Prerequisites: None

Homework: Recommended

Infinite Ramsey Theory (A.S., Susan, 1–2 days)

Suppose you throw a party and invite six people, and some of those people know each other and some don't. A well known result from graph theory tells us that we can guarantee a group of three mutual friends, or three mutual strangers. But suppose you wanted to throw a much, much bigger party, with infinitely many attendees? Can you guarantee an infinite group of friends or strangers? Come to this class to find out!

Class format: Interactive Lecture

Prerequisites: None

Homework: Recommended

Susan Teaches Ben's Class (, Susan, 1 day)

If this class runs, Ben will email Susan a slide deck five minutes before class starts. Good luck, Susan!

Class format: Revenge slideshow telephone

Prerequisites: None. There is already too much risk of Susan figuring things out.

Homework: Recommended

ZACH'S CLASSES

How many trees exist with vertices $\{1, 2, ..., n\}$? The (perhaps surprising) answer is n^{n-2} , and there are soooo many neat ways to show this! Last year we filled 3 days with many different proofs, but I still have more! Let's look at two more proofs: one using a pure bijection that turns trees inside-out at a parking lot, and another that runs backwards from infinity.

Last year's class is *not* a prereq.

Class format: Lecture Prerequisites: None. Homework: Optional

Generating PIE (A), Zach, 1 day)

In this talk we will learn a simple recipe for PIE—the Principle of Inclusion-Exclusion—whose main ingredient is Generating Functions. Mix together one dash of theoretical derivation linking the topics, a generous handful of examples (preferably tough for proper consistency), a few leaves from your favorite labelled tree, and a pinch of humor (to taste). Let sit for an hour, and enjoy! This PIE recipe is sugar-free but hopefully still pretty sweet.

Class format: Lecture Prerequisites: None. Homework: None

Permuting Conditionally Convergent Series (A.A., Zach, 2 days)

The starting point for this class is the famous Riemann's Rearrangement Theorem: given any conditionally convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers—that is, the sum converges but $\sum_{n=1}^{\infty} |a_n|$ does not—you can permute the terms so that the new sum converges to any desired value. But what if the a_n are instead allowed to be *complex*? Can you permute to obtain any *complex* sum? Not always: for the sequence $a_n = (-1)^n \cdot i/n$, the achievable sums lie on the line $i \cdot \mathbb{R} \subset \mathbb{C}$. What other subsets of \mathbb{C} arise in this way? And what about series in \mathbb{R}^k ? We'll discuss and prove this beautiful classification.

We'll also consider the problem from the permutations' perspective: what are the permutations on \mathbb{N} that always turn convergent series into different convergent series (not necessarily with the same sum)? Amazingly, there are some permutations that do better than \mathbb{N} itself: they turn convergent series into convergent series, but they also transform some *divergent* series into convergent ones!

Class format: Lecture

Prerequisites: Prereq: ϵ - δ definition of limit

Homework: Recommended

Summation by Parts on the Battlefield (A.A., Zach, 1 day)

Integration by parts provides a handy method to integrate a product $f(x) \cdot g(x)$ by instead considering the integral of $f'(x) \cdot G(x)$, where f' is the derivative of f and G is an antiderivative of g. Its discrete analog, known as "Summation by Parts" or "Abel Summation", deals instead with differences and partial sums, and it is a simple but unusually powerful weapon to have in your arsenal—especially, perhaps, when doing battle with some olympiad problems.

In this class you will receive basic weapons training, learning how to use this innocuous-seeming hand grenade to blow some enemies (enemy sums?) into, well, parts. You will also learn to recognize when such a maneuver could be add-vantageous. Finally, we will apply this training in simulated combat against incomplete units of Egyption fractions.

Class format: Lecture Prerequisites: N/A Homework: Optional

 $^{^3}$ Yes, it is really called "Abel Summation". I did not just name it after myself. Some Norwegian mathematician named it after me instead...